Solutions - HW4

§2.6 #4. $(2xy^2+2y)+(2x^2y+2x)y'=0$

$M = 2xy^2+2y \qquad \frac{\partial M}{\partial y} = 4xy+2$

$N = 2x^2y+2x \qquad \frac{\partial N}{\partial x} = 4xy+2$

$\Rightarrow$ exact.

Look for solution as $\psi(x,y)=C$ where

$\frac{\partial \psi}{\partial x} = M = 2xy^2+2y \Rightarrow \psi = \int M\,dx = x^2y^2+2xy+g(y)$

$\frac{\partial \psi}{\partial y} = N = 2x^2y+2x \qquad \frac{\partial \psi}{\partial y} = 2x^2y+2x+g'(y) = 2x^2y+2x$

$\Rightarrow g'(y)=0$

so $g(y)=0$: $\boxed{x^2y^2+2xy = C}$

#5. $\frac{dy}{dx} = -\frac{ax+by}{bx+cy} \Rightarrow (bx+cy)\frac{dy}{dx} = -(ax+by) \Rightarrow (ax+by)+(bx+cy)\frac{dy}{dx}=0$

$M = ax+by \Rightarrow \frac{\partial M}{\partial y} = b$

$N = bx+cy \qquad \frac{\partial N}{\partial x} = b$

$\Rightarrow$ exact.

Look for solution as $\psi(x,y)=C$ where

$\frac{\partial \psi}{\partial x} = M = ax+by \Rightarrow \psi = \int M\,dx = \frac{a}{2}x^2+bxy+g(y)$

$\frac{\partial \psi}{\partial y} = N = bx+cy \qquad \frac{\partial \psi}{\partial y} = bx+g'(y) = bx+cy$

$\Rightarrow g'(y)=cy$

so $g(y)=\frac{c}{2}y^2$: $\boxed{\frac{a}{2}x^2+bxy+\frac{c}{2}y^2 = C}$

#6. $\frac{dy}{dx} = -\frac{ax-by}{bx-cy} \Rightarrow (bx-cy)\frac{dy}{dx} = -(ax-by) \Rightarrow (ax-by)+(bx-cy)\frac{dy}{dx}=0$

$M = ax-by \qquad \frac{\partial M}{\partial y} = -b$

$N = bx-cy \qquad \frac{\partial N}{\partial x} = b$

$\Rightarrow$ not exact.

#20. $\frac{\sin y}{y} - 2e^{-x}\sin x + \frac{\cos y + 2e^{-x}\cos x}{y}\frac{dy}{dx} = 0$

$M = \frac{\sin y}{y} - 2e^{-x}\sin x \qquad \frac{\partial M}{\partial y} = \frac{y\cos y - \sin y}{y^2}$

$N = \frac{\cos y + 2e^{-x}\cos x}{y} \qquad \frac{\partial N}{\partial x} = \frac{2}{y}\left(-e^{-x}\cos x - e^{-x}\sin x\right)$

$\Bigg\} \Rightarrow$ not exact

$\mu = ye^x: \quad \left(e^x\sin y + 2y\sin x\right) + \left(e^x\cos y + 2\cos x\right)\frac{dy}{dx} = 0$

$\mu M = e^x\sin y + 2y\sin x \qquad \frac{\partial}{\partial y}(\mu M) = e^x\cos y + 2\sin x$

$\mu N = e^x\cos y + 2\cos x \qquad \frac{\partial}{\partial x}(\mu N) = e^x\cos y + 2\sin x$

$\Bigg\}$ exact.

Look for solution $\psi(x,y) = C$ where

$\frac{\partial\psi}{\partial x} = \mu M = e^x\sin y - 2y\sin x \implies \psi = \int \mu M\,dx = e^x\sin y + 2y\cos x + g(y)$

$\frac{\partial\psi}{\partial y} = \mu N = e^x\cos y + 2\cos x \qquad \frac{\partial\psi}{\partial y} = e^x\cos y + 2\cos x + g'(y)$

$\implies g'(y) = 0 \quad$ so $g(y) = 0$.

$\therefore$ solution is $\boxed{e^x\sin y + 2y\cos x = C.}$

#25. $\left(3x^2y + 2xy + y^3\right) + (x^2+y^2)\frac{dy}{dx} = 0$

$M = 3x^2y + 2xy + y^3 \qquad \frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$

$N = x^2 + y^2 \qquad \frac{\partial N}{\partial x} = 2x$

$\Bigg\}$ not exact but... $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3x^2+3y^2}{x^2+y^2} = 3$

so there is an integrating factor depending on x: $\mu(x)$

$\mu(x)\left(3x^2y + 2xy + y^3\right) + \mu(x)\left(x^2+y^2\right)\frac{dy}{dx} = 0$

will be exact when $\frac{\partial}{\partial y}\left(\mu(x)(3x^2y + 2xy + y^3)\right) = \frac{\partial}{\partial x}\left(\mu(x)(x^2+y^2)\right)$

$\mu(x)\left(3x^2 + \cancel{2x} + 3y^2\right) = \mu'(x)\left(x^2+y^2\right) + \cancel{\mu(x)(2x)}$

$\mu(x)\left(\frac{3x^2+3y^2}{x^2+y^2}\right) = \mu'(x)$

$3\mu(x) = \mu'(x) \implies \mu(x) = e^{3x}$

$e^{3x}\left(3x^2y + 2xy + y^3\right) + e^{3x}\left(x^2+y^2\right) = 0$ will be exact; $\psi(x,y) = C$ where

$\frac{\partial\psi}{\partial x} = e^{3x}\left(3x^2y + 2xy + y^3\right)$

$\frac{\partial\psi}{\partial y} = e^{3x}\left(x^2+y^2\right) \implies \psi = \int e^{3x}(x^2+y^2)\,dy = e^{3x}\left(x^2y + \frac{1}{3}y^3\right) + g(x)$

$\frac{\partial\psi}{\partial x} = e^{3x}(2xy) + 3e^{3x}\left(x^2y + \frac{1}{3}y^3\right) + g'(x)$

$= e^{3x}\left(2xy + 3x^2y + y^3\right) + g'(x)$

$\implies g'(x) = 0 \implies g(x) = 0$

$\therefore \boxed{e^{3x}\left(x^2y + \frac{1}{3}y^3\right) = C.}$