§3.2 #20.

The system of equations can be written in matrix form as

$$\begin{bmatrix} c & -2 & 2 \\ -2 & c & -2 \\ 2 & -2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}.$$

To answer (a) we could find the determinant of the coefficient matrix but it will be less work overall if we just put the augmented matrix into ~~reduced~~ row echelon form and look to see when the system is inconsistent (an equation that amounts to $0 = 1$):

$$\left[\begin{array}{ccc|c} c & -2 & 2 & 0 \\ -2 & c & -2 & 4 \\ 2 & -2 & c & 0 \end{array}\right] \xrightarrow{③ \leftrightarrow ①} \left[\begin{array}{ccc|c} 2 & -2 & c & 0 \\ -2 & c & -2 & 4 \\ c & -2 & 2 & 0 \end{array}\right] \xrightarrow[③ - \frac{c}{2}① \to ③]{② + ① \to ②} \left[\begin{array}{ccc|c} 2 & -2 & c & 0 \\ 0 & c-2 & c-2 & 4 \\ 0 & -2+c & 2-\frac{c^2}{2} & 0 \end{array}\right]$$

$$\xrightarrow{③ - ② \to ③} \left[\begin{array}{ccc|c} 2 & -2 & c & 0 \\ 0 & c-2 & c-2 & 4 \\ 0 & 0 & 2-\frac{c^2}{2}-(c-2) & -4 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & -2 & c & 0 \\ 0 & c-2 & c-2 & 4 \\ 0 & 0 & 4-c-\frac{c^2}{2} & -4 \end{array}\right].$$

This system is singular when $0 = 4 - c - \frac{c^2}{2} = -\frac{1}{2}(c+4)(c-2)$, that is, when $c = 2$ or $c = -4$. For all other values of c the solution is found by back substitution:

$$-\tfrac{1}{2}(c+4)(c+2)\, z = -4 \implies z = \frac{8}{(c+4)(c-2)}$$

$$(c-2)\,y = 4 - (c-2)z = 4 - \frac{8}{c+4} = \frac{4(c+4)-8}{c+4} = \frac{4c+8}{c+4} \implies y = \frac{4(c+2)}{(c+4)(c-2)}$$

$$2x = 0 - cz + 2y = \frac{-8c + 2\cdot 4(c+2)}{(c+4)(c-2)} = \frac{\cancel{8}\,16}{(c+4)(c-2)} \implies x = \frac{8}{(c+4)(c-2)}.$$

For all $c \neq 2$ and $c \neq -4$, the unique solution is

$$x = \frac{8}{(c+4)(c-2)}, \quad y = \frac{4(c+2)}{(c+4)(c-2)}, \quad z = \frac{8}{(c+4)(c-2)}.$$