MATH 520 (Section 001)
Prof. Meade

Exam 2
March 7, 2008

University of South Carolina
Spring 2008

Name: Key

Instructions:

1. There are a total of 5 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive full credit for a correct answer. Correct answers with no supporting work will be eligible for at most half-credit.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.
5. Check your work. If I see *clear evidence* that you checked your answer (when possible) and you *clearly indicate* that your answer is incorrect, you will be eligible for more points than if you had not checked your work.

Problem	Points	Score
1	30	
2	10	
3	18	
4	15	
5	27	
Total	100	

I hope you have a great Spring Break!

Honor Code Statement

I understand that it is the responsibility of every member of the Carolina community to uphold and maintain the University of South Carolina's Honor Code.

As a Carolinian, I certify that I have neither given nor received unauthorized aid on this exam.

Signature / Date

1. (30 points) [10 points each]

(a) Find the general solution to $y'' - 2y' - 3y = 0$.

Assume $y = e^{rt}$: $y' = re^{rt}$

$y'' = r^2 e^{rt}$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = 3, \; r = -1$

$y = c_1 e^{3t} + c_2 e^{-t}$

(b) Find the general solution to $y'' + 2y' + 5y = 0$.

$r^2 + 2r + 5 = 0$

$r = \frac{1}{2}\left(-2 \pm \sqrt{4 - 4 \cdot 5}\right)$

$= \frac{1}{2}(-2 \pm \sqrt{-16})$

$= \frac{1}{2}(-2 \pm 4i)$

$= -1 \pm 2i.$

$y = c_1 e^{-t} \cos(2t) + c_2 e^{-t} \sin(2t).$

(c) Find the general solution to $y''' - 4y'' + 4y' = 0$.

$r^3 - 4r^2 + 4r = 0$

$r(r^2 - 4r + 4) = 0$

$r(r+2)^2 = 0$

$r = 0, \; r = -2, \; r = -2$

$y = c_1 e^{0t} + c_2 e^{-2t} + c_3 t e^{-2t}$

$= c_1 + c_2 e^{-2t} + c_3 t e^{-2t}.$

2. (10 points) [5 points each] Let $y_1 = \ln(x)$ and $y_2 = x\ln(x)$.

(a) Find the Wronskian of y_1 and y_2.

$$W[y_1, y_2] = \det \begin{pmatrix} \ln(x) & x\ln(x) \\ \frac{1}{x} & \ln(x) + x \cdot \frac{1}{x} \end{pmatrix}$$

$$= \det \begin{pmatrix} \ln(x) & x\ln(x) \\ \frac{1}{x} & \ln(x) + 1 \end{pmatrix}$$

$$= \ln(x)\left(\ln(x)+1\right) - \frac{1}{x}\, x\ln(x)$$

$$= \left(\ln(x)\right)^2 + \ln(x) - \ln(x)$$

$$= \left(\ln(x)\right)^2.$$

(b) Explain why y_1 and y_2 could *not* be linearly independent solutions to a homogeneous second-order linear differential equation on the interval $(0, \infty)$.

By Abel's Theorem, if y_1 & y_2 are linearly independent solutions to a homogeneous 2^{nd}-order linear DE on $(0,\infty)$ then $W[y_1,y_2](x) \neq 0$ for all $x \in (0,\infty)$.

But, $W[y_1,y_2](1) = (\ln(1))^2 = 0$

Therefore y_1 & y_2 cannot be linearly independent solutions to a homogeneous 2^{nd}-order DE on $(0,\infty)$.

3. (18 points) Consider the differential equation

$$\sin(x)y'' + (1 + \sin(x))y' + y = 0.$$

(a) [8 points] Show that any solution of the form $y = e^{rx}$ must satisfy

$$(r^2 + r)\sin(x) + (r + 1) = 0.$$

$y' = re^{rx}$
$y'' = r^2 e^{rx}$

$\sin(x)\, r^2 e^{rx} + (1+\sin(x))\, re^{rx} + e^{rx} = 0$

$e^{rx}\left(r^2 \sin(x) + r(1+\sin(x)) + 1\right) = 0$

$(r^2+r)\sin(x) + r+1 = 0.$

because $e^{rx} \neq 0$ for all x.

