Solutions: HW6

§3.1 #4. $2y''-3y'+y=0$ 2nd order linear, const.-coef, homog.

$y=e^{rt}$: $2r^2-3r+1=(2r-1)(r-1)=0 \implies r=\frac{1}{2}$ or $r=1$.

$y_1=e^{1/2\,t}$, $y_2=e^t$ & $\underline{y=c_1e^{1/2\,t}+c_2e^t.}$

#5. $y''+5y'=0$

$y=e^{rt}$: $r^2+5r=r(r+5)=0 \implies r=0,\ r=-5$.

$y_1=e^{0t}=1$, $y_2=e^{-5t}$ & $\underline{y=c_1+c_2e^{-5t}}$

#15. $y''+8y'-9y=0$, $y(1)=1$, $y'(1)=0$.

$y=e^{rt}$: $r^2+8r-9=(r+9)(r-1)=0 \implies r=-9$ or $r=1$.

$y_1=e^{-9t}$, $y_2=e^t \implies y=c_1e^{-9t}+c_2e^t$

$y'=-9c_1e^{-9t}+c_2e^t$

To determine c_1 & c_2:

$y(1)=1$: $1=c_1e^{-9}+c_2e$

$y'(1)=0$: $0=-9c_1e^{-9}+c_2e$

$1=10c_1e^{-9} \implies c_1=\frac{1}{10}e^9$

$c_2=\dfrac{9c_1e^{-9}}{e}=\dfrac{9\cdot\frac{1}{10}e^9\cdot e^{-9}}{e}=\frac{9}{10}e^{-1}.$

so $y=\frac{1}{10}e^9e^{-9t}+\frac{9}{10}e^{-1}e^t=\frac{1}{10}e^{-9(t-1)}+\frac{9}{10}e^{t-1}.$

#20. $2y''-3y'+y=0$, $y(0)=2$, $y'(0)=\frac{1}{2}$.

$y=e^{rt}$: $2r^2-3r+1=(2r-1)(r-1)=0 \implies r=1/2$ or $r=1$.

$y_1=e^{t/2}$, $y_2=e^t \implies y=c_1e^{t/2}+c_2e^t$

$y'=\frac{c_1}{2}e^{t/2}+c_2e^t$

To determine c_1 & c_2: $y(0)=2$: $2=c_1+c_2$

$y'(0)=\frac{1}{2}$: $\frac{1}{2}=\frac{c_1}{2}+c_2$

$\frac{3}{2}=\frac{c_1}{2} \implies c_1=3 \implies c_2=2-c_1=2-3=-1.$

$\therefore\ y=3e^{t/2}-e^t.$

This solution is zero when $y=0$: $0=3e^{t/2}-e^t \implies e^t=3e^{t/2} \implies e^{t/2}=3.$

$t/2=\ln 3$ so $t=2\ln 3=\ln 9$ ← location of crossing.

$y'=\frac{3}{2}e^{t/2}-e^t=0 \implies \frac{3}{2}e^{t/2}=e^t \implies \frac{3}{2}=e^{t/2} \implies \ln(3/2)=\frac{t}{2}$

so the local maximum must occur at $t=2\ln(3/2)=\ln(9/4)$

A local max occurs at $t=2\ln(3/\cancel{4}\,2)=\ln(\cancel{81/16}\ 9/4)$ because y' changes from positive to negative here.

§3.2 #4. $W[x, xe^x] = \det \begin{bmatrix} x & xe^x \\ 1 & (x+1)e^x \end{bmatrix} = x(x+1)e^x - xe^x = x^2 e^x$

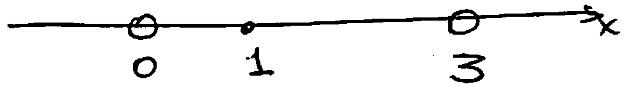
#11. $(x-3)y'' + xy' + (\ln|x|)y = 0, \quad y(1) = 0, \quad y'(1) = 1$

Put this 2^{nd} order linear homogeneous DE in standard form:

$$y'' + \frac{x}{x-3}y' + \frac{\ln|x|}{x-3}y = 0.$$

$p(x) = \dfrac{x}{x-3}$ ← continuous for $x \neq 3$.

