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The Truth of It All

The objective of mathematicians is to discover and to communicate certain truths. *Mathematics* is the language of mathematicians, and a *proof* is a method of communicating a mathematical truth to another person who also "speaks" the language. A remarkable property of the language of mathematics is its precision. Properly presented, a proof contains no ambiguity—there will be no doubt about its correctness. Unfortunately, many proofs that appear in textbooks and journal articles are presented for someone who already knows the language of mathematics. Thus, to understand and present a proof, you must learn a new language, a new method of thought. This book explains much of the basic grammar, but as in learning any new language, a lot of practice is needed to become fluent.

1.1 THE OBJECTIVES OF THIS BOOK

The approach taken here is to categorize and to explain the various **proof techniques** that are used in *all* proofs, regardless of the subject matter. One objective is to teach you how to read and understand a written proof by identifying the techniques that are used. Learning to do so enables you to study almost any mathematical subject on your own—a desirable goal in itself.

A second objective is to teach you to develop and to communicate your own proofs of known mathematical truths. Doing so requires you to use a certain amount of creativity, intuition, and experience. Just as there are many ways xpress the same idea in any language, so there are different proofs for same mathematical fact. The techniques presented here are designed to you started and to guide you through a proof. Consequently, this book cribes not only *how* the techniques work but also *when* each one is likely to used and *why*. Often you will be able to choose a correct technique based the form of the problem under consideration. Therefore, when attempting reate your own proof, *learn to select a technique consciously* before wasting urs trying to figure out what to do. The more aware you are of your thought cesses, the better.

The ultimate objective, however, is to use your newly acquired skills and guage to discover and communicate previously unknown mathematical ths. The first step in this direction is to reach the level of being able to d proofs and develop your own proofs of already-known facts. This alone l give you a much deeper and richer understanding of the mathematical verse around you.

Anyone with a good knowledge of high school mathematics can read this ok. Advanced students who have seen proofs before can read the first two pters, skip to the summary Chapter 15, and subsequently read any of the pendices to see how all the techniques fit together in a specific mathematical ject. Each chapter on a particular technique also contains a brief summary the end that describes how and when to use the technique. The remainder this chapter explains the types of relationships to which proofs are applied. ditional books on proofs and advanced mathematical reasoning are listed the bibliography at the end of this book.

2 WHAT IS A PROOF?

proof is a convincing argument expressed in the language of mathematics at a statement is true. All of the foregoing words are important to your derstanding of what a proof is. For example, in mathematics, a **statement** a sentence that is either true or false. Some examples follow:

1. Two parallel lines in a plane have the same slope.
2. $1 = 0$.
3. The real number $x \not> 0$ (x is not greater than 0).
4. There is an angle t such that $\cos(t) = t$.

bserve that statement (1) is true, (2) is false, and (3) is either true or false, pending on the value of a variable.

It is perhaps not as obvious that statement (4) is also true. Furthermore, statement that appears to be true can, in fact, be false. This is why it is cessary to do proofs—you will know that a statement is true only when

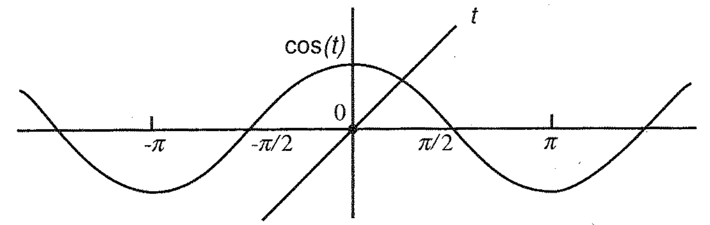

Fig. 1.1 A proof that there is an angle t such that $\cos(t) = t$.

you have *proved* it to be true. In this and other books, proofs are often given for what seem to be obviously true statements. One reason for doing so is to provide examples that are easy to follow so that you can eventually prove more difficult statements.

A proof should contain enough mathematical details to be convincing to the person(s) to whom the proof is addressed. A proof of statement (4) that is meant to convince a mathematics professor might consist of nothing more than Figure 1.1; whereas a proof directed toward a high school student would require more details, perhaps even the definition of cosine. Your proofs should contain enough details to be convincing to someone else at your own mathematical level (for example, a classmate). It is the lack of sufficient detail that often makes a proof difficult to read and understand. One objective of this book is to teach you to decipher "condensed" proofs that typically appear in textbooks and other mathematical literature.

