Solutions - HW 3.

§2.1 #3. (c) $y' + y = te^{-t} + 1$: 1st order linear (in std. form)

Find $\mu(t)$ s.t. $\mu(y'+y) = \frac{d}{dt}(\mu y) = \mu y' + \mu' y \Rightarrow \mu y = \mu' y$

$\Rightarrow \mu = \mu' \Rightarrow \mu = e^{t}$.

Then: $\frac{d}{dt}(e^{t} y) = e^{t}(te^{-t}+1) = t + e^{t}$

so $e^{t} y = \frac{1}{2}t^2 + e^{t} + C$

and $y = e^{-t}\left(\frac{1}{2}t^2 + e^{t} + C\right) = \frac{1}{2}t^2 e^{-t} + 1 + Ce^{-t}$

As $t \to \infty$, $y \to 1$ for all values of C.

#8. (c) $(1+t^2)y' + 4ty = (1+t^2)^{-2}$ 1st order linear (not in std. form)

$y' + \frac{4t}{1+t^2} y = (1+t^2)^{-3}$

Find $\mu(t)$ s.t. $\mu\left(y' + \frac{4t}{1+t^2} y\right) = \frac{d}{dt}(\mu y) = \mu y' + \mu' y$

$\Rightarrow \mu \frac{4t}{1+t^2} y = \mu' y \Rightarrow \mu' = \frac{4t}{1+t^2}\mu$ (separable)

$\frac{d\mu}{\mu} = \frac{4t}{1+t^2} dt \Rightarrow \ln|\mu| = 2\ln(1+t^2) = \ln\left((1+t^2)^2\right)$

$\Rightarrow \mu = (1+t^2)^2$.

Then: $\frac{d}{dt}\left((1+t^2)^2 y\right) = (1+t^2)^2 (1+t^2)^{-3} = \frac{1}{1+t^2}$

so $(1+t^2)^2 y = \arctan t + C$.

$y = \frac{\arctan t}{(1+t^2)^2} + \frac{C}{(1+t^2)^2}$

As $t \to \infty$, $\arctan t \to \pi/2$ and $1+t^2 \to \infty$ so $y \to 0$ for all C

#16. $y' + \frac{2}{t} y = \frac{\cos t}{t^2}$, $y(\pi) = 0$ $(t>0)$: 1st order linear, (std form)

Find $\mu(t)$ s.t. $\mu\left(y' + \frac{2}{t} y\right) = \frac{d}{dt}(\mu y) = \mu y' + \mu' y$

$\Rightarrow \mu \frac{2}{t} y = \mu' y \Rightarrow \mu' = \frac{2\mu}{t} \xrightarrow{\text{separable}} \frac{d\mu}{\mu} = \frac{2}{t} dt$

$\Rightarrow \ln|\mu| = 2\ln|t| = \ln t^2 \Rightarrow \mu = t^2$

Then $\frac{d}{dt}(t^2 y) = \cos t$

$t^2 y = \sin t + C$

$y(\pi) = 0$: $\pi^2 \cdot 0 = \sin \pi + C$

$0 = 0 + C$

$C = 0$

$\therefore t^2 y = \sin t$

$y = \frac{\sin t}{t^2}$.

#31. $y' - \frac{3}{2}y = 3t + 2e^t$, $y(0) = y_0$ $\quad$ 1^{st} order linear (std. form)

Find $\mu(t)$ s.t. $\mu(y' - \frac{3}{2}y) = \frac{d}{dt}(\mu y) = \mu y' + \mu' y \Rightarrow -\frac{3}{2}\mu y = \mu' y$

$\Rightarrow \mu' = -\frac{3}{2}\mu \Rightarrow \mu = e^{-3/2 t}$

Then $\frac{d}{dt}\left(e^{-3/2 t} y\right) = e^{-3/2 t}(3t + 2e^t) = 3te^{-3/2 t} + 2e^{-t/2}$

so $e^{-3/2 t} y = \int \underbrace{3te^{-3/2 t}}_{\text{int by parts.}} dt - 4e^{-t/2} + C$

$u = 3t \quad dv = e^{-3/2 t}$

$du = 3dt \quad v = -\frac{2}{3}e^{-3/2 t}$

$= 3t\left(-\frac{2}{3}e^{-3/2 t}\right) - \int -\frac{2}{3}e^{-3/2 t} \cdot 3\, dt - 4e^{-t/2} + C$

