§7.8 HW Solutions

#1 a) $A = \begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}$ $\quad \det(A-\lambda I) = \det\begin{pmatrix}3-\lambda & -4\\ 1 & -1-\lambda\end{pmatrix} = (3-\lambda)(-1-\lambda)+4$

$$= \lambda^2 - 2\lambda + 1 = (\lambda-1)^2.$$

$\lambda = 1$: $(A-I)\vec\xi = \vec 0$ $\quad \begin{pmatrix}2 & -4\\ 1 & -2\end{pmatrix}\vec\xi = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad \vec\xi = \begin{pmatrix}2\\1\end{pmatrix}$.

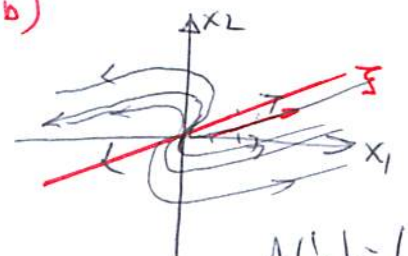
$\vec x^{(1)} = \begin{pmatrix}2\\1\end{pmatrix}e^t$

$\vec x^{(2)} = \begin{pmatrix}2\\1\end{pmatrix}te^t + \vec\eta e^t$ $\quad$ where $(A-I)\vec\eta = \begin{pmatrix}2\\1\end{pmatrix}$

$\left(\begin{array}{cc|c}2 & -4 & 2\\ 1 & -2 & 1\end{array}\right) \to \begin{pmatrix}1 & -2 & 1\\ 0 & 0 & 0\end{pmatrix}$ $\quad \vec\eta = \begin{pmatrix}1+2\eta_2\\ \eta_2\end{pmatrix} = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\vec\eta} + \eta_2\begin{pmatrix}2\\1\end{pmatrix}$

so $\vec x^{(2)} = \begin{pmatrix}2\\1\end{pmatrix}te^t + \begin{pmatrix}1\\0\end{pmatrix}e^t$.

b)



all solutions become parallel to the line from $\begin{pmatrix}0\\0\end{pmatrix}$ to $\begin{pmatrix}2\\1\end{pmatrix}$.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$

$A\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$

#7 b) $\vec x' = \begin{pmatrix}1 & -4\\ 4 & -7\end{pmatrix}\vec x$, $\quad \vec x(0) = \begin{pmatrix}3\\2\end{pmatrix}$

$\det(A-\lambda I) = \det\begin{pmatrix}1-\lambda & -4\\ 4 & -7-\lambda\end{pmatrix} = (1-\lambda)(-7-\lambda)+16 = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0.$

$\lambda = -3$: $(A+3I)\vec\xi = \begin{pmatrix}4 & -4\\ 4 & -4\end{pmatrix}\vec\xi = \begin{pmatrix}0\\0\end{pmatrix}$ $\quad \vec\xi = \begin{pmatrix}1\\1\end{pmatrix}$ $\quad \vec x^{(1)} = \begin{pmatrix}1\\1\end{pmatrix}e^{-3t}$

$\vec x^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \eta e^{-3t}$

$(A+3I)\vec\eta = \vec\xi^{(1)}$ $\quad \begin{pmatrix}4 & -4 & 1\\ 4 & -4 & 1\end{pmatrix} \to \begin{pmatrix}4 & -4 & 1\\ 0 & 0 & 0\end{pmatrix}$ $\quad \vec\eta = \begin{pmatrix}\frac14+\eta_2\\ \eta_2\end{pmatrix}$

$$= \underbrace{\begin{pmatrix}1/4\\0\end{pmatrix}}_{\vec\eta} + \eta_2\begin{pmatrix}1\\1\end{pmatrix}.$$

$\vec x^{(2)} = \begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1/4\\0\end{pmatrix}e^{-3t}.$

