Homework Solutions - §3.6

#11. $y'' - 5y' + 6y = g(t)$

Homog: $r^2 - 5r + 6 = (r-3)(r-2) = 0 \quad \therefore \; y_1 = e^{3t}, \; y_2 = e^{2t}$.

Var. of Param: $y_p = u_1 e^{3t} + u_2 e^{2t}$ where $\begin{bmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$.

Row 2 − 2·Row 1 gives $e^{3t} u_1' = g(t)$ so $u_1' = g(t) e^{-3t} \Rightarrow u_1 = \int g(s) e^{-3s}\, ds$.

Row 2 − 3·Row 1 gives $-e^{2t} u_2' = g(t)$ so $u_2' = -g(t) e^{-2t} \Rightarrow u_2 = -\int g(s) e^{-2s}\, ds$.

Thus $y_p = e^{3t} \int_0^t g(s) e^{-3s}\, ds - e^{2t} \int^t g(s) e^{-2s}\, ds$ ← if you stop here, which is fine, you can leave everything in terms of t

[This means to evaluate the antiderivative at $s=t$.] $= \int^t g(s) \left(e^{3(t-s)} - e^{2(t-s)} \right) ds$

The general sol'n to the nonhomogeneous DE is:

$$y = c_1 e^{3t} + c_2 e^{2t} + \int^t g(s) \left(e^{3(t-s)} - e^{2(t-s)} \right) ds.$$

Verify y_1: $y_1 = y_1' = y_1'' = e^x$

$(1-x)y_1'' + xy_1' - y_1 = (1-x)e^x + xe^x - e^x = 0$.

Verify y_2: $y_2 = x, \; y_2' = 1, \; y_2'' = 0$

$(1-x)y_2'' + xy_2' - y_2 = (1-x)0 + x(1) - x = 0$

#19. $(1-x)y'' + xy' - y = g(x) \quad (0 < x < 1) \quad y_1 = e^x, \; y_2 = x$

std. form: $y'' + \frac{x}{1-x} y' - \frac{1}{1-x} y = \frac{g(x)}{1-x}$

Var. of Param: $y_p = u_1 e^x + x u_2$ where $\begin{bmatrix} e^x & x \\ e^x & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{g(x)}{1-x} \end{bmatrix}$

Row 2 − $\frac{1}{x}$·Row 1 gives: $\left(e^x - \frac{e^x}{x} \right) u_1' = \frac{g(x)}{1-x}$

$$\left(1 - \tfrac{1}{x}\right) e^x u_1' = \frac{g(x)}{1-x}$$

$$\frac{x-1}{x} e^x u_1' = \frac{g(x)}{1-x} \Rightarrow u_1' = \frac{x}{x-1} e^{-x} \frac{g(x)}{1-x} = -\frac{x e^{-x} g(x)}{(1-x)^2}$$

$$u_1(x) = \int^t -\frac{s e^{-s} g(s)}{(1-s)^2}\, ds$$

Row 2 − Row 1 gives $(1-x) u_2' = \frac{g(x)}{1-x} \Rightarrow u_2' = \frac{g(x)}{(1-x)^2}$

$$u_2(x) = \int^t \frac{g(s)}{(1-s)^2}\, ds$$

Thus $y_p = e^x \int^t -\frac{s e^{-s} g(s)}{(1-s)^2}\, ds + x \int^t \frac{g(s)}{(1-s)^2}\, ds = \int^t \left(x - s e^{x-s} \right) \frac{g(s)}{(1-s)^2}\, ds$

The gen'l sol'n to the nonhomogeneous DE is:

$$y = c_1 e^x + c_2 x + \int^t \left(x - s e^{x-s} \right) \frac{g(s)}{(1-s)^2}\, ds$$