Exam 3 - Solutions

#1 a) $W[\vec{x}^{(1)}, \vec{x}^{(2)}](t) = \det\begin{pmatrix} t & e^{-t} \\ 1 & -e^{-t}\end{pmatrix} = -te^{-t} - 1e^{-t} = (-t-1)e^{-t}$

$= -(t+1)e^{-t} = 0 \iff t = -1$

b) These solutions are linearly independent on any interval that does not contain $t=-1$, i.e. $(-\infty,-1) \cup (-1,\infty)$.

c) In $\vec{x}' = A(t)\vec{x}$ the entries in A will be continuous everywhere except $t=-1$.

d) $\vec{x}' = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\vec{x}$ for both $\vec{x}^{(1)}$ & $\vec{x}^{(2)}$.

$\vec{x} = \vec{x}^{(1)}$: $\vec{x}^{(1)\prime} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\vec{x}^{(1)} \Rightarrow \begin{pmatrix} 1 \\ 0\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} t \\ 1\end{pmatrix}$

$$\Rightarrow \begin{cases} ta + b = 1 \\ tc + d = 0\end{cases}$$

$\vec{x} = \vec{x}^{(2)}$: $\vec{x}^{(2)\prime} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\vec{x}^{(2)} \Rightarrow \begin{pmatrix} -e^{-t} \\ e^{-t}\end{pmatrix} = \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} e^{-t} \\ -e^{-t}\end{pmatrix}$.

$$\Rightarrow \begin{cases} e^{-t}a - e^{-t}b = -e^{-t} \\ e^{-t}c - e^{-t}d = e^{-t}\end{cases}$$

$$\Rightarrow \begin{cases} a - b = -1 \\ c - d = 1\end{cases}$$

Solve: $ta + b = 1$
$+ (a - b = -1)$
$(t+1)a = 0 \Rightarrow a = 0$
$b = 1$

$tc + d = 0$
$+ (c - d = 1)$
$(t+1)c = 1 \Rightarrow c = \frac{1}{t+1}$
$d = -tc = \frac{-t}{t+1}$

So $\vec{x}^{(1)}$ & $\vec{x}^{(2)}$ are both solutions to

$$\boxed{\vec{x}' = \begin{pmatrix} 0 & 1 \\ \frac{1}{t+1} & \frac{-t}{t+1}\end{pmatrix}\vec{x}.}$$

#2. $A = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix}$ $\quad \det(A - \lambda I) = (1-\lambda)(-1-\lambda) + 10 = \lambda^2 - 1 + 10 = \lambda^2 + 9.$

$\lambda = +3i$: $(A - 3iI)\vec{\xi} = \vec{0}$: $\begin{pmatrix} 1-3i & 2 \\ -5 & -1-3i \end{pmatrix}\vec{\xi} = \vec{0}.$ $\quad \vec{\xi} = \begin{pmatrix} 2 \\ -1+3i \end{pmatrix}.$

$$\vec{x} = e^{\lambda t}\vec{\xi} = (\cos 3t + i \sin 3t)\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + i\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + i\begin{pmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}$$

Then $\vec{x}^{(1)} = \begin{pmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix}$, $\vec{x}^{(2)} = \begin{pmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}$

and $\vec{x}(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}.$

Applying the IC: $\begin{pmatrix} 8 \\ 10 \end{pmatrix} = c_1\begin{pmatrix} 2 \\ -1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$8 = 2c_1 \quad \Rightarrow c_1 = 4$

$10 = -c_1 + 3c_2 \quad 3c_2 = 10 + c_1 = 10 + 4 \Rightarrow c_2 = 14/3.$

So $\vec{x}(t) = 4\begin{pmatrix} 2\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + \frac{14}{3}\begin{pmatrix} 2\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix}$

$$= \begin{pmatrix} 8\cos 3t + \frac{28}{3}\sin 3t \\ 10\cos 3t - \frac{50}{3}\sin 3t \end{pmatrix}.$$

#3. $A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}$ $\quad \lambda_1 = -3,\ \vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

so $\vec{x}^{(1)} = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Look for $\vec{x}^{(2)} = te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \eta e^{-3t}$ where $(A + 3I)\vec{\eta} = \vec{\xi}.$

$\begin{pmatrix} 4 & -4 \\ 4 & -4 \end{pmatrix}\vec{\eta} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow 4\eta_1 - 4\eta_2 = 1 \Rightarrow \eta_1 = \frac{1}{4} + \eta_2 \Rightarrow \vec{\eta} = \begin{pmatrix} \frac{1}{4} + \eta_2 \\ \eta_2 \end{pmatrix}$

$\eta_2 = \eta_2$

$= \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} + \eta_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Choosing $\eta_2 = 0$: $\vec{\eta} = \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}$ and $\vec{x}^{(2)} = te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}e^{-3t}$

General solution: $\vec{x}(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)}$

$$= c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\left(te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix}e^{-3t}\right)$$

Note: if you chose $\eta_2 = -1/4$, then $\vec{\eta} = \begin{pmatrix} 0 \\ -1/4 \end{pmatrix}$ and $\vec{x}^{(2)} = te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t}\begin{pmatrix} 0 \\ -1/4 \end{pmatrix}.$

#4. Given $\lambda_1 = -3,\ \vec{\xi}^{(1)} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \Rightarrow \vec{x}^{(1)} = e^{-3t}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$

$\lambda_2 = 2,\ \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{x}^{(2)} = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Homogeneous solution: $\vec{x}_c = c_1 \vec{x}^{(1)} + c_2 \vec{x}^{(2)} = \Psi(t)\,\vec{c}$

where $\Psi(t) = \begin{pmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{pmatrix}.$

