Homework Solutions - §4.3

#12. $y^{(4)} + 2y''' + y'' + 8y' - 12y = 12\sin t - e^{-t}$ $\quad y(0)=3,\ y'(0)=0,\ y''(0)=-1,\ y'''(0)=2$

Homog: $r^4 + 2r^3 + r^2 + 8r - 12 = (r-1)(r^3+3r^2+4r+12) = (r-1)(r+3)(r^2+4) = 0$

$r = 1, r = -3, r = \pm 2i \Rightarrow y_1 = e^t,\ y_2 = e^{-3t},\ y_3 = \cos 2t,\ y_4 = \sin 2t$

Undet. Coef: $g_1(t) = 12\sin t \Rightarrow Y_1 = A\cos t + B\sin t$

$Y_1' = -A\sin t + B\cos t$

$Y_1'' = -A\cos t - B\sin t$

$Y_1''' = A\sin t - B\cos t$

$Y_1^{(4)} = A\cos t + B\sin t$

$Y_1^{(4)} + 2Y_1''' + Y_1'' + 8Y_1' - 12Y_1 = (A - 2B - A + 8B - 12A)\cos t + (B + 2A - B - 8A - 12B)\sin t$

$= (-12A + 6B)\cos t + (-6A - 12B)\sin t$

$= 12\sin t$

cost: $(-12A + 6B = 0) \times 2$

sint: $-6A - 12B = 12$

$-30A = 12 \Rightarrow A = -\frac{12}{30} = -\frac{2}{5} \qquad B = \frac{12A}{6} = 2A = -\frac{4}{5}$

$\therefore Y_1 = -\frac{2}{5}\cos t - \frac{4}{5}\sin t.$

$g_2(t) = -e^{-t} \Rightarrow Y_2 = Ae^{-t}$

$Y_2' = -Ae^{-t},\ Y_2'' = Ae^{-t},\ Y_2''' = -Ae^{-t},\ Y_2^{(4)} = Ae^{-t}$

$Y_2^{(4)} + 2Y_2''' + Y_2'' + 8Y_2' - 12Y_2 = (A - 2A + A - 8A - 12A)e^{-t} = -20Ae^{-t}$

$= -e^{-t}$

e^{-t}: $-20A = -1 \Rightarrow A = \frac{1}{20} \qquad \therefore Y_2 = \frac{1}{20}e^{-t}.$

General Solution: $y = c_1e^t + c_2e^{-3t} + c_3\cos(2t) + c_4\sin(2t) - \frac{2}{5}\cos(t) - \frac{4}{5}\sin(t) + \frac{1}{20}e^{-t}$

$y' = c_1e^t - 3c_2e^{-3t} - 2c_3\sin(2t) + 2c_4\cos(2t) + \frac{2}{5}\sin t - \frac{4}{5}\cos t - \frac{1}{20}e^{-t}$

$y'' = c_1e^t + 9c_2e^{-3t} - 4c_3\cos(2t) - 4c_4\sin(2t) + \frac{2}{5}\cos t + \frac{4}{5}\sin t + \frac{1}{20}e^{-t}$

$y''' = c_1e^t - 27c_2e^{-3t} + 8c_3\sin(2t) - 8c_4\cos(2t) - \frac{2}{5}\sin t + \frac{4}{5}\cos t - \frac{1}{20}e^{-t}$

$y(0) = c_1 + c_2 + c_3 - \frac{2}{5} + \frac{1}{20} = 3 + \frac{2}{20} = \frac{8/20}{20}$

$y'(0) = c_1 - 3c_2 + 2c_4 - \frac{4}{5} - \frac{1}{20} = 0 + \frac{12}{20} = \frac{17}{20}$

$y''(0) = c_1 + 9c_2 - 4c_3 + \frac{2}{5} + \frac{1}{20} = -1 - \frac{5}{20} = \frac{-29}{20}$

$y'''(0) = c_1 - 27c_2 - 8c_4 + \frac{4}{5} - \frac{1}{20} = 2 + \frac{15}{20} = \frac{25}{20}$

$\Rightarrow \quad c_1 = \frac{81}{40},\ c_2 = \frac{73}{520},\ c_3 = \frac{77}{65},\ c_4 = -\frac{49}{130}$

Final Solution: $y = \frac{81}{40}e^t + \frac{73}{520}e^{-3t} + \frac{77}{65}\cos(2t) - \frac{49}{130}\sin(2t) - \frac{2}{5}\cos t - \frac{4}{5}\sin t + \frac{1}{20}e^{-t}$