Solutions - HW #12

§7.1 #4. $u^{(4)} - u = 0$

Let
$$x_1 = u(t), \quad x_2 = u'(t), \quad x_3 = u''(t), \quad x_4 = u'''(t)$$

Then
$$x_1' = u' = x_2$$
$$x_2' = u'' = x_3$$
$$x_3' = u''' = x_4$$
$$x_4' = u^{(4)} = u = x_1$$

So the equivalent 1st order system is

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ x_4' &= x_1 \end{aligned}}$$

#5. $u'' + 0.25u' + 4u = 2\cos 3t \qquad u(0) = 1, u'(0) = -2.$

Let
$$x_1 = u(t), \quad x_2 = u'(t)$$

Then
$$x_1' = u' = x_2$$
$$x_2' = u'' = -\tfrac{1}{4}u' - 4u + 2\cos 3t$$
$$= -\tfrac{1}{4}x_2 - 4x_1 + 2\cos 3t$$

Also $x_1(0) = u(0) = 1$ & $x_2(0) = u'(0) = -2.$

So the equivalent 1st order linear system is

$$\boxed{\begin{aligned} x_1' &= x_2 & \qquad x_1(0) &= 1 \\ x_2' &= -4x_1 - \tfrac{1}{4}x_2 + 2\cos 3t & \qquad x_2(0) &= -2. \end{aligned}}$$

§7.2 #2. $A = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} \quad B = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}$

a) $A - 2B = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} - \begin{pmatrix} 2i & 6 \\ 4 & -4i \end{pmatrix} = \begin{pmatrix} 1-i & -7+2i \\ -1+2i & 2+3i \end{pmatrix}$

b) $3A + B = \begin{pmatrix} 3+3i & -3+6i \\ 9+6i & 6-3i \end{pmatrix} + \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix} = \begin{pmatrix} 3+4i & 6i \\ 11+6i & 6-5i \end{pmatrix}$

c) $AB = \begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix}\begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix} = \begin{pmatrix} i-1-2+4i & 3+3i+2i+4 \\ 3i-2+4-2i & 9+6i-4i-2 \end{pmatrix}$

$= \begin{pmatrix} -3+5i & 7+5i \\ 2+i & 7+2i \end{pmatrix}$

d) $BA = \begin{pmatrix} i & 3 \\ 2 & -2i \end{pmatrix}\begin{pmatrix} 1+i & -1+2i \\ 3+2i & 2-i \end{pmatrix} = \begin{pmatrix} i-1+9+6i & -i-2+6-3i \\ 2+2i-6i+4 & -2+4i-4i-2 \end{pmatrix}$

$= \begin{pmatrix} 8+7i & 4-4i \\ 6-4i & -4 \end{pmatrix}$

#8. $\bar{x} = \begin{pmatrix} 2 \\ 3i \\ 1-i \end{pmatrix}$, $\bar{y} = \begin{pmatrix} -1+i \\ 2 \\ 3-i \end{pmatrix}$. (Note: $\vec{a}^T\vec{b} = \vec{a}\cdot\vec{b}$)

a) $\vec{x}^T\vec{y} = -2+2i+6i+(3-i-3i-1) = \underline{0+4i}$

b) $\vec{y}^T\vec{y} = (-1+i)^2+2^2+(3-i)^2 = 1-1+2(-1)i+4+9-1+2(3)(-1)i$
$= \underline{12-8i}$

c) $(\vec{x},\vec{y}) = \vec{x}^T\bar{\vec{y}} = \vec{x}\cdot\bar{\vec{y}} = 2(-1-i)+3i(2)+(1-i)(3+i)$
$= -2-2i+6i+3+i-3i+1 = \underline{2+2i}$

d) $(\vec{y},\vec{y}) = \vec{y}^T\bar{\vec{y}} = \vec{y}\cdot\bar{\vec{y}} = (-1+i)(-1-i)+2^2+(3-i)(3+i)$
$= 1+1+4+9+1 = \underline{16}$

#12. $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ To attempt to find A^{-1}:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 5 & 0 & 1 & 0 \\ 3 & 5 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(3)-3(1)\to(3)]{(2)-2(1)\to(2)} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & 1 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{(2)\leftrightarrow(3)} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 0 & 1 \\ 0 & 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow{\substack{-(2) \\ -(3)}} \left[\begin{array}{cccccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 0 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[(1)-3(3)\to(1)]{(2)-3(3)\to(2)} \left[\begin{array}{cccccc} 1 & 2 & 0 & -5 & 3 & 0 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{(1)-2(2)\to(1)} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -3 & 2 \\ 0 & 1 & 0 & -3 & 3 & -1 \\ 0 & 0 & 1 & 2 & -1 & 0 \end{array}\right] = [I \mid A^{-1}]$$

so $A^{-1} = \underline{\begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}}$

Check: $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

#21. $A(t) = \begin{bmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{bmatrix}$ $\quad B(t) = \begin{bmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{bmatrix}$

a) $A+3B = \begin{bmatrix} 7e^t & 5e^{-t} & 10e^{2t} \\ -e^t & 7e^{-t} & 2e^{2t} \\ 8e^t & 0 & -e^{2t} \end{bmatrix}$

b) $AB = \begin{bmatrix} 2e^{2t} - 2 + 3e^{3t} & 1 + 4e^{-2t} - e^t & 3e^{3t} + 2e^t - e^{4t} \\ 4e^{2t} - 1 - 3e^{3t} & 2 + 2e^{-2t} + e^t & 6e^{3t} + e^t + e^{4t} \\ -2e^{2t} - 3 + 6e^{3t} & -1 + 6e^{-2t} - 2e^t & -3e^t + 3e^t - 2e^{4t} \end{bmatrix}.$

c) $\dfrac{dA}{dt} = \begin{bmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{bmatrix}.$

d) $\displaystyle\int_0^1 A(t)\,dt = \begin{bmatrix} e-1 & -2(e^{-1}-1) & \frac{1}{2}(e^2-1) \\ 2(e-1) & -(e^{-1}-1) & -\frac{1}{2}(e^2-1) \\ -(e-1) & -3(e^{-1}-1) & (e^2-1) \end{bmatrix}.$