Math 141 HANDOUT

Fundamental Theorem of Calculus (FTC)

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function.
Let $F\colon [a,b] \to \mathbb{R}$ be a function.

- If F is an antiderivative of f on $[a,b]$ (i.e. $F'(x) = f(x)$ for each $x \in [a,b]$), then

$$\int_a^b f(x)\,dx \;\equiv\; \int_a^b F'(x)\,dx \;=\; F(b) - F(a)\ .$$

- If $F(x) = \int_a^x f(t)\,dt$ for each $x \in [a,b]$, then F is an antiderivative of f on $[a,b]$, i.e.

$$F'(x) \equiv D_x\left[\int_a^x f(t)\,dt\right] \;=\; f(x)\ .$$

Basic Differentiation Rules

If the functions $y = f(x)$ and $y = g(x)$ are differentiable at x and a and b are constants, then:

1. $D_x\left[af(x) \;+\; bg(x)\right] \;=\; af'(x) \;+\; bg'(x)$
2. $D_x\left[f(x) \;\cdot\; g(x)\right] \;=\; f'(x) \;\cdot\; g(x) \;+\; f(x) \;\cdot\; g'(x)$
3. $D_x\left[\frac{f(x)}{g(x)}\right] \;=\; \frac{f'(x)\,\cdot\,g(x)\;-\;f(x)\,\cdot\,g'(x)}{[g(x)]^2}$ provided $g(x) \neq 0$
4. $D_x\left[f(x)\right]^r \;=\; r\left[f(x)\right]^{r-1}\; f'(x)$ provided $r \in \mathbb{Q}$

If f is differentiable at x and g is differentiable at $f(x)$, then:

5. $D_x\left[g(f(x))\right] \;=\; g'(f(x))\; f'(x)$

Generalized Exponential and Logarithmic Functions with base a where $a > 0$ but $a \neq 1$

DERIVATIVES	$\overset{\text{FTC}}{\Longrightarrow}$	INTEGRALS
$D_x\ \ln\lvert u\rvert \overset{u\neq 0}{=} \frac{1}{u}\,\frac{du}{dx}$		$\int \frac{du}{u} \overset{u\neq 0}{=} \ln\lvert u\rvert \;+\; C$
$D_x\ e^u \;=\; e^u\,\frac{du}{dx}$		$\int e^u\,du \;=\; e^u \;+\; C$
$D_x \log_a\lvert u\rvert \overset{u\neq 0}{=} \frac{1}{u}\,\frac{1}{\ln a}\,\frac{du}{dx}$		
$D_x a^u = a^u\ \ln a\,\frac{du}{dx}$		$\int a^u\,du = \frac{a^u}{\ln a} \;+\; C$

$\boxed{\text{TRIG and CALCULUS}}$

DERIVATIVES $\overset{\text{FTC}}{\Longrightarrow}$ INTEGRALS

$$D_x \sin u = \cos u \ \frac{du}{dx} \qquad \int \cos u \, du = \sin u \ + \ C$$

$$D_x \tan u = \sec^2 u \ \frac{du}{dx} \qquad \int \sec^2 u \, du = \tan u \ + \ C$$

$$D_x \sec u = \sec u \tan u \ \frac{du}{dx} \qquad \int \sec u \tan u \, du = \sec u \ + \ C$$

$$D_x \cos u = -\sin u \ \frac{du}{dx} \qquad \int \sin u \, du = -\cos u \ + \ C$$

$$D_x \cot u = -\csc^2 u \ \frac{du}{dx} \qquad \int \csc^2 u \, du = -\cot u \ + \ C$$

$$D_x \csc u = -\csc u \cot u \ \frac{du}{dx} \qquad \int \csc u \cot u \, du = -\csc u \ + \ C$$

MORE INTEGRALS

$$\int \tan u \, du = -\ln|\cos u| \ + \ C = \ln|\sec u| \ + \ C$$

$$\int \cot u \, du = \ln|\sin u| \ + \ C = -\ln|\csc u| \ + \ C$$

$$\int \sec u \, du = \ln|\sec u \ + \ \tan u| \ + \ C = -\ln|\sec u \ - \ \tan u| \ + \ C$$

$$\int \csc u \, du = -\ln|\csc u \ + \ \cot u| \ + \ C = \ln|\csc u \ - \ \cot u| \ + \ C$$

DERIVATIVES $\overset{\text{FTC}}{\Longrightarrow}$ INTEGRALS $(a > 0)$

$$D_x \sin^{-1} u = \frac{1}{\sqrt{1-u^2}} \ \frac{du}{dx} \qquad \int \frac{1}{\sqrt{a^2-u^2}} \, du = \sin^{-1} \frac{u}{a} \ + \ C$$

$$D_x \tan^{-1} u = \frac{1}{1+u^2} \ \frac{du}{dx} \qquad \int \frac{1}{a^2+u^2} \, du = \frac{1}{a} \tan^{-1} \frac{u}{a} \ + \ C$$

$$D_x \sec^{-1} u = \frac{1}{|u| \sqrt{u^2-1}} \ \frac{du}{dx} \qquad \int \frac{1}{u\sqrt{u^2-a^2}} \, du = \frac{1}{a} \sec^{-1} \frac{|u|}{a} \ + \ C$$

$$D_x \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \ \frac{du}{dx}$$

$$D_x \cot^{-1} u = \frac{-1}{1+u^2} \ \frac{du}{dx}$$

$$D_x \csc^{-1} u = \frac{-1}{|u| \sqrt{u^2-1}} \ \frac{du}{dx}$$

Natural Logarithm Fn. $y = \ln x$ AND Natural Exponential Fn. $y = \exp x$

$\ln x \overset{x>0}{\equiv} \int_1^x \frac{dt}{t}$ | nat. exp. fn. $\equiv$ inverse of the nat. log. fn.