$q(x) = \dfrac{\ln|x|}{x-3}$ ← continuous for $x \neq 3$ and $x \neq 0$.



The interval we seek is the one that includes the initial point $x=1$, that is, on $(0, 3)$.

#14. $y_1 = 1, \; y_2 = t^{1/2}$. $\qquad yy'' + (y')^2 = 0.$

$y_1 = 1: \; y_1' = 0, \; y_1'' = 0 \qquad 1(0) + 0^2 = 0$ ✓

$y_2 = t^{1/2}: \; y_2' = \frac{1}{2}t^{-1/2}, \; y_2'' = -\frac{1}{4}t^{-3/2} \qquad t^{1/2}\left(-\frac{1}{4}t^{-3/2}\right) + \left(\frac{1}{2}t^{-1/2}\right)^2$

$= -\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = 0$ ✓

$y = c_1 + c_2 t^{1/2}: \; y_2' = \frac{c_2}{2}t^{-1/2}$

$y_2'' = -\frac{c_2}{4}t^{-3/2}$

$$yy'' + (y')^2 = \left(c_1 + c_2 t^{1/2}\right)\left(-\frac{c_2}{4}t^{-3/2}\right) + \left(\frac{c_2}{2}t^{-1/2}\right)^2$$

$$= -\frac{c_1c_2}{4}t^{-3/2} - \cancel{\frac{c_2^2}{4}t^{-1}} + \cancel{\frac{c_2^2}{4}t^{-1}} = -\frac{c_1c_2}{4}t^{-3/2} \neq 0.$$

This does not contradict Theorem 3.2.2.
because the DE is not linear.

#17. Given $W[f,g] = 3e^{4t}$ and $f(t) = e^{2t}$

we find g as follows:

$$3e^{4t} = \det\begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = \begin{bmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{bmatrix} = e^{2t}g'(t) - 2e^{2t}g(t)$$

So
$$e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$$
$$g'(t) - 2\,g(t) = 3e^{2t}$$
$$\underline{\mu g'(t) - 2\mu g(t)} = 3\mu e^{3t}.$$

Find the integrating factor:
$$\frac{d}{dt}(\mu g(t)) = \mu g'(t) + \mu' g(t) = \mu g'(t) - 2\mu g(t)$$
$$\mu' \cancel{g(t)} = -2\mu \cancel{g(t)}$$
$$\frac{d\mu}{\mu} = -2\,dt \Rightarrow \ln|\mu| = -2t + C$$
$$\Rightarrow |\mu| = e^{-2t+c}$$
$$\Rightarrow \mu = Ce^{-2t}. \quad (\text{choose } C=1).$$

So
$$\frac{d}{dt}\left(e^{-2t}g(t)\right) = 3e^{-2t}e^{2t} = 3 \cdot 0.$$
$$e^{-2t}g(t) = 3t + C$$
$$g(t) = \cancel{3(t+C)}\ (3t+C)e^{2t}$$

#25. $y'' - 2y' + y = 0 \ : \ y_1 = e^t \ , y_2 = te^t$

$y_1 = e^t;\ y_1' = e^t,\ y_1'' = e^t$ $\qquad y_1'' - 2y_1' + y_1 = e^t - 2e^t + e^t = 0.$

$y_2 = te^t : y_2' = (t+1)e^t, y_2'' = (t+2)e^t$ $\quad y_2'' - 2y_2' + y_2 = (t+2)e^t + 2(t+1)e^t + te^t$

$= (t - 2t + t)e^t$

$+ (2-2)e^t$

$= 0.$

To show that y_1 & y_2 form a fundamental set of solutions to $y'' - 2y' + y = 0$, look at the Wronskian: $W[f,g] = \det\begin{pmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{pmatrix} = (\cancel{t}+1)e^{2t} - \cancel{t}e^{2t}$

$= e^{2t}$

$\neq 0.$