MATH 142 (Section 502)
Prof. Meade

Exam 3
November 10, 2014

University of South Carolina
Fall 2014

Name: Key
Section 502

Instructions:

1. There are a total of 8 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

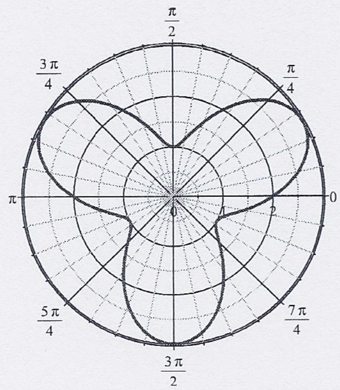
Problem	Points	Score
1	18	
2	6	
3	11	
4	10	
5	20	
6	8	
7	20	
8	7	
Total	100	

Good Luck!

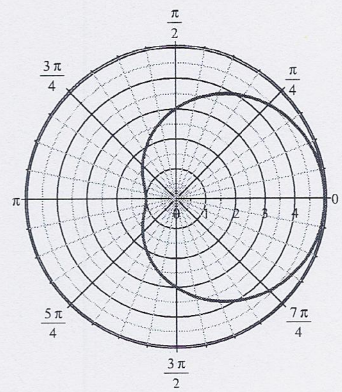
There is no exam content on this page.

1. (18 points) [3 points each] Match polar equations a)-e) with the graphs labelled I–VI.

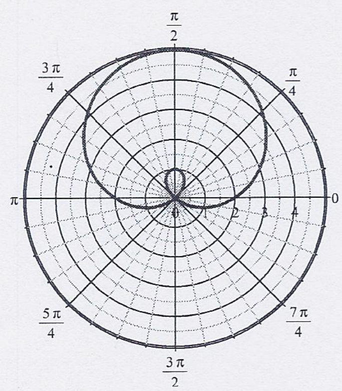
a)	III	$r = \sin 3\theta$	c)	I	$r = 2 + \sin 3\theta$	e) IV	$r = 3 + 2\sin\theta$
b)	V	$r = \cos 3\theta$	d)	II	$r = 2 + 3\cos\theta$	f) VI	$r = 2 + 2\cos\theta$



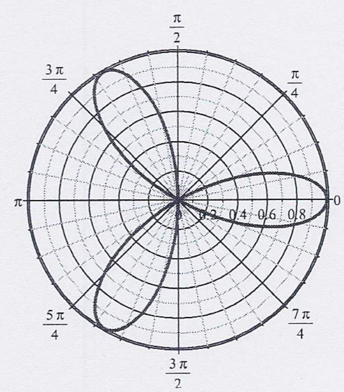
I



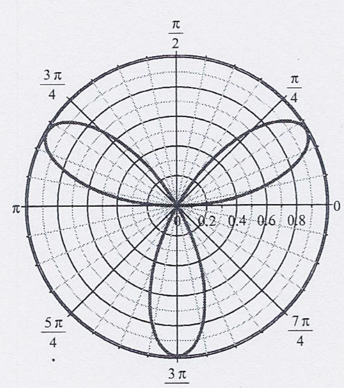
IV



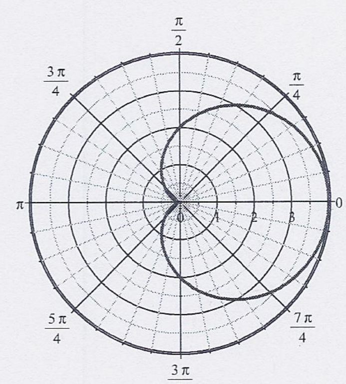
II



V



III



VI

2. (6 points) Find a polar equation for the curve given by the Cartesian equation $x^2 - y^2 = 4$.

$(r\cos\theta)^2 - (r\sin\theta)^2 = 4$

$r^2\cos^2\theta - r^2\sin^2\theta = 4$

$r^2(\cos^2 - \sin^2\theta) = 4$

$r^2 = \dfrac{4}{\cos^2\theta - \sin^2\theta}$ so $r = \pm\dfrac{2}{\sqrt{\cos^2\theta - \sin^2\theta}}$.

Note that

$\cos^2\theta - \sin^2\theta = \cos(2\theta)$

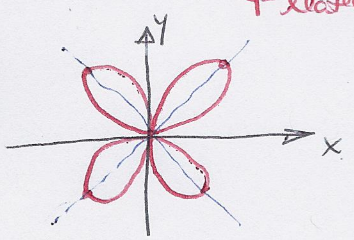
and $\dfrac{1}{\cos 2\theta} = \sec 2\theta$

so you might write the function as

$r = \pm 2\sqrt{\sec(2\theta)}$

3. (11 points) Find a definite integral for the area of the region enclosed by one loop of the polar curve $r = \sin 2\theta$. *Do not attempt to evaluate this integral.*

