Solutions - HW7

§3.3 #4. $e^{2-\frac{\pi}{2}i} = e^2 e^{-\frac{\pi}{2}i} = e^2\left(\cos\left(\frac{-\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) = e^2(0 + i(-1)) = \underline{-ie^2}$

#11. $y'' + 6y' + 13y = 0$

$r^2 + 6r + 13 = 0 \Rightarrow r = \frac{-6 \pm \sqrt{36 - 4(13)}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$

so $y_1 = e^{-3t}\cos 2t$ & $y_2 = e^{-3t}\sin 2t$

The gen'l sol'n is $y = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t.$

#17. $y'' + 4y = 0 \qquad y(0) = 0,\ y'(0) = 1.$

$r^2 + 4 = 0 \Rightarrow r = \pm 2i$

$y_1 = \cos 2t \qquad y_2 = \sin 2t$

Gen'l sol'n: $y = c_1\cos 2t + c_2 \sin 2t.$ $\qquad y(0) = c_1 = 0$

Apply ICs $y' = -2c_1\sin 2t + 2c_2\cos 2t \qquad y'(0) = 2c_2 = 1$

$\therefore c_1 = 0$
$c_2 = 1/2.$

$y = 0\cos 2t + 1/2\sin 2t = \frac{1}{2}\sin(2t).$

period oscillations w/ amplitude 1/2 & period π.

#23. $3u'' - u' + 2u = 0 \quad u(0) = 2,\ u'(0) = 0$

(a) $3r^2 - r + 2 = 0 \Rightarrow r = \frac{1}{3\cdot 2}\left(1 \pm \sqrt{(-1)^2 - 4(3)(2)}\right)$

$= \frac{1}{6}(1 \pm \sqrt{23}\,i) = \frac{1}{6} \pm \frac{\sqrt{23}}{6}i$

Gen'l sol'n: $u = c_1 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + c_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$

$u(0) = c_1 = 2$

$u' = \frac{1}{6}c_1 e^{t/6}\left(\cos\frac{\sqrt{23}}{6}t\right) + c_1 e^{t/6}\left(-\frac{\sqrt{23}}{6}\right)\sin\left(\frac{\sqrt{23}t}{6}\right)$
$+ \frac{1}{6}c_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + c_2 e^{t/6}\left(\frac{\sqrt{23}}{6}\right)\cos\left(\frac{\sqrt{23}}{6}t\right)$

so $u(0) = c_1$ so $c_1 = 2 \qquad u'(0) = \frac{c_1}{6} + \frac{c_2\sqrt{23}}{6} = 0$
$\quad\ \ = 2$

$\frac{2}{6} + c_2\frac{\sqrt{23}}{6} = 0.$

$\Rightarrow c_2 = \frac{-2}{\sqrt{23}}$

so $u(t) = 2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)$

(b) With a graphing calculator or computer software,
The first time $|u(t)| = 10$ occurs when $t = 10.7598$.

#31. Claim: For any constant $r = \lambda + i\mu$.

$$\frac{d}{dt} e^{rt} = \frac{d}{dt}\left(e^{\lambda t}(\cos\mu t + i\sin\mu t)\right)$$
$$= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + e^{\lambda t}(-\mu\sin\mu t + \mu i\cos\mu t)$$
$$= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + \mu e^{\lambda t}(-\sin\mu t + i\cos\mu t)$$
$$= \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t) + i\mu\, e^{\lambda t}(\cos\mu t + i\sin\mu t)$$
$$= (\lambda + i\mu)e^{\lambda t}(\cos\mu t + i\sin\mu t)$$
$$= r e^{rt}. \quad \square$$

§3.4 #6. $y'' - 6y' + 9y = 0$

$r^2 - 6r + 9 = (r-3)^2 = 0. \implies r = 3$ (repeated)

$y = c_1 e^{3t} + c_2 t e^{3t}.$

$\cancel{y' = 3c_1 e^{3t} + 3c_2 t e^{3t} + c_2 e^{3t}}$

#11. $9y'' - 12y' + 4y = 0 \quad y(0) = 2,\ y'(0) = -1$

$9r^2 - 12r + 4 = 0 \longrightarrow (3r-2)^2 = 0 \implies r = \frac{2}{3}$ (multiplicity)