Given two statements A and B, each of which may be either true or false, a fundamental problem of interest in mathematics is to show that the following **conditional statement**—also called an **implication**—is true:

If A is true, then B is true.

One reason for wanting to prove that an implication is true is when B is a statement that you would like to be true but whose truth is not easy to verify. In contrast, suppose that A is a statement whose truth is relatively easy to verify. If you have proved that "If A is true, then B is true," and if you can verify that A is in fact true, then you will know that B is true. For brevity, the statement "If A is true, then B is true" is shortened to "If A, then B" or simply "A **implies** B." Mathematicians have developed a symbolic shorthand notation and would write "$A \Rightarrow B$" instead of "A implies B." For the most part, textbooks do not use the symbolic notation, but teachers often do, and eventually you might find it useful, too. Therefore, notational symbols are included in this book but are not used in the proofs. A complete list of symbols is presented in the glossary at the end of this book.

When working with the implication "A implies B," it is important to realize that there are three separate statements: the statement A which is called the **hypothesis**, the statement B which is called the **conclusion**, and the

Table 1.1 The Truth of "A Implies B."

A	B	A implies B
True	True	True
True	False	False
False	True	True
False	False	True

tement "A implies B." To prove that "A implies B" is true, you must actly what it means for such a statement to be true. In particular, the nditions under which "A implies B" are true depend on whether A and B emselves are true. Thus, there are four possible cases to consider:

1. A is true and B is true.
2. A is true and B is false.
3. A is false and B is true.
4. A is false and B is false.

Suppose, for example, that your friend made the statement,

$$\text{If } \underbrace{\text{you study hard}}_{A}\text{, then } \underbrace{\text{you will get a good grade.}}_{B}\text{"}$$

determine when this statement "A implies B" is false, ask yourself in which the four foregoing cases you would be willing to call your friend a liar. In e first case—that is, when you study hard (A is true) and you get a good ade (B is true)—your friend has told the truth. In the second case, you udied hard, and yet you did not get a good grade, as your friend said you ould. Here your friend has not told the truth. In cases 3 and 4, you did ot study hard. You would not want to call your friend a liar in these cases ecause your friend said that something would happen only when you did udy hard. Thus, the statement "A implies B" is true in each of the four ses except the second one, as summarized in Table 1.1.

Table 1.1 is an example of a **truth table**, which is a method for determining hen a complex statement (in this case, "A implies B") is true by examining l possible truth values of the individual statements (in this case, A and B). ther examples of truth tables appear in Chapter 3.

According to Table 1.1, when trying to show that "A implies B" is true, u might attempt to determine the truth of A and B individually and then e the appropriate row of the table to determine the truth of "A implies B." r example, to determine the truth of the statement,

$$\text{if } 1 < 2, \text{ then } 4 < 3,$$

u can easily see that the hypothesis A (that is, $1 < 2$) is true and the nclusion B (that is, $4 < 3$) is false. Thus, using the second row of Table 1.1 orresponding to A being true and B being false) you can conclude that in

$$\text{if } 2 < 1, \text{ then } 3 < 4,$$

is true according to the third row of the table because A (that is, $2 < 1$) is false and B (that is, $3 < 4$) is true.

Now suppose you want to prove that the following statement is true:

$$\text{if } x > 2, \text{ then } x^2 > 4.$$

The difficulty with using Table 1.1 for this example is that you cannot determine whether A (that is, $x > 2$) and B (that is, $x^2 > 4$) are true or false because the truth of the statements A and B depend on the variable x, whose value is not known. Nonetheless, you can still use Table 1.1 by reasoning as follows:

> Although I do not know the truth of A, I do know that the statement A must be either true or false. Let me assume, for the moment, that A is false (subsequently, I will consider what happens when A is true). When A is false, either the third or the fourth row of Table 1.1 is applicable and, in either case, the statement "A implies B" is true—thus I would be done. Therefore, I need only consider the case in which A is true.
>
> When A is true, either the first or the second row of Table 1.1 is applicable. However, because I want to prove that "A implies B" is true, I need to be sure that the first row of the truth table is applicable, and this I can do by establishing that B is true.

