Chapter 9 Solutions
Math 300 – Spring 2014
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1(a). Assume that l, m, n are three consecutive integers and 24 does divide $l^2 + m^2 + n^2 + 1$.
(b). Assume that integer n>2, $x^n+y^n = z^n$ has an integer solution for x,y,z.
(c). Assume that f and g are two functions with $g \geq f$ and f is unbounded above, then g is bounded above.

2(a). Assume that n is an integer with n>2 and there does exist positive integers x,y, and z such that $x^n + y^n = z^n$.
(b). Assume that a is a positive real number and there exist real numbers b, c and M, with M>0, such that $ax^2+bx+c>M$.
(c). Assume that the matrix M is not singular, and the rows of M are linearly dependent.

3(a). By contradiction, assume a and b are integers with b, odd, and +1,-1 are roots of ax^2+bx+a. Work backwards from the conclusion, or the statement that "+1,-1 are roots of ax^2+bx+a.", i.e. when x=1, b=0 and when x = -1, 2a+b=0. So b=0 or 2a+b=0.
(b).Yes I would agree with the student. If the integer b is found to be even, this is a contradiction of the assumption that b is odd .I think it would be better if he/she acknowledged the contradiction.

4(a). There are not a finite number of primes.
(b). 1 and p are not the only positive integers that divide the positive integer p (that is, p is not prime).

5(a). The real number ad-bc is greater than or less than zero
(b). The triangle ABC is scalene or isosceles.

6(b). Because of the quantifier "for all", you should use the choose method to arbitrarily choose an element $s \in S$ in order to show that there does not exist $t \in T$ such that s>t.

Because of the "no" in the backward statement, you should use the contradiction method. Assume that a real number $s \in S$ and that there does exist a real number $t \in T$ such that s>t

7(b). As there are no "not"'s in this statement, contradiction is not a likely candidate for a proof of the claim in this problem.

Because of the quantifier "there is", use construction method to construct an element y>0 in order to show that for every element $x \in S$, f(x)<y.

Because of the quantifier "for every", apply specialization to the forward statement that $x \in S$, f(x)<y.

Or instead of specialization you can use contradiction to assume that a real number $y>0$, a real number $x \in S$, and $f(x) \geq y$.

13. Claim: If p and q are integers with $p \neq q$ and p is prime and divides q, then q is not prime.

    By contradiction, assume that p and q are integers with $p \neq q$, p prime and p divides q, and q is prime.

    H1: p,q are integers
    H2: $p \neq q$
    H3: p is prime
    H4: p divides q
    A1: q is prime

Proof: Let p, q be integers with $p \neq q$, p prime, and p divides q. By assumption, we assume that q is prime. For q to be prime, by definition it can only be divisible by one and itself. By hypothesis we are given that p divides q or $q=pa$ (a an integer). For our assumption to remain true, either $p = q$ or $p = 1$. By hypothesis H2, $p \neq q$. And, by hypothesis H3, p is prime. Therefore $p \neq 1$ because one is not prime. This contradicts the hypothesis that p divides q; therefore, q cannot be prime.

15. Claim: There are at least two people on the planet who were born on the same second of the same hour of the same day of the same year in the twentieth century. (assume at least 4 billion was the population in said century).

    By contradiction assume that there were not at least two people born on the same second of the same hour of the same day of the same year.

Proof: Let it be given that in the 20^{th} century there were at least 4 billion people and by contradiction we are going to assume that there were less than two people born on the same second of the same hour of the same day of the same year. First we must calculate how many birthdays, down to the second that there are in 100 years. We do this by: 100yrs * 365days * 24 hrs * 60min * 60sec = 3,153,600,000. That is a little over 3 billion. From the hypothesis we are given that 4 billion people had birthdays in the 20^{th} century, but only 3,153,600,000 seconds are in a century, so for everyone to have different birthdays down to the second, there would have to be 3,153,600,000 people or less in population. This contradicts the hypothesis that there were 4 billion people alive during that time period so by contradiction there would have to be at least 2 people on the planet with the same birthdays down to the same second.

17. Claim: If x,y are real numbers such that $x,y \geq 0$, and $x+y=0$ then $x=0$ and $y=0$.

By contradiction, assume that x,y are real numbers with $x,y \geq 0$ and $x+y=0$ and $x \neq 0$ or $y \neq 0$

    H1: x,y are real numbers with $x,y \geq 0$
    H2: $x+y = 0$

H3 $x \neq 0$ or $y \neq 0$
A1: by H1 and H3, either $x > 0$ or $y>0$
A2: suppose $x>0$
A3: by H2 $x+y=0$ becomes $y=-x$
A4: Since x is positive, y must be negative, which contradicts the hypothesis that $y \geq 0$
A5:The only other possibility is $y>0$
A6: suppose $y>0$
A7: by H2, $x+y = 0$ becomes $x=-y$
A8: Since $y>0$, x must be negative, but this contradicts the hypothesis that $x\geq0$

Proof: Let real numbers x and y be given such that $x,y\geq 0$ and $x+y = 0$. By contradiction we assume that $x \neq 0$ or $y \neq 0$. If $x \neq 0$ and $x,y \geq 0$, then $x>0$. Then $x+y=0$ becomes $y=-x$, so y is negative – which contradicts $y\geq0$. Similarly, suppose $y\neq 0$ and $y>0$. Then, $x+y=0$ becomes $x=-y$, so x is negative - which contradicts $y\geq0$. Since neither $x \neq 0$ or $y \neq 0$ is possible, the initial assumption must be false. Thus, by contradiction, both $x=0$ and $y=0$.

19. Claim: If $a>0$ is a real number, then $f(x) = a^x$ is a one-to-one function.

Definition: A function f is one-to-one if and only if for all real numbers x and y with $x\neq y$, $f(x)\neq f(y)$

(a). In the first sentence the author uses the definition of one-to-one to apply it to this specific problem
(b). The author then recognizes the "not" in the conclusion and decides to use contradiction.
(c). The author use the hypothesis that $a>0$ when he/she uses logarithms to rewrite $\log(a^x) = \log(a^y)$ as $x \log(a)=y \log(a)$ because you cannot take the logarithm of zero or a negative number.
(d). The proof is not complete because for one to divide log(a) into another number, they would have to be sure that it does not equal zero, but log(a) equals zero at $a=1$. Therefore the author would need to specify that a must be greater than 1.