Seek a particular solution: $\vec{x}_p = \Psi(t)\vec{u}(t)$ where $\Psi(t)\vec{u}' = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$

$$\vec{u}' = \Psi(t)^{-1}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$= -\frac{1}{5e^{-t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -4e^{-3t} & -e^{-3t} \end{pmatrix}\begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}$$

$$= \frac{-e^{t}}{5}\begin{pmatrix} 1 + 2e^{3t} \\ -4e^{-5t} + 2e^{-2t} \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} e^{t} + 2e^{4t} \\ -4e^{-4t} + 2e^{-t} \end{pmatrix}$$

Integrate: $\vec{u} = \int u'\,dt = -\frac{1}{5}\begin{pmatrix} e^{t} + \frac{1}{2}e^{4t} \\ e^{-4t} - 2e^{-t} \end{pmatrix}$

Then $\vec{x}_p = \Psi(t)\vec{u} = -\frac{1}{5}\begin{pmatrix} -e^{-3t} & e^{2t} \\ 4e^{-3t} & e^{2t} \end{pmatrix}\begin{pmatrix} e^{t} + \frac{1}{2}e^{4t} \\ e^{-4t} - 2e^{-t} \end{pmatrix}$

$$= -\frac{1}{5}\begin{pmatrix} \cancel{-e^{-2t}} - \frac{1}{2}e^{t} + \cancel{e^{-2t}} - 2e^{t} \\ 4e^{-2t} + \cancel{2e^{t}} + e^{-2t} - \cancel{2e^{t}} \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} -\frac{5}{2}e^{t} \\ 5e^{-2t} \end{pmatrix}$$

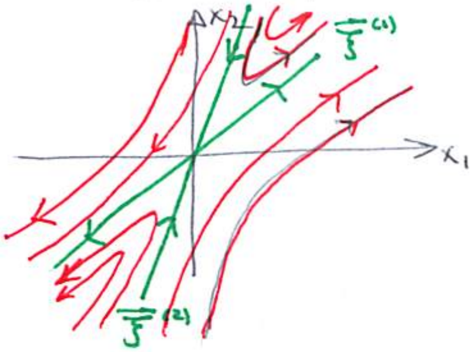
$$= \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix}$$

General solution: $\vec{x}(t) = \vec{x}_c + \vec{x}_p$

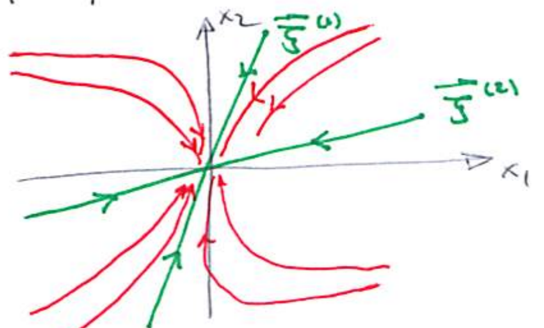
$$= c_1 e^{-3t}\begin{pmatrix} -1 \\ 4 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2}e^{t} \\ -e^{-2t} \end{pmatrix}.$$

#5.

a) $\lambda_1 > 0,\ \lambda_2 < 0 \implies$ saddle point

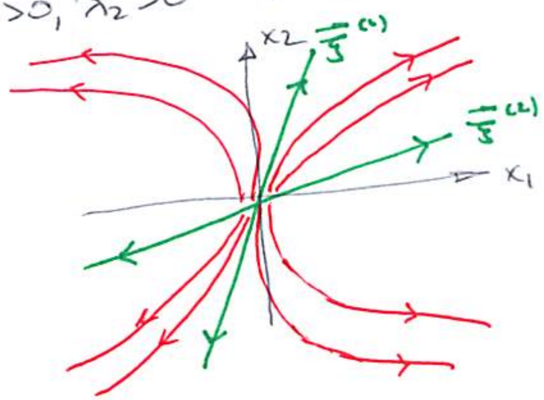


b) $\lambda_1 < 0,\ \lambda_2 < 0 \implies$ asymptotically stable node.

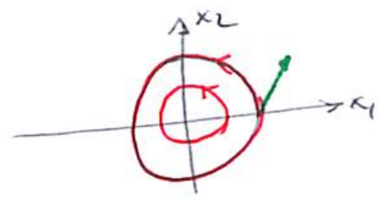


larger e-value is λ_1 so solution curves approach $\begin{pmatrix}0\\0\end{pmatrix}$ along $\vec{\xi}^{(1)}$.

c) $\lambda_1 > 0,\ \lambda_2 > 0 \implies$ unstable node



larger e-value is λ_2 so solution curves diverge from $\begin{pmatrix}0\\0\end{pmatrix}$ on a path that becomes parallel to $\vec{\xi}^{(2)}$.

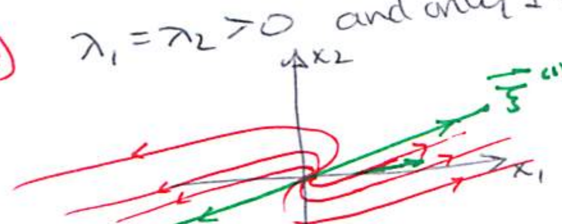


$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix}$

d) $Re(\lambda_1) = Re(\lambda_2) = 0 \implies$ center

e) $\lambda_1 = \lambda_2 > 0$ and only 1 e-vector $\implies$ unstable improper node

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1/2\end{pmatrix}$



f) $Re(\lambda_1) = Re(\lambda_2) = \sqrt{6} > 0 \implies$ unstable spiral.

$A\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{4}{3}\sqrt{6}\\ \frac{1}{3}\sqrt{6}\end{pmatrix}$