Exam 3 - Solutions

#1. $y'' - xy' - y = 0, \quad x_0 = 0$

a) $y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$0 = y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= \left(2 \cdot 1 \cdot a_2 x^0 - a_0\right) + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} - n a_n - a_n\right) x^n$$

$$= \left(2a_2 - a_0\right) x^0 + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} - (n+1) a_n\right) x^n$$

So $\boxed{\underline{n=0}\quad 2a_2 - a_0 = 0}$

$\underline{n=1,2,\dots}\quad (n+2)(n+1) a_{n+2} - (n+1) a_n = 0$

$(n+1)\left((n+2) a_{n+2} - a_n\right) = 0$

$$\boxed{a_{n+2} = \frac{a_n}{n+2} \qquad (n = 1, 2, \dots)}$$

b)
$n=0: \quad a_2 = \frac{1}{2} a_0$

$n=1: \quad a_3 = \frac{1}{3} a_1$

$n=2: \quad a_4 = \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0$

$n=3: \quad a_5 = \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$

$n=4: \quad a_6 = \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0$

$n=5: \quad a_7 = \frac{1}{7} a_5 = \frac{1}{7 \cdot 5 \cdot 3} a_1$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \dots$$

$$= a_0 + \frac{1}{2} a_0 x^2 + \frac{1}{8} a_0 x^4 + \frac{1}{48} a_0 x^6 + \dots$$
$$a_1 x + \frac{1}{3} a_1 x^3 + \frac{1}{15} a_1 x^5 + \frac{1}{105} a_1 x^7 + \dots$$

$$= a_0 \underbrace{\left(1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \dots\right)}_{y_1}$$
$$+ a_1 \underbrace{\left(x + \frac{x^3}{3} + \frac{x^5}{15} + \frac{x^7}{105} + \dots\right)}_{y_2}$$

c) It's easiest to evaluate at the ~~Wrons~~ center of the series: $\underline{x = x_0 = 0}$

$$W[y_1, y_2](0) = \det \begin{bmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \neq 0$$

$y_1 = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \dots \qquad y_2 = x + \frac{x^3}{3} + \dots$

$y_1' = x + \frac{x^3}{2} + \dots \qquad y_2' = 1 + x^2 + \dots$

Because $W[y_1, y_2](0) \neq 0$, the Wronskian is never zero.

As a result, the solutions y_1 & y_2 are a fundamental set of solutions for this DE.

2. $x^2 y'' + (1+x) y' + 3(\ln x) y = 0$

$y(1) = 2$

$y'(1) = 0$

a) plug in $x=1$: $1\, y''(1) + 2y'(1) + 3 \ln 1\, y(1) = y''(1) + 0 + 0 = \underline{y''(1) = 0}$

differentiate: $x^2 y''' + 2xy'' + (1+x)y'' + y' + 3(\ln x) y' + \frac{3}{x} y = 0$

plug in $x=1$: $1\, y'''(1) + 2\cdot 0 + 2 \cdot 0 + 0 + 0\cdot 0 + 3 \cdot 2 = 0$

$y'''(1) + 6 = 0$

$\underline{y'''(1) = -6}$

differentiate: $x^2 y^{(4)} + 2xy''' + 2xy''' + 2y'' + (1+x)y''' + y'' + y'' + 3(\ln x) y'' + \frac{3}{x} y'$
$+ \frac{3}{x} y' - \frac{3}{x^2} y = 0$

plug in $x=1$: $y^{(4)}(1) + 2(-6) + 2(-6) + 2\cdot 0 + 2\cdot(-6) + 0 + 0 + 3(\ln 1) 0 + \frac{3}{1}\cdot 0$
$+ \frac{3}{1} \cdot 0 - \frac{3}{1^2}(2) = 0$

$y^{(4)}(1) - 12 - 12 - 12 - 6 = 0$

$\underline{y^{(4)}(1) = 42}$

b) We know that $y = \sum_{n=0}^{\infty} \frac{y^{(n)}(1)}{n!} (x-1)^n$

so the power series solution $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

must have $a_n = \frac{y^{(n)}(1)}{n!}$

With $n=0$: $a_0 = \frac{y(1)}{0!} = \frac{2}{1} = 2$

$n=1$: $a_1 = \frac{y'(1)}{1!} = \frac{0}{1} = 0$

$n=2$: $a_2 = \frac{y''(1)}{2!} = \frac{0}{2} = 0$

$n=3$: $a_3 = \frac{y'''(1)}{3!} = \frac{-6}{6} = -1$

$n=4$: $a_4 = \frac{y^{(4)}(1)}{4!} = \frac{42}{24} = \frac{7}{4}$

so $y = a_0 (x-1)^0 + a_1 (x-1)^1 + a_2 (x-1)^2 + a_3 (x-1)^3 + a_4 (x-1)^4 + \ldots$

$= 2 - (x-1)^3 + \frac{7}{4} (x-1)^4 + \ldots$

3. $x^2y'' - 3xy' + 4y = 0$ (Euler eqn.)

