§1.3 HW Solutions

#9. $ty' - y = t^2$

To verify that $y(t) = 3t + t^2$ solves this DE, compute $y' = 3 + 2t$.

Then insert into the LHS:

$$ty' - y = t(3+2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2 \quad ✓$$

#12. $t^2 y'' + 5ty' + 4y = 0 \quad (t > 0)$ ← assures us that we won't divide by 0 or take ln of a negative or zero.

To verify that $y_1(t) = t^{-2}$ satisfies this DE, compute $y_1' = -2t^{-3}$, $y_1'' = 6t^{-4}$.

Then, inserting into the LHS:

$$t^2 y_1'' + 5ty_1' + 4y_1 = t^2(6t^{-4}) + 5t(-2t^{-3}) + 4t^{-2}$$
$$= 6t^{-2} - 10t^{-2} + 4t^{-2} = 0 \quad ✓$$

Likewise for $y_2 = t^{-2}\ln t$: $y_2' = t^{-2}\cdot\frac{1}{t} - 2t^{-3}\ln t = t^{-3}(1 - 2\ln t)$.

$$y_2'' = t^{-3}\left(-\frac{2}{t}\right) - 3t^{-4}(1 - 2\ln t) = t^{-4}(-5 + 6\ln t).$$

Inserting into the LHS:

$$t^2 y_2'' + 5ty_2' + 4y_2 = t^2\cdot t^{-4}(-5 + 6\ln t) + 5t\cdot t^{-3}(1 - 2\ln t) + 4t^{-2}\ln t$$
$$= t^{-2}(-5 + 6\ln t + 5 - 10\ln t + 4\ln t)$$
$$= 0. \quad ✓$$

#14. $y' - 2ty = 1$.

To verify that $y = e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}$ is a sol'n to this DE,

we compute $y' = e^{t^2}\left(e^{-t^2}\right) + 2te^{t^2}\int_0^t e^{-s^2}\,ds + 2te^{t^2}$

$$= 1 + 2te^{t^2}\int_0^t e^{-s^2}\,ds + 2te^{t^2}.$$

Recall (FTC): $\frac{d}{dt}\int_0^t f(s)\,ds = f(t)$

Now: $y' - 2ty = 1 + 2te^{t^2}\int_0^t e^{-s^2}\,ds + 2te^{t^2} - 2t\left(e^{t^2}\int_0^t e^{-s^2}\,ds + e^{t^2}\right) = 1.$ ✓

#28. $\alpha^2 u_{xx} = u_t$

Let $u(t,x) = \left(\frac{\pi}{t}\right)^{1/2} e^{-\frac{x^2}{4\alpha^2 t}}$. Then $u_t = \left(\frac{\pi}{t}\right)^{1/2} e^{-\frac{x^2}{4\alpha^2 t}}\left(\frac{x^2}{4\alpha^2 t^2}\right) + \frac{1}{2}\frac{\pi^{1/2}}{t^{3/2}} e^{-\frac{x^2}{4\alpha^2 t}}$

and $u_x = \left(\frac{\pi}{t}\right)^{1/2} e^{-\frac{x^2}{4\alpha^2 t}}\left(-\frac{2x}{4\alpha^2 t}\right) = -\frac{1}{2\alpha^2}\left(\frac{\pi^{1/2}}{t^{3/2}}\right) x e^{-\frac{x^2}{4\alpha^2 t}}$

and $u_{xx} = -\frac{1}{2\alpha^2}\frac{\pi^{1/2}}{t^{3/2}}\left(x e^{-x^2/4\alpha^2 t}\cdot\frac{-2x}{4\alpha^2 t} + 1\cdot e^{-\frac{x^2}{4\alpha^2 t}}\right) = -\frac{1}{2\alpha^2}\frac{\pi^{1/2}}{t^{3/2}}\left(\frac{-2x^2}{4\alpha^2 t} + 1\right) e^{-x^2/4\alpha^2 t}$

Then $\alpha^2 u_{xx} - u_t = \cancel{\alpha^2}\left(-\frac{1}{2\cancel{\alpha^2}}\frac{\pi^{1/2}}{t^{3/2}}\left(-\frac{x^2}{2\alpha^2 t} + 1\right)e^{-x^2/4\alpha^2 t}\right) - \left(\left(\frac{\pi}{t}\right)^{1/2} e^{-x^2/4\alpha^2 t}\left(\frac{x^2}{4\alpha^2 t}\right) + \frac{1}{2}\frac{\pi^{1/2}}{t^{3/2}} e^{-\frac{x^2}{4\alpha^2 t}}\right)$

$$= \frac{\pi^{1/2}}{4t^{3/2}}\frac{x^2}{\alpha^2 t} e^{-x^2/4\alpha^2 t} - \frac{\pi^{1/2}}{2t^{3/2}} e^{-x^2/4\alpha^2 t} - \frac{\pi^{1/2}x^2}{4\alpha^2 t^{5/2}} e^{-x^2/4\alpha^2 t} + \frac{\pi^{1/2}}{2t^{3/2}} e^{-\frac{x^2}{4\alpha^2 t}}$$

$= 0.$ So $\alpha^2 u_{xx} = u_t$.