(b) [10 points] Because the above condition must be satisfied <u>for all x</u>, the only solution is $r = -1$. That is, $y_1 = e^{-x}$. A second solution can be found in the form $y_2(x) = v(x)y_1(x)$. Find the first-order differential satisfied by $u = v'$. *Do not attempt to solve this first-order differential equation.*

$y_2 = ve^{-x}$

$y_2' = v'e^{-x} - ve^{-x}$

$y_2'' = v''e^{-x} - v'e^{-x} - v'e^{-x} + ve^{-x} = v''e^{-x} - 2v'e^{-x} + ve^{-x}.$

Substituting back into the DE:

$0 = \sin(x)\, y_2'' + (1+\sin(x))\, y_2' + y_2$

$= \sin(x)\left(v''e^{-x} - 2v'e^{-x} + ve^{-x}\right) + (1+\sin(x))\left(v'e^{-x} - ve^{-x}\right) + ve^{-x}$

$= e^{-x}\left(\sin(x)\, v'' + \left(-2v'\sin(x) + (1+\sin(x))v'\right) + \sin(x)v - (1+\sin(x))v + v\right)$

$= e^{-x}\left(\sin(x)\, v'' + (1-\sin(x))\, v'\right)$

So $\sin(x)\, v'' + (1-\sin(x))v' = 0$

Let $u = v'$, then $u' = v''$ and so $\underline{\sin(x)\, u' + (1-\sin(x))\, u = 0}$

4. (15 points) Consider the initial value problem for the Cauchy–Euler equation

$$x^2y'' - 2y = 0 \qquad y(1) = \alpha \quad y'(1) = 3.$$

(a) [8 points] Find the general solution.

HINT: *Look for solutions in the form $y = x^r$.*

$y = x^r$
$y' = rx^{r-1}$
$y'' = r(r-1)x^{r-2}$

$x^2 r(r-1)x^{r-2} - 2x^r = 0$
$r(r-1)x^r - 2x^r = 0$
$(r(r-1) - 2)x^r = 0$
$r^2 - r - 2 = 0$
$(r-2)(r+1) = 0$
$r = 2, \ r = -1$

$\underline{y = c_1 x^2 + c_2 x^{-1}}$.

(b) [5 points] Find the solution to the initial value problem.

$y = c_1 x^2 + c_2 x^{-1}$
$y' = 2c_1 x - c_2 x^{-2}$

$y(1) = c_1 + c_2 = \alpha$
$y'(1) = 2c_1 - c_2 = 3$
$3c_1 = 3 + \alpha \implies c_1 = \frac{3+\alpha}{3} = 1 + \frac{\alpha}{3}$.

$c_2 = \alpha - c_1$
$= \alpha - (1 + \frac{\alpha}{3})$
$= \frac{2}{3}\alpha - 1$

$$y = (1 + \tfrac{\alpha}{3})x^2 + (\tfrac{2}{3}\alpha - 1)x^{-1}.$$

(c) [2 points] For what value(s) of α does the solution approach 0 as $x \to \infty$?

$\lim_{x \to \infty} x^2 = +\infty$

$\lim_{x \to \infty} x^{-1} = 0$

so $\lim_{x \to \infty} y = \lim_{x \to \infty} \left((1 + \frac{\alpha}{3})x^2 + (\frac{2}{3}\alpha - 1)x^{-1} \right) = 0$

only if $1 + \frac{\alpha}{3} = 0$
$\frac{\alpha}{3} = -1$
$\underline{\alpha = -3}$

5. (27 points) [9 points each] Find a suitable trial solution for each of the following differential equations. *Do not find a particular solution.*

   (a) $y' + 3y = e^{3t} + \sin(3t) + t^2$

   $r = -3$

   HINT: *The complementary solution is* $y = c_1 e^{-3t}$.

$$Y = \underbrace{Ae^{3t}}_{\text{from } e^{3t}} + \underbrace{\left(B\sin(3t) + C\cos(3t)\right)}_{\text{from } \sin(3t)} + \underbrace{\left(Dt^2 + Et + F\right)}_{\text{from } t^2}$$

   (b) $y''' + 4y'' + 5y' + 2y = 3t^2 e^{-t}$

   $r = -1$ (mult. 2)
   $r = -2$

   HINT: *The complementary solution is* $y = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^{-2t}$.

$$Y = \left(At^2 + Bt + C\right) e^{-t} \cdot t^2$$
$$= \left(At^4 + Bt^3 + Ct^2\right) e^{-t}.$$

   (c) $y'' + 4y = t\cos(2t) + \sin(2t)$

   $r = \pm 2i$

   HINT: *The complementary solution is* $y = c_1 \cos(2t) + c_2 \sin(2t)$.

$$Y = \left(\left(At + B\right)\cos(2t) + \left(Ct + D\right)\sin(2t)\right) t$$
$$= \left(At^2 + Bt\right)\cos(2t) + \left(Ct^2 + Dt\right)\sin(2t).$$