$\vec x(t) = c_1\vec x^{(1)} + c_2\vec x^{(2)}$ $\quad$: @ $t=0$: $\begin{pmatrix}3\\2\end{pmatrix} = \vec x(0) = c_1\begin{pmatrix}1/4\\0\end{pmatrix} + c_2\begin{pmatrix}1\\1\end{pmatrix}$

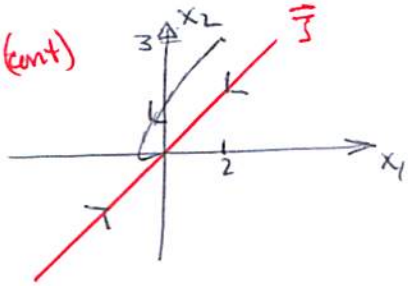
$3 = \frac{c_1}{4} + c_2 \qquad \frac{c_1}{4} = 1 \Rightarrow c_1 = 4.$

$2 = c_2 \Rightarrow c_2 = 2$

$\vec x(t) = 4\vec x^{(1)}(t) + 2\vec x^{(2)}(t)$

$= 4\begin{pmatrix}1\\1\end{pmatrix}e^{-3t} + 2\left(\begin{pmatrix}1\\1\end{pmatrix}te^{-3t} + \begin{pmatrix}1/4\\0\end{pmatrix}e^{-3t}\right) = e^{-3t}\begin{pmatrix}4+2t+\frac12\\ 4+2t\end{pmatrix} = e^{-3t}\begin{pmatrix}2t+\frac92\\ 2t+4\end{pmatrix}$

7b (cont)



\#11 a.) $A = \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 6 & 2 \end{pmatrix}$

Note: triangular, so e-values are the diagonal entries.

$\lambda_1 = \lambda_2 = 1$: $(A - I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 0 & 0 & 0 \\ -4 & 0 & 0 \\ 3 & 6 & 1 \end{pmatrix} \vec{\xi} = \vec{0} \Rightarrow \vec{\xi} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

$(A - I)\vec{\eta} = \vec{\xi}$: $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 1 \\ 3 & 6 & 1 & -6 \end{array}\right) \Rightarrow \vec{\eta} = \begin{pmatrix} -\frac{1}{4} \\ 0 \\ -21/4 \end{pmatrix} + \eta_2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix}$

$\therefore \vec{x}^{(1)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t$

$\vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t.$

$\lambda_3 = 2$: $(A - 2I)\vec{\xi} = \vec{0}$: $\begin{pmatrix} -1 & 0 & 0 \\ -4 & -1 & 0 \\ 3 & 6 & 0 \end{pmatrix} \vec{\xi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{\xi} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \eta_2.$

$\therefore \vec{x}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}.$

General soln: $\vec{x} = c_1 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t + c_2 \left(\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t \right) + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{2t}$

$\begin{pmatrix} -1 \\ 2 \\ -30 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} + c_2 \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$-1 = -\frac{1}{4} c_2 \Longrightarrow c_2 = 4$

$2 = c_1 \Longrightarrow c_1 = 2$

$-30 = -6c_1 - \frac{21}{4} c_2 + c_3 \Longrightarrow c_3 = -30 + 6c_1 + \frac{21}{4} c_2$

$= -30 + 12 + 21 = 3.$

$\therefore \vec{x}(t) = 2 \begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} e^t + 4 \left(\begin{pmatrix} 0 \\ 1 \\ -6 \end{pmatrix} te^t + \begin{pmatrix} -1/4 \\ 0 \\ -21/4 \end{pmatrix} e^t \right) + 3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^t$

$= e^t \begin{pmatrix} -0 \quad -1+0 \\ 2 + 4t \quad +0 \\ -12 - 24t_{-21} + 3 \end{pmatrix} = \cancel{\begin{pmatrix} -1 \\ 4t+2 \\ -24t - 3 \end{pmatrix}}$

$= e^t \begin{pmatrix} -1 \\ 2 + 4t \\ -30 - 24t \end{pmatrix}$