From the foregoing reasoning, when trying to prove that "A implies B" is true, *you can assume that A is true; your job is to conclude that B is true.*

Note that a proof of the statement "A implies B" is not an attempt to verify whether A and B themselves are true but rather to show that B is a logical result of having assumed that A is true. Your ability to show that B is true depends on the fact that you have assumed A to be true; ultimately, you have to discover the relationship between A and B. Doing so requires a certain amount of creativity. The techniques presented here are designed to get you started and guide you along the path.

The first step in doing a proof is to identify the hypothesis A and the conclusion B. This is easy to do when the implication is written in the form "If A, then B" because everything after the word "if" and before the word "then" is A and everything after the word "then" is B. Unfortunately, implications are not always written in this specific form. In such cases, everything that you are assuming to be true is the hypothesis A; everything that you are trying to prove is true is the conclusion B. You might have to interpret the meaning of symbols from the context in which they are used and even introduce your own notation sometimes. Consider the following examples.

ımple 1: The sum of the first n positive integers is $n(n+1)/2$.

Hypothesis: n is a positive integer. (Note that this is implied for the statement to make sense.)
Conclusion: The sum of the first n positive integers is $n(n+1)/2$.

ımple 2: The quadratic equation $ax^2 + bx + c = 0$ has two real roots vided that $b^2 - 4ac > 0$, where $a \neq 0$, b, and c are given real numbers.

Hypothesis: a, b, and c are real numbers with $a \neq 0$ and $b^2 - 4ac > 0$.
Conclusion: The quadratic equation $ax^2+bx+c=0$ has two real roots.

ample 3: Two lines tangent to the endpoints of the diameter of a circle parallel.

Hypothesis: L_1 and L_2 are two lines that are tangent to the endpoints of the diameter of a circle.
Conclusion: L_1 and L_2 are parallel.

ample 4: There is a real number x such that $x = 2^{-x}$.

Hypothesis: None, other than your previous knowledge of mathematics.
Conclusion: There is a real number x such that $x = 2^{-x}$.

ore starting a proof, always be clear what you are assuming—that is, the othesis A—and what you are trying to show—that is, the conclusion B.

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roof is a convincing argument, expressed as a sequence of proof techniques, t a statement is true. Of particular interest is an implication in which A l B are given statements that are either true or false. The problem is to ve that "A implies B" is true. According to Table 1.1, after identifying hypothesis A and conclusion B, you should assume that A is true and use s assumption to reach the conclusion that B is true.

ercises

te: Solutions to those exercises marked with a W are located on the web http://www.wiley.com/college/solow/.

l Which of the following are mathematical statements?

. $ax^2 + bx + c = 0$.

b. $(-b + \sqrt{b^2 - 4ac})/(2a)$.

. Triangle XYZ is similar to triangle RST.

d. $3 + n + n^2$.

. For every angle t, $\sin^2(t) + \cos^2(t) = 1$.

1.2 Which of the following are mathematical statements?

a. There is an even integer n that, when divided by 2, is odd.

b. {integers n such that n is even}.

c. If x is a positive real number, then $\log_{10}(x) > 0$.

d. $\sin(\pi/2) < \sin(\pi/4)$.

^{W}**1.3** For each of the following problems, identify the hypothesis (what you can assume is true) and the conclusion (what you are trying to show is true).

a. If the right triangle XYZ with sides of lengths x and y and hypotenuse of length z has an area of $z^2/4$, then the triangle XYZ is isosceles.

b. n^2 is an even integer provided that n is an even integer.

c. Let a, b, c, d, e, and f be real numbers. You can solve the two linear equations $ax + by = e$ and $cx + dy = f$ for x and y when $ad - bc \neq 0$.

1.4 For each of the following problems, identify the hypothesis (what you can assume is true) and the conclusion (what you are trying to show is true).

a. r is irrational if r is a real number that satisfies $r^2 = 2$.

b. If p and q are positive real numbers with $\sqrt{pq} \neq (p+q)/2$, then $p \neq q$.

c. Let $f(x) = 2^{-x}$ for any real number x. Then $f(x) = x$ for some real number x with $0 \leq x \leq 1$.

