HW Solution for §11.3

#27. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

We apply the integral test: $\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\infty} u^{-p}\,du = \frac{u^{-p+1}}{-p+1}\Big|_{\ln 2}^{\infty} = \lim_{A\to\infty} \frac{\frac{1}{A^{p-1}} - \frac{1}{(\ln 2)^{p-1}}}{1-p}$

For this limit to be finite, $\lim_{A\to\infty} \frac{1}{A^{p-1}}$ must exist; this requires $p>1$.
(When $p=1$ the integral is a little different, but the improper integral diverges because $\lim_{A\to\infty} \ln(\ln(A)) = \infty$.)

In conclusion, when $p>1$ the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges by the Integral Test.

#38. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^2}$

(a) By the Integral Test, $\int_1^{\infty} \frac{(\ln x)^2}{x^2}\,dx = \lim_{A\to\infty} \left(-\frac{\ln(n)^2}{n} - \frac{2\ln(n)}{n} - \frac{2}{n}\right)\Big|_1^A$ ← from Maple.

$$= \lim_{A\to\infty}\left(-\frac{\ln(A)^2}{A} - \frac{2\ln(A)}{A} - \frac{2}{A}\right) + 2 = 2$$

so the series converges.

(b) An upper bound for $R_n = s - s_n$ is $\int_n^{\infty} \frac{(\ln x)^2}{x^2}\,dx = \frac{\ln(N)^2 + 2\ln(N) + 2}{N}$ ← from Maple.

(c) To find the smallest value of n with $R_n < 0.05$ we solve

$$\frac{\ln(N)^2 + 2\ln(N) + 2}{N} = 0.05$$

Maple gives 5 answers, but 4 are complex-valued and the only real-valued solution is

Maple ⟶ $n = 1372.909\ldots$

This means S_{1373} is the first partial sum that is within 0.05 of the exact value.

(d) $s_{1373} = \sum_{n=1}^{1373} \frac{(\ln n)^2}{n^2} \approx 1.93929$ ← Maple

Note that $s = 1.98928$ ← Maple. so that $R_{1373} = s - s_{1373} = 0.0499\ldots < 0.05$

and $s_{1372} \approx 1.93926$ $\quad R_{1372} = s - s_{1372} = 0.05001\ldots > 0.05$.