$y(-1) = 2$

$y'(-1) = 3$

Sol'n in the form $y = x^r$ leads to

$$r(r-1) - 3r + 4 = 0$$
$$r^2 - r - 3r + 4 = 0$$
$$r^2 - 4r + 4 = 0$$
$$(r-2)^2 = 0$$
$$r = 2, 2. \implies y_1 = x^2$$
$$y_2 = x^2 \ln|x|.$$

Gen'l sol'n: $y = c_1 x^2 + c_2 x^2 \ln|x|.$

$$y' = 2c_1 x + 2c_2 x \ln|x| + c_2 x^2 \cdot \frac{1}{x}$$

$$y(-1) = c_1(-1)^2 + c_2(-1)^2 \ln|-1| = c_1 = 2$$

$$y'(-1) = 2c_1(-1) + 2c_2(-1)\ln|-1| + c_2(-1) = -2c_1 - c_2 = 3.$$

$$c_2 = -2c_1 - 3$$
$$= -2(2) - 3 = -7$$

$\therefore$ $\boxed{y = 2x^2 - 7x^2 \ln|x|.}$

4. $t^2 u'' + tu' + (t^2 - 4)u = 0 \implies u'' = -\frac{1}{t^2}\left(tu' + (t^2-4)u\right)$

$$= -\frac{1}{t}u' + \left(\frac{4}{t^2} - 1\right)u$$

Let $x_1 = u$

$x_2 = u'$

Then $x_1' = u' = x_2$

and $x_2' = u'' = -\frac{1}{t}u' + \left(\frac{4}{t^2} - 1\right)u = -\frac{1}{t}x_2 + \left(\frac{4}{t^2} - 1\right)x_1$

so

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{1}{t}x_2 + \left(\frac{4}{t^2} - 1\right)x_1. \end{aligned}}$$

5. $A = \begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$

To check if A is nonsingular, try to reduce A to I:

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 8 & 0 & 1 & 0 \\ 1 & -2 & -7 & 0 & 0 & 1 \end{array}\right) \xrightarrow[(3)-(1)\to(3)]{(2)+2(1)\to(2)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 5 & 10 & 2 & 1 & 0 \\ 0 & -4 & -8 & -1 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[+\frac{1}{4}(3)\to(3)]{\frac{1}{5}(2)\to(2)} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & -1 & -2 & -1/4 & 0 & 1/4 \end{pmatrix} \xrightarrow[(3)+(2)\to(3)]{} \begin{pmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2/5 & 1/5 & 0 \\ 0 & 0 & \boxed{0} & 3/20 & 1/5 & 1/4 \end{pmatrix}$$

Because there is not a pivot in the $(3,3)$ location, A is singular.

Alternatively, you could find

$$\det A = 1(-7+16) - 2(14-8) + 1(4-1)$$
$$= 9 - 12 + 3$$
$$= 0$$

which also shows that A is singular.

6. We are given $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + 2\begin{pmatrix} 1 \\ 1 \end{pmatrix} te^t = \begin{pmatrix} e^t + 2te^t \\ 2te^t \end{pmatrix}$.

Then $\vec{x}\,' = \begin{pmatrix} e^t + 2e^t + 2te^t \\ 2e^t + 2te^t \end{pmatrix} = \begin{pmatrix} 3e^t + 2te^t \\ 2e^t + 2te^t \end{pmatrix}$.

and $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} e^t + 2te^t \\ 2te^t \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$

$$= \begin{pmatrix} 2(e^t + 2te^t) - 2te^t \\ 3(e^t + 2te^t) - 4te^t \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$= \begin{pmatrix} 2e^t + 2te^t \\ 3e^t + 2te^t \end{pmatrix} + \begin{pmatrix} e^t \\ -e^t \end{pmatrix}$$

$$= \begin{pmatrix} 3e^t + 2te^t \\ 2e^t + 2te^t \end{pmatrix} = \vec{x}\,'$$

so this $\vec{x}$ is a solution to the DE in this problem.