Homework Solution

§3.5 #12

$$
\begin{aligned}
&W[e^{rt}, e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t)] \\
&= \det \begin{bmatrix} e^{rt} & e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ re^{rt} & e^{\alpha t}\left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right) & e^{\alpha t}\left(\alpha\sin(\beta t) + \beta\cos(\beta t)\right) \\ r^2e^{rt} & e^{\alpha t}\left(\alpha^2\cos(\beta t) - \alpha\beta\sin(\beta t) - \alpha\beta\sin(\beta t) - \beta^2\cos(\beta t)\right) & e^{\alpha t}\left(\alpha^2\sin(\beta t) + \alpha\beta\cos(\beta t) + \alpha\beta\cos(\beta t) - \beta^2\sin(\beta t)\right) \end{bmatrix} \\
&= \det \begin{bmatrix} e^{rt} & e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ re^{rt} & e^{\alpha t}\left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right) & e^{\alpha t}\left(\alpha\sin(\beta t) + \beta\cos(\beta t)\right) \\ r^2e^{rt} & e^{\alpha t}\left((\alpha^2-\beta^2)\cos(\beta t) - 2\alpha\beta\sin(\beta t)\right) & e^{\alpha t}\left((\alpha^2-\beta^2)\sin(\beta t) + 2\alpha\beta\cos(\beta t)\right) \end{bmatrix} \\
&= e^{rt}\det \begin{bmatrix} e^{\alpha t}\left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right) & e^{\alpha t}\left(\alpha\sin(\beta t) + \beta\cos(\beta t)\right) \\ e^{\alpha t}\left((\alpha^2-\beta^2)\cos(\beta t) - 2\alpha\beta\sin(\beta t)\right) & e^{\alpha t}\left((\alpha^2-\beta^2)\sin(\beta t) + 2\alpha\beta\cos(\beta t)\right) \end{bmatrix} \\
&\quad - re^{rt}\det \begin{bmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\left((\alpha^2-\beta^2)\cos(\beta t) - 2\alpha\beta\sin(\beta t)\right) & e^{\alpha t}\left((\alpha^2-\beta^2)\sin(\beta t) + 2\alpha\beta\cos(\beta t)\right) \end{bmatrix} \\
&\quad + r^2e^{rt}\det \begin{bmatrix} e^{\alpha t}\cos(\beta t) & e^{\alpha t}\sin(\beta t) \\ e^{\alpha t}\left(\alpha\cos(\beta t) - \beta\sin(\beta t)\right) & e^{\alpha t}\left(\alpha\sin(\beta t) + \beta\cos(\beta t)\right) \end{bmatrix}
\end{aligned}
$$

$$= e^{rt}\left(e^{2\alpha t}\left(\left(\alpha\cos(\beta t)-\beta\sin(\beta t)\right)\left((\alpha^2-\beta^2)\sin(\beta t)+2\alpha\beta\cos(\beta t)\right)\right.\right.$$
$$\left.\left. - \left(\alpha\sin(\beta t)+\beta\cos(\beta t)\right)\left((\alpha^2-\beta^2)\cos(\beta t)-2\alpha\beta\sin(\beta t)\right)\right)\right)$$

$$-re^{rt}\left(e^{2\alpha t}\left(\cos(\beta t)\left((\alpha^2-\beta^2)\sin(\beta t)+2\alpha\beta\cos(\beta t)\right)\right.\right.$$
$$\left.\left. - \sin(\beta t)\left((\alpha^2-\beta^2)\cos(\beta t)-2\alpha\beta\sin(\beta t)\right)\right)\right)$$

$$+r^2e^{rt}\left(e^{2\alpha t}\left(\cos(\beta t)\left(\alpha\sin(\beta t)+\beta\cos(\beta t)\right)\right.\right.$$
$$\left.\left. - \sin(\beta t)\left(\alpha\cos(\beta t)-\beta\sin(\beta t)\right)\right)\right)$$

$$= e^{rt}e^{2\alpha t}\Big(\big(\cancel{\alpha(\alpha^2-\beta^2)} - \cancel{2\alpha\beta^2} - \cancel{\alpha(\alpha^2-\beta^2)} + \cancel{2\alpha\beta}\big)\cos(\beta t)\sin(\beta t)$$
$$+ \left(2\alpha^2\beta - \beta(\alpha^2-\beta^2)\right)\cos^2(\beta t) + \left(-\beta(\alpha^2-\beta^2)+2\alpha^2\beta\right)\sin^2(\beta t)\Big)$$

$$= \left(2\alpha^2\beta-\beta(\alpha^2-\beta^2)\right)\left(\cos^2(\beta t)+\sin^2(\beta t)\right)$$
$$= 2\alpha^2\beta-\beta(\alpha^2-\beta)$$
$$= \beta(2\alpha^2-(\alpha^2-\beta)) = \beta(\alpha^2+\beta^2)$$

$$-re^{rt}e^{2\alpha t}\Big(\overbrace{\left((\alpha^2-\beta^2)-(\alpha^2-\beta^2)\right)}^{=0}\cos(\beta t)\sin(\beta t) + 2\alpha\beta\cos^2(\beta t) + 2\alpha\beta\sin^2(\beta t)\Big) = 2\alpha\beta\left(\cos^2(\beta t)+\sin^2(\beta t)\right)$$
$$= 2\alpha\beta$$

$$+r^2e^{rt}e^{2\alpha t}\Big(\overbrace{(\alpha-\alpha)}^{=0}\cos(\beta t)\sin(\beta t) + \beta\cos^2(\beta t)+\beta\sin^2(\beta t)\Big) = \beta$$

$$= e^{rt}e^{2\alpha t}\left(\beta(\alpha^2+\beta^2) - 2\alpha\beta r + \beta r^2\right)$$

$$= e^{rt}e^{2\alpha t}\beta\left(\alpha^2+\beta^2-2\alpha r + r^2\right)$$

$$= e^{rt}e^{2\alpha t}\beta\left((\alpha-r)^2+\beta^2\right).$$