4-leafed rose



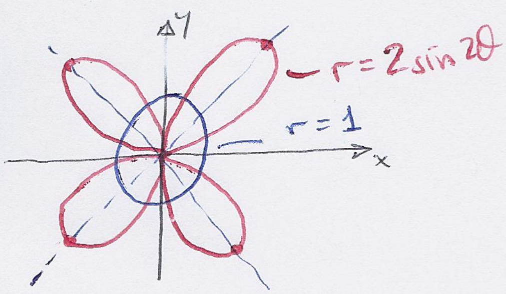
$r = 0$ when $2\theta = 0, \pi, 2\pi, \ldots, k\pi$

$\theta = 0, \frac{\pi}{2}, \pi, \ldots, \frac{k}{2}\pi$

one leaf is enclosed when $0 \le \theta \le \pi/2$

$$\text{area} = \int_0^{\pi/2} \frac{1}{2} r^2 \, d\theta = \int_0^{\pi/2} \frac{1}{2}\sin^2(2\theta)\, d\theta$$

4. (10 points) Find all points of intersection of the polar curves $r = 2\sin 2\theta$ and $r = 1$.



↑4-leafed rose ↑circle w/ radius 1 center (0,0)

graphically, there should be 8 intersections.

$2\sin 2\theta = 1$

$\sin 2\theta = \frac{1}{2}$

$2\theta = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$

so $\theta = \frac{\pi}{12} + k\pi$ or $\frac{5\pi}{12} + k\pi$.

These give 4 angles:

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

But, sometimes $r < 0$ so we also have to look at

$2\sin 2\theta = -1$

$\sin 2\theta = -\frac{1}{2}$

so

$2\theta = \frac{7\pi}{6} + 2k\pi$ or $\frac{11\pi}{6} + 2k\pi$

$\theta = \frac{7\pi}{12} + k\pi$ or $\frac{11\pi}{12} + k\pi$

These give 4 more angles:

$\theta = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}$

5. (20 points) [5 points each] Determine whether each sequence converges or diverges. *Find the limit of each sequence that converges.*

(a) $a_n = \dfrac{3+5n^3}{n+n^2}$ $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3/n^2 + 5n}{1/n + 1} = \infty$

This sequence diverges.

(b) $a_n = \sqrt{\dfrac{n+1}{9n^2+1}}$ $\lim_{n\to\infty} a_n = \sqrt{\lim_{n\to\infty} \dfrac{1/n + 1/n^2}{9 + 1/n^2}} = \sqrt{0} = 0.$

(c) $a_n = n\underbrace{\sin(n\pi)}_{=0 \text{ for every integer } n.}$ $\lim_{n\to\infty} a_n = 0.$

(d) $a_1 = 0,\ a_{n+1} = \sqrt{2 + a_n}$

Assume $\lim_{n\to\infty} a_n = L$.

Then $L = \sqrt{2+L}$

(Note: $L \geq 0$)

$L^2 = 2 + L$

$L^2 - L - 2 = 0$

$(L-2)(L+1) = 0$

$L = 2$ or $L = -1$

Since $L \geq 0$, only $L = 2$ is a realistic possibility for the limit of this sequence. ($\lim_{n\to\infty} a_n = 2$).

6. (8 points) Let $a_n = \dfrac{2n+1}{3n+1}$.

(a) Determine whether $\{a_n\}$ is convergent or divergent.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2 + 1/n}{3 + 1/n} = \dfrac{2}{3}$ (convergent)

(b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

Because $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

7. (20 points) [5 points each] Determine whether each series converges or diverges. *Do not attempt to find the sum of any series that is convergent.*

(a) $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converge (p-series, $p = 5 > 1$)

(b) $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ $a_n = f(n)$ where $f(x) = \frac{1}{x^2+1}$.

$$\int_0^\infty f(x)dx = \lim_{A\to\infty} \int_0^A \frac{dx}{x^2+1} = \lim_{n\to\infty} \left(\arctan x \Big|_0^A\right)$$

$$= \lim_{A\to\infty} (\arctan A - 0) = \pi/2 \quad \therefore \sum_{n=0}^{\infty} \frac{1}{n^2+1} \text{ converges (by Integral Test).}$$

(c) $\sum_{n=1}^{\infty} \frac{n+2}{n+1}$ diverges because $\lim_{n\to\infty} \frac{n+2}{n+1} = 1 \neq 0$.

(d) $\sum_{n=3}^{\infty} 2n^{-0.85}$ diverges (p-series, $p = 0.85 < 1$).

8. (7 points) Find the value of c such that $\sum_{n=0}^{\infty} e^{nc} = 4$.

$$4 = \sum_{n=0}^{\infty} e^{nc} = \sum_{n=0}^{\infty} (e^c)^n = \frac{1}{1-e^c} \quad \text{so } 1 - e^c = \frac{1}{4}$$

{geometric: $a = 1$, $r = e^c$}

$$e^c = \frac{3}{4}$$

$$c = \ln(3/4) = \ln(3) - \ln(4).$$