Homework Solutions – §3.2

#14. $yy'' + (y')^2 = 0$, $t > 0$

To verify that $y_1 = 1$ is a solution:

$y_1' = 0$, $y_1'' = 0$ so $y_1 y_1'' + (y_1')^2 = 1\cdot 0 + 0^2 = 0$ ✓

To verify that $y_2 = t^{1/2}$ is a solution:

$y_2' = \frac{1}{2}t^{-1/2}$, $y_2'' = -\frac{1}{4}t^{-3/2}$ so $y_2 y_2'' + (y_2')^2 = t^{1/2}\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2$

$= -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$ ✓

To see if (when) $y = c_1 + c_2 t^{1/2}$ $\left(= c_1 y_1 + c_2 y_2\right)$ is a solution:

$y' = \frac{c_2}{2}t^{-1/2}$ $\quad y'' = -\frac{c_2}{4}t^{-3/2}$ so $yy'' + (y')^2 = (c_1 + c_2 t^{1/2})\left(-\frac{c_2}{4}t^{-3/2}\right) + \left(\frac{c_2}{2}t^{-1/2}\right)^2$

$= -\frac{c_1c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} = -\frac{c_1c_2}{4}t^{-3/2}$

The linear combinations of y_1 & y_2 are a solution only when $c_1 = 0$ or $c_2 = 0$.
This does not contradict the Principle of Superposition (Thm. 3.2.2) because the DE is nonlinear (the Thm. applies only to linear DEs).

#28. $y'' - y' - 2y = 0$

(a) To see that $y_1 = e^{-t}$ and $y_2 = e^{2t}$ are solutions we note that the characteristic equation is $r^2 - r - 2 = (r-2)(r+1) = 0$ iff $r = 2$ or $r = -1$.

To see that they are a fundamental set of solutions look at the Wronskian:

$$W[y_1, y_2] = \det\begin{pmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{pmatrix} = e^{-t}(2e^{2t}) - (-e^{-t})e^{2t} = 2e^{t} + e^{t} = 3e^{t} \neq 0.$$

(b) $y_3 = -2e^{2t} = 0y_1 - 2y_2$

$y_4 = y_1 + 2y_2$

$y_5 = 2y_1 - 2y_3 = 2y_1 + 4y_2$

Since all three of these functions are linear combinations of y_1 & y_2, they are solutions to the DE.

(c) To determine if 2 sol'ns form a fundamental set of solutions, look at their Wronskian:

$W[y_1, y_3] = \det\begin{pmatrix} e^{-t} & -2e^{2t} \\ -e^{-t} & -4e^{2t} \end{pmatrix} = -4e^{t} - 2e^{t} = -6e^{-t} \neq 0$ ∴ fundamental set of sol'ns.

$W[y_2, y_3] = \det\begin{pmatrix} e^{2t} & -2e^{2t} \\ 2e^{2t} & -4e^{2t} \end{pmatrix} = -4 + 4 = 0$ ∴ not a fund. set of sol'ns.

$W[y_1, y_4] = \det\begin{pmatrix} e^{-t} & e^{-t} + 2e^{2t} \\ -e^{-t} & -e^{-t} + 4e^{2t} \end{pmatrix} = -e^{-2t} + 4e^{t} - (-e^{-2t} - 2e^{t}) = 6e^{t} \neq 0$ ∴ fundamental set of sol'ns.

$W[y_4, y_5] = \det\begin{pmatrix} e^{-t} + 2e^{2t} & 2e^{-t} + 4e^{2t} \\ -e^{-t} + 4e^{2t} & -2e^{-t} + 8e^{2t} \end{pmatrix} = -2e^{-2t} + 8e^{t} - 4e^{t} + 16e^{4t} - (-2e^{-2t} - 4e^{t} + 8e^{t} + 16e^{4t})$

$= 0$ ∴ not a fund. set of sol'ns.