MATH 300 (Section 001)
Prof. Meade

Exam 2
20 March 2014

University of South Carolina
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Name: Key

Instructions:

1. There are a total of 7 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. No electronic or other inanimate objects can be used during this exam. All questions have been designed with this in mind and should not involve unreasonable manual calculations.
3. Be sure you answer the questions that are asked.
4. To be eligible for full credit, explanations and justifications must be written in complete English sentences.

Problem	Points	Score
1	15	
2	20	
3	10	
4	10	
5	15	
6	15	
7	15	
Total	100	

Good Luck!

1. (15 points) State the definition of each of the following terms.

   (a) Let m and n be integers. $n|m$ (n divides m) if and only if

   there is an integer k such that $nk = m$

   (b) An integer n is even if and only if

   there is an integer k such that $2k = n$

   (c) A real number r is rational if and only if

   there exist integers p, q with $q \neq 0$ such that $\frac{p}{q} = r$

2. (20 points) Rewrite each of the following statements so that the quantifier ("there-is" or "for-all") appears in standard form. Then identify the $\underset{O}{\text{object}}$, the certain $\underset{P}{\text{property}}$, and the something that happens. ← S

   (a) If θ is an angle, then $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$.

   $\forall$ angles $\underset{O}{\theta}$, $\underbrace{\cos(2\theta) = \cos^2\theta - \sin^2\theta}_{S}$

   (there is no P)

   (b) The equation $f(x) = 3$ has a solution in the inteval $[1, 4)$.

   $\exists\ \underset{O}{x}$ with $\underbrace{1 \le x < 4}_{P}$ such that $\underbrace{f(x) = 3}_{S}$.

   (c) Some element of the set S is < 2.

   $\exists\ \underset{S}{s}$ with $\underbrace{s \in S}_{P}$ such that $\underbrace{s < 2}_{S}$

   (d) The square root of the product of two nonnegative real numbers p and q is not less than the average of the two numbers.

   $\underbrace{\forall \text{ real numbers } p, q}_{O}$ with $\underbrace{p \ge 0 \text{ and } q \ge 0}_{P}$, $\underbrace{pq \ge \frac{1}{2}(p+q)}_{S}$.

   (e) The integer $n > 1$ can be divided by some integer p with $1 < p < n$.

   Let $n > 1$ be given.

   $\exists$ $\underset{O}{\text{integer } p}$ with $\underbrace{1<p<n}_{P}$ such that $\underbrace{p|n}_{S}$.

3. (10 points) For the following "for-all" statements, what properties must the given object satisfy so that you can apply specialization? Given that the object does satisfy those properties, what can you conclude about the object?

Statement: If a and b are real numbers with $a < 0$ and $y = -b/(2a)$, then for all real numbers x, $ax^2 + bx \leq ay^2 + by$.

Given object: $4x - x^2$

To apply the statement to $4x - x^2$ we must identify appropriate values for a, b, and y.

$b = 4$ $a = -1$

With $a = -1$ (which satisfies $a<0$) and $y = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$

the statement applies to yield:

$-x^2 + 4x \leq -y^2 + 4y$ for all real numbers x

4. (10 points) To what specific object would you specialize the following "for-all" statement so that the result of specialization leads to the desired conclusion. Verify that the object to which you are applying specialization satisfies the certain property in the "for-all" statement so that you can apply specialization.

Statement: f is a function of one real variable such that, for all x, y, and t with $0 \leq t \leq 1$, $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. $= f(y) + t(f(x) - f(y))$

Desired conclusion: the function satisfies $f(u) \leq f(0) + u(f(1) - f(0))$ for all $u \in [0, 1]$.

For specialization of this statement to produce the desired conclusion we must have the following correspondences:

$u = tx + (1-t)y$

$0 = y$ } substituting these into yields $u = t(1) + (1-t)0 = t$

$1 = x$

Applying the statement with $x = 1, y = 0,$ and $t = u$ yields:

$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) \quad \forall\ 0 \leq t \leq 1.$

or $f(u) \leq u f(1) + (1-u) f(0)$

$= f(0) + u(f(1) - f(0)) \quad \forall\ 0 \leq u \leq 1.$

5. (15 points) Consider the following statement: $\exists k$ such that $ak = c$.

if a, b, and c are integers with $a|(b+c)$ and $a|b$, then $a|c$.

