Chapter 7 Solutions
Math 300 – Spring 2014
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1a. Quantifier "there is"

Object: real number y

Certain Property: none

Something Happens:for every real number x, $f(x) \leq y$

Quantifier "for every"

Object: real number x

Certain Property: none

Something Happens: $f(x) \leq y$

b. Quantifier "there is"

Object: real number M

Certain Property: $M > 0$

Something Happens: for all elements $x \in S$, $|x| < M$

Quantifier "for all"

Object: element x

Certain Property: $x \in S$

Something Happens: $|x| < M$

c. Quantifier "for every"

Object: real number $\in$

Certain Property: $\in > 0$

Something Happens: there exists a real number $\delta > 0$ such that for all real numbers y with $|x-y| < \delta$, $|f(x) - f(y)| < \in$

Quantifier "there exists"

Object: real number δ

Certain Property: $\delta > 0$

Something Happens: for all real numbers y with $|x-y| < \delta$, $|f(x) - f(y)| < \in$

Quantifier "for all"

Object: real number y

Certain Property: none

Something Happens: if $|x-y| < \delta$ then $|f(x) - f(y)| < \in$

d. Quantifier "for all"

Object: real number $\in$

Certain Property: $\in > 0$

Something Happens: there exists an integer $j \geq 1$ such that for every integer $k > j$, $|x_k - x| < \in$

Quantifier "there exists"

Object: an integer j

Certain Property: $j \geq 1$

Something Happens: for every integer $k > j$, $|x_k - x| < \in$

Quantifier "for all"

Object: integer k

Certain Property: none

Something Happens: if k>j, then $|x_k - x| < \in$

2 a. For a set S of real numbers, for all elements x∈S, there exists another element y, y∈S with y>x
b. A function f of one real variable has the property that there is a real number y such that for all numbers, $|f(x)| < y$

3 a. Both S1 and S2 are true.
When you apply the choose method to each statement, in both statements you will choose real numbers x and y with $0 \le x \le 1$ and $0 \le y \le 2$ for which you can then show that $2x^2 + y^2 \le 6$.

b. S1 and S2 are different.
The two statements describe different regions.
The difference is like horizontal and vertical slices for finding the value of a double integral.

4 a. The statements are both the same and they are both true because the statements use the quantifier "there is" for the same set of (x,y) values: There are real numbers $x \ge 2$ and $y \ge 1$ such that $x^2 + y^2 < 9$. For example: let x=2 and y=1 : $2^2+2(1)^2<9$, $4+2<9$, $6<9$

b. The statements are not equivalent because they quantifiers describe different regions in the plane: $0 \le y \le 2x$, $0 \le x \le 1$ and $0 \le x \le 2y$, $0 \le y \le 1$.
S1 is false: by replacing the y with 2x to maximize the outcome, $2x^2 + 4x^2 > 6$ or $6x^2 > 6$. When x=1, 6(1)>6 is false. There are no numbers that will make S1 true. A similar demonstration applies to S2.

5 a. Recognizing the first quantifier "for all", the first step in the backward process is to choose an object X with a certain property P for which it must be shown that there is an object Y with property Q such that something happens. For the second quantifier "there is", we must then construct object Y with property Q such that something happens.

b. Recognizing the first quantifier "there is", the first step in the backward process is to construct an object X with property P. After X is constructed, the choose method is then used to show that for the constructed X, it is true that for all objects Y with property Q, that something happens. Then recognizing the quantifier "for all", we would then use the choose method to choose an object Y with property Q and show that something happens.

7 a. First: construct a real number M, M>0
Second: choose t∈T
b. First: choose a real number M, M>0
Second: construct t∈T
c. First: choose real number ∈, ∈>0
Second: construct real number δ, δ >0
Third: choose real number s and t

8 a. See Exercise 7 (this section

9 a. If S is a subset of a set T of real numbers and T is bounded, then S is bounded.

Key Question: How to show a set is bounded?

Definition: a set of real numbers S is bounded if and only if there is a real number M>0 such that for all elements $x \in S$, $|x|<M$

Answer: There exists a real number M>0 such that for all real numbers $x \in S$, $|x|<M$

H1: There exists a real number N>0 such that for all real numbers $v \in T$, $|v|<N$

A1: construct a real number N, N>0 such that $|v|<N$ for all $v \in S$
A2: Let v be given arbitrarily (choose)

B2: For all $v \in S$, $|v|<N$
B1: There exists a real number M, M>0 such that for all $x \in S$, $|x|<M$
B: S is bounded

b. If the functions f and g are onto, then the function $f \circ g$ is onto where $(f \circ g) = f(g(x))$

Key Question: How to show a function is onto?

Definition: a function f from the set of real numbers to the set of real numbers is onto if and only if for real numbers y, there exists a real number x such that f(x)=y

Answer: For all real numbers y, there exists a real number x such that f(x)= y

H1: f is onto
H2: g is onto
A1: let a real number v be given arbitrarily (choose)
A2: construct u such that f(g(u)) =v

B2: There exists a real number u such that f(g(u)) = v
B1: For all real numbers y, there exists a real number x such that f(x)=y
B: $f \circ g$ is onto

c. If f and g are functions of one real variable for which $g \geq f$ on the set of real numbers and g is bounded above, then f is bounded above.

Key Question: How to show a function is bounded?

Definition: the function f of one real variable is bounded above if and only if there exists a real number x such that $f(x) \leq y$

Answer: There must exist a real number x such that $f(x)\leq y$

H1: f and g are functions of one real variable
H2: $g\geq f$
H3: g is bounded above
A1: construct a real number y
A2: Let a real number t be given arbitrarily (choose)

B3: $f(t)\leq y$
B2: For all t, $f(t)\leq y$
B1: There exists a real number y such that for all real numbers x, $f(x)\leq y$
B: f is bounded above

11. For all real numbers $\in>0$ and $a>0$, there exists an integer $n>0$ such that $(a/n) <\in$

A1: let real numbers $\in>0$ and $a>0$ be given arbitrarily (choose)
A2: construct $m>0$ such that $(a/m)< \in$

B1: There exists an $n>0$ such that $(a/n)< \in$
B: For all $\in>0$ and $a>0$, $(a/n)< \in$

Proof: Let $\in>0$ and $a>0$ be real numbers. It is necessary to show that there exists an integer $n>0$ such that $(a/n)< \in$. Now we must construct an integer $n>0$ such that $(a/n)< \in$. Because $n>0$ we can multiply it on both sides resulting in a new inequality, $a <n\in$. Because $\in>0$ we can also divide both sides by $\in$ resulting in $(a/\in)<n$. In other words, choose n to be the first positive integer strictly greater than $a/\in>0$. Because $\in$ and y were chosen arbitrarily, we know that for all $\in$ and a, $m>0$, $(a/\in)<m$, in particular, for any $\in>0$ and $a>0$: $a/n<\in$.