$\ln\colon (0,\infty) \to (-\infty,\infty)$ | $\exp\colon (-\infty,\infty) \to (0,\infty)$

$$y = \ln x \iff x = \exp y$$

$\exists$ a unique $e \in \mathbb{R}$ so that $\ln e = 1$ | $e^x \overset{x\in\mathbb{R}}{\equiv} \exp(x)$

$x, y > 0$ & $r \in \mathbb{Q}$	$x, y \in \mathbb{R}$ & $r \in \mathbb{Q}$
$e^{\ln x} = x$	$\ln(e^x) = x$
$\ln 1 = 0$	$e^0 = 1$
$\ln(xy) = \ln x + \ln y$	$e^x\, e^y = e^{x+y}$
$\ln(\frac{x}{y}) = \ln x - \ln y$	$\frac{e^x}{e^y} = e^{x-y}$
$\ln(x^r) = r(\ln x)$	$(e^x)^r = e^{xr}$

Generalized Exponential $y = a^x$ and Logarithmic $y = \log_a x$ Functions

with base a where $a > 0$ but $a \neq 1$ and $b > 0$ but $b \neq 1$

$\boxed{\log_e \equiv \ln}$

$f(x) = a^x \equiv e^{x \ln a}$ $\quad : (-\infty,\infty) \to (0,\infty)$

$g(x) = \log_a x \equiv$ the inverse of the fn. $f(x) = a^x$ $\quad : (0,\infty) \to (-\infty,\infty)$

$$y = \log_a x \iff x = a^y$$

$$(\log_a b)(\log_b c) = \log_a c \implies \log_a x = \frac{\ln x}{\ln a}$$

$x, y > 0$ & $r \in \mathbb{R}$	$x, y \in \mathbb{R}$ & $r \in \mathbb{R}$ & $0 < b \neq 1$
$a^{\log_a x} = x$	$\log_a(a^x) = x$
$\log_a 1 = 0$	$a^0 = 1$
$\log_a(xy) = \log_a x + \log_a y$	$a^x\, a^y = a^{x+y}$
$\log_a(\frac{x}{y}) = \log_a x - \log_a y$	$\frac{a^x}{a^y} = a^{x-y}$
$\log_a(x^r) = r(\log_a x)$	$(a^x)^r = a^{xr}$
	$(ab)^x = a^x b^x$ and $(\frac{a}{b})^x = \frac{a^x}{b^x}$

Basic Trig

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

Basic Inverse Trig Functions

$y = \sin\theta$	$\Leftrightarrow$	$\theta = \sin^{-1} y$	where	$-1 \le y \le 1$	and	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
$y = \cos\theta$	$\Leftrightarrow$	$\theta = \cos^{-1} y$	where	$-1 \le y \le 1$	and	$0 \le \theta \le \pi$
$y = \tan\theta$	$\Leftrightarrow$	$\theta = \tan^{-1} y$	where	$y \in \mathbb{R}$	and	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$y = \cot\theta$	$\Leftrightarrow$	$\theta = \cot^{-1} y$	where	$y \in \mathbb{R}$	and	$0 < \theta < \pi$
$y = \sec\theta$	$\Leftrightarrow$	$\theta = \sec^{-1} y$	where	$\lvert y\rvert \ge 1$	and	$0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$
$y = \csc\theta$	$\Leftrightarrow$	$\theta = \csc^{-1} y$	where	$\lvert y\rvert \ge 1$	and	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$

Math 142

Integration by Parts: $\int u\,dv = uv - \int v\,du$

Trig Identities: $(*)$ do not have to memorize but should be able to use if given

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \sin(2x) = 2\sin x\cos x$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos(s+t) \overset{(*)}{=} \cos s\cos t - \sin s\sin t \qquad \sin(s+t) \overset{(*)}{=} \sin s\cos t + \cos s\sin t$$

$$\cos(s-t) \overset{(*)}{=} \cos s\cos t + \sin s\sin t \qquad \sin(s-t) \overset{(*)}{=} \sin s\cos t - \cos s\sin t$$

Trig Substitution

IF INTEGRAND INVOLVES	THEN MAKE THE SUBSTITUTION	RESTRICTION ON θ
$a^2 - u^2$	$u = a\sin\theta \leftrightsquigarrow \theta = \sin^{-1}\frac{u}{a}$	$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$
$a^2 + u^2$	$u = a\tan\theta \leftrightsquigarrow \theta = \tan^{-1}\frac{u}{a}$	$\frac{-\pi}{2} < \theta < \frac{\pi}{2}$
$u^2 - a^2$	$u = a\sec\theta \leftrightsquigarrow \theta = \sec^{-1}\frac{u}{a}$	$0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$