$= -2te^{-3/2 t} + 2\int e^{-3/2 t} dt - 4e^{-t/2} + C$

$= -2te^{-3/2 t} - \frac{4}{3}e^{-3/2 t} - 4e^{-t/2} + C.$

$\therefore y = e^{3/2 t}\left(-2te^{-3/2 t} - \frac{4}{3}e^{-3/2 t} - 4e^{-t/2} + C\right)$

$= -2t - \frac{4}{3} - 4e^t + Ce^{3/2 t}$

$y(0) = y_0$: $y_0 = -2 \cdot 0 - \frac{4}{3} - 4e^0 + Ce^0 = -\frac{4}{3} - 4 + C = -\frac{16}{3} + C$

$\Rightarrow C = y_0 + \frac{16}{3}.$

so $y = -2t - \frac{4}{3} - 4e^t + \left(y_0 + \frac{16}{3}\right)e^{3/2 t}.$

The dominant term is $\left(y_0 + \frac{16}{3}\right)e^{3/2 t}$

and $\left(y_0 + \frac{16}{3}\right)e^{3/2 t} \to +\infty$ when $y_0 + \frac{16}{3} > 0$ & $\to -\infty$ when $y_0 + \frac{16}{3} < 0$.

so, $y_0 = -\frac{16}{3}$ is the value of y_0 where the behavior of the solutions changes.

§2.4 #4. $(4-t^2)y' + 2ty = 3t^2$, $y(-3) = 1$.

In standard form: $y' + \frac{2t}{4-t^2} y = \frac{3t^2}{4-t^2}$

This linear DE has a solution on the interval where $p(t) = \frac{2t}{4-t^2}$ and $g(t) = \frac{3t^2}{4-t^2}$ are continuous and contains $t = -3$. The coefficients are continuous on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$. The interval with $t = -3$ is $\boxed{(-\infty, -2).}$

#5. $(4-t^2)y' + 2ty = 3t^2$, $y(1) = -3$.

This is the same DE as #4, just $t = 1$ (not $t = -3$).

The interval with $t = 1$ is $\boxed{(-2, 2).}$

#22. (a) $y_1 = 1 - t$; $y_1' = -1$; $\stackrel{?}{=}$ $\frac{-t + (t^2 + 4(1-t))^{1/2}}{2} = \frac{-t + (t^2 - 4t + 4)^{1/2}}{2}$

$[y_1(2) = 1 - 2 = -1]$ $= \frac{-t + (t-2)}{2} = -\frac{2}{2} = -1 = y_1'$

$y_2 = -\frac{t^2}{4}$: $y_2' = -\frac{2t}{4} = -\frac{t}{2}$; $\frac{-t + (t^2 + 4(-t^2/4))^{1/2}}{2} = \frac{-t + 0}{2} = -\frac{t}{2} = y_2'$

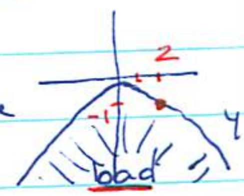
$[y_2(2) = -\frac{4}{4} = -1]$

To have 2 sol'ns to the same IVP (1st order), the hypotheses must not be satisfied for the existence & uniqueness thm must not be satisfied.

$f(t, y) = \frac{-t + (t^2 + 4y)^{1/2}}{2}$ and $\frac{\partial f}{\partial y} = \frac{\frac{1}{2}(t^2 + 4y)^{-1/2}(4)}{2} = \frac{1}{(t^2 + 4y)^{1/2}}$

are not continuous when $t^2 + 4y \leq 0$ $(y \leq -t^2/4)$

The initial point, $(2, -1)$, is on this parabola, so there is no rectangle containing $(2, -1)$ on which f & $\frac{\partial f}{\partial y}$ are continuous.



#23. (a) $y = \varphi(t) = e^{2t}$: $y' = \varphi' = 2e^{2t}$, $y' - 2y = 2e^{2t} - 2e^{2t} = 0$.

$y = c\varphi(t) = ce^{2t}$: $y' = 2ce^{2t}$, $y' - 2y = 2ce^{2t} - 2ce^{2t} = 0$.

(b) $y = \varphi(t) = t^{-1}$: $y' = -t^{-2}$, $y' + y^2 = -t^{-2} + (t^{-1})^2 = -t^{-2} + t^{-2} = 0$.

$y = c\varphi(t) = ct^{-1}$: $y' = -ct^{-2}$, $y' + y^2 = -ct^{-2} + (ct^{-1})^2 = -ct^{-2} + c^2t^{-2}$

$= (c^2 - c)t^{-2} = 0$

only if $c^2 - c = 0$

that is, $c = 0$ or $c = 1$