^{W}**1.5** For each of the following problems, identify the hypothesis (what you can assume is true) and the conclusion (what you are trying to show is true).

a. Suppose that A and B are sets of real numbers with $A \subseteq B$. For any set C of real numbers, it follows that $A \cap C \subseteq B \cap C$.

b. For a positive integer n, define the following function:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$$

Then for any positive integer n, there is an integer $k > 0$ such that $f^k(n) = 1$, where $f^k(n) = f^{k-1}(f(n))$, and $f^1(n) = f(n)$.

c. When x is a real number, the minimum value of $x(x-1) \geq -1/4$.

^{W}**1.6** "If I do not get my car fixed, I will miss my job interview," says Jack. Later, you come to know that Jack's car was repaired but that he missed his job interview. Was Jack's statement true or false? Explain.

"If I get my car fixed, I will not miss my job interview," says Jack. ;er, you come to know that Jack's car was repaired but that he missed his interview. Was Jack's statement true or false? Explain.

; Suppose someone says to you that the following statement is true: "If :k is younger than his father, then Jack will not lose the contest." Did Jack 1 the contest? Why or why not? Explain.

9 Determine the conditions on the hypothesis A and conclusion B under .ich the following statements are true and false and give your reason.

a. If $2 > 7$, then $1 > 3$. b. If $x = 3$, then $1 < 2$.

10 Determine the conditions on the hypothesis A and conclusion B under ιich the following statements are true and false and give your reason.

a. If $2 < 7$, then $1 < 3$. b. If $x = 3$, then $1 > 2$.

11 If you are trying to prove that "A implies B" is true and you know at B is false, do you want to show that A is true or false? Explain.

12 By considering what happens when A is true and when A is false, was decided that to prove the statement "A implies B" is true, you can sume that A is true and your goal is to show that B is true. Use the same pe of reasoning to derive another approach for proving that "A implies B" true by considering what happens when B is true and when B is false.

.13 Using Table 1.1, prepare a truth table for "A implies (B implies C)."

14 Using Table 1.1, prepare a truth table for "(A implies B) implies C."

15 Using Table 1.1, prepare a truth table for "$B \Rightarrow A$." Is this statement ue under the same conditions for which "$A \Rightarrow B$" is true?

.16 Suppose you want to show that $A \Rightarrow B$ is false. According to Table 1, how should you do this? What should you try to show about the truth A and B? (Doing this is referred to as a **counterexample** to $A \Rightarrow B$.)

.17 Apply your answer to Exercise 1.16 to show that each of the following atements is false by constructing a counterexample.

a. If $x > 0$, then $\log_{10}(x) > 0$.

b. If n is a positive integer, then $n^3 \geq n!$ (where $n! = n(n-1)\cdots 1$).

.18 Apply your answer to Exercise 1.16 to show that each of the following ;atements is false by constructing a counterexample.

a. If n is a positive integer, then $3^n \geq n!$ (where $n! = n(n-1)\cdots 1$).

b. If x is a positive real number between 0 and 1, then the first three decimal digits of x are not equal to the first three decimal digits of 2^{-x}.

2 The Forward-Backward Method

The purpose of this chapter is to describe the fundamental proof techniques the **forward-backward method**. Special emphasis is given to the material of this chapter because all other proof techniques rely on this method.

Recall from Chapter 1 that, when proving "A implies B," you can assume that A is true and you must use this information to reach the conclusion that B is true. In attempting to reach the conclusion that B is true, you will go through a **backward process**. When you make specific use of the information contained in A, you will go through a **forward process**. Both of these processes are described in detail now using the following example.

Proposition 1 *If the right triangle XYZ with sides of lengths x and y and hypotenuse of length z has an area of $z^2/4$, then the triangle XYZ is isosceles (see Figure 2.1).*

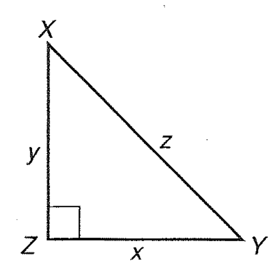

Fig. 2.1 The right triangle XYZ.