(a) What method (construction, choose, or specialization) would be used in the proof of this statement?

(b) Prove the statement.

① H1. $a | (b+c) \iff \exists\ m$ such that $am = b+c$

② H2. $a | b \iff \exists\ n$ such that $an = b$

⑤ A1. subtract: $c = (b+c) - b = am - an = a(m-n)$

⑥ A2. construct $k = m-n$

④ B1. $ak = c$

③ B. $\exists k$ such that $ak = c$

Proof: Assume a, b, and c are integers with $a | (b+c)$ and $a | b$. By the definition of divisibility, there are integers m and n for which $am = b+c$ and $an = b$. Thus, by subtracting, $c = (b+c) - b = am - an = a(m-n)$. Let $k = m-n$, then $c = ak$, which confirms that $a | c$. ∎

6. (15 points) Consider the following statement:

> if a and b are real numbers, then the set $C = \{\text{real numbers } x\text{: } ax < b\}$ is a convex set.

(a) What method (construction, choose, or specialization) would be used in the proof of this statement?

(b) Prove the statement.

0. {H1. a and b are real numbers}

4. A1. choose $x^*, y^* \in C$ and $0 \le t^* \le 1$ arbitrarily

5. A2. $ax^* < b$ (by def'n of $x \in C$)

6. A3. $ay^* < b$ (by def'n of $y \in C$)

7. A4. $t^*(ax^*) + (1-t^*)(ay^*) < tb + (1-t)b = tb + b - tb = b$

3. B2. $a(t^* x^* + (1-t^*)y^*) < b$

2. B1. $\forall\ x, y \in C$ & $0 \le t \le 1$, $tx + (1-t)y \in C$

1. B. $C = \{x : ax < b\}$ is convex.

Proof: We will be working to show that

$$C = \{x : ax < b\} \quad \text{(where } a \text{ \& } b \text{ are real numbers)}$$

is a convex set.

Choose $x^*, y^* \in C$ and $0 \le t^* \le 1$ arbitrarily.

Because $x^* \in C$ we know $ax^* < b$.

Similarly, $ay^* < b$.

Then
$$\begin{aligned} a(t^* x^* + (1-t^*)y^*) &= t^*(ax^*) + (1-t^*)(ay^*) \\ &< t^* b + (1-t^*)b \\ &= t^* b + b - t^* b \\ &= b. \end{aligned}$$

So that $t^* x^* + (1-t^*)y^* \in C$.

Since this holds for all $x^*, y^* \in C$ and $0 \le t^* \le 1$, C is convex. $\boxtimes$

Recall that a set S is convex if and only if for all $x, y \in S$ and all $0 \le t \le 1$, $tx + (1-t)y \in S$.

7. (15 points) Consider the following statement:

> For real numbers L and M, if L is a lower bound for a set of real numbers S and $L \geq M$, then M is a lower bound for S.

(a) What method (construction, choose, or specialization) would be used in the proof of this statement?

also?

(b) Prove the statement.

1. H1. L is a lowerbound for S

2. H2. $L \geq M$

6. A1. By def'n of lower bound: $\forall s \in S,\ L \leq s$.

7. A2. Choose $s^* \in S$ arbitrarily.

8. A3. $L \leq s^*$ (specialization of A1 to s^*)

9. A4. $M \leq L \leq s^*$

5. B2 $M \leq s^*$

4. B1 $\forall x \in S,\ M \leq x$ (def'n of lower bound)

3. B. M is a lower bound for S

Proof: Choose $s^* \in S$ arbitrarily.
Then $M \leq L \leq s^*$.
Since this ($M \leq s^*$) for all $s^* \in S$, M is a lower bound for S. ∎

A number L is a lower bound for a set S if and only if $L \leq x$ for all $x \in S$.