$y_1 = e^{\frac{2}{3}t} \qquad y_2 = t e^{\frac{2}{3}t}.$

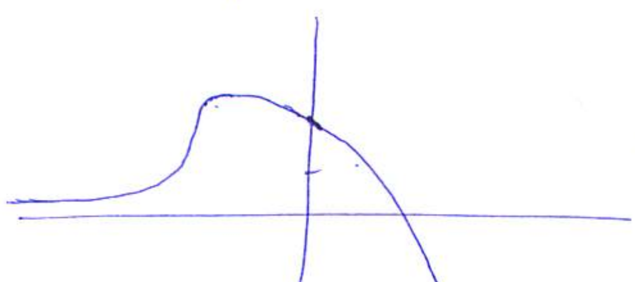
$y = c_1 e^{\frac{2}{3}t} + c_2 t e^{\frac{2}{3}t}$

$y' = \frac{2}{3}c_1 e^{\frac{2}{3}t} + c_2 e^{\frac{2}{3}t} + c_2 t \cdot \frac{2}{3} e^{\frac{2}{3}t}$

$y(0) = c_1 = 2$

$y'(0) = \frac{2}{3}c_1 + c_2 = -1 \implies c_2 = -1 - \frac{2}{3}c_1 = -1 - \frac{4}{3} = -\frac{7}{3}.$

$y = 2e^{\frac{2}{3}t} - \frac{7}{3} t e^{\frac{2}{3}t}$



~~$\lim_{t\to\infty} y(t) =$~~

$\lim_{t\to+\infty} y(t) = -\infty.$

#16. $y'' - y + 0.25y = 0$ $\quad y(0) = 2,\ y'(0) = b.$

$r^2 - r + 0.25 = 0 \Rightarrow r = \dfrac{+1 \pm \sqrt{1-1}}{2} = \frac{1}{2}$ (repeated)

$y_1 = e^{t/2}$ so $y_2 = te^{t/2}$.

Gen'l soln: $y = c_1 e^{t/2} + c_2 t e^{t/2}$ $\qquad y(0) = \underline{c_1 = 2}$

$y' = \frac{1}{2} c_1 e^{t/2} + \frac{c_2}{2} t e^{t/2} + c_2 e^{t/2}$ $\qquad y'(0) = \frac{c_1}{2} + c_2 = b$

$\Rightarrow \underline{c_2 = b-1}$

so $y = 2e^{t/2} + (b-1)te^{t/2}$

$$\lim_{t\to\infty} y(t) = \begin{cases} +\infty & \text{if } b \geq 1 \\ -\infty & \text{if } b < 1. \end{cases}$$

The transition from solutions that $\to +\infty$ from those that $\to -\infty$ occurs when $t = 1$.

#23. $t^2y'' - 4ty' + 6y = 0$, $t > 0$

Given $y_1 = t^2$, use reduction of order to find $y_2 = vy_1 = t^2 v(t)$

$y_2 = t^2 v$

$y_2' = t^2 v' + 2tv$

$y_2'' = t^2 v'' + 2tv' + 2tv' + 2v = t^2v'' + 4tv' + 2v$

Now

$$\begin{aligned} t^2y'' - 4ty' + 6y &= t^2(t^2v'' + 4tv' + 2v) \\ &\quad - 4t(t^2v' + 2tv) \\ &\quad + 6(t^2v) \\ &= t^4v'' + (\cancelto{0}{4t^3 - 4t^3})v' + (\cancelto{0}{2t^2 - 8t^2 + 6t^2})v \\ &= t^2v'' = 0 \end{aligned}$$

Since $t > 0$, we conclude that $v'' = 0. \Rightarrow v' = c \Rightarrow v = ct + d$.

so $y_2 = ct^3 + \underbrace{dt^2}_{\text{already in } y_1}$ & choose $c = 1, d = 0$: $\underline{y_2 = t^3}$