MATH 142 (Honors)
Prof. Meade

Exam 2
October 13, 2014

University of South Carolina
Fall 2014

Name: Key
Section H01

Instructions:

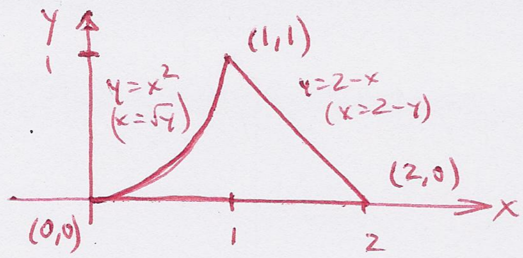
1. There are a total of 5 problems on 4 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	30	
2	20	
3	10	
4	20	
5	10	
Total	90	

Good Luck!

1. (30 points) Let $\mathcal{R}$ be the region in the first quadrant bounded by $y = x^2$, $y = 2 - x$, and $y = 0$.

   (a) (5 points) Sketch $\mathcal{R}$, clearly labeling all intersections between two of the above curves.



$y = x^2 \iff x = \sqrt{y} \quad (x > 0)$

$y = 2-x \iff x = 2-y$

$x^2 = 2-x$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0 \qquad x = -2 \text{ or } \underline{\underline{x = 1}}$

   (b) (10 points) Express the area of $\mathcal{R}$ as one or two definite integrals using vertical slices.

$$A = \int_0^1 x^2\,dx + \int_1^2 2-x\,dx$$

   (c) (10 points) Express the area of $\mathcal{R}$ as one or two definite integrals using horizontal slices.

$$A = \int_0^1 (2-y) - \sqrt{y}\,dy.$$

   (d) (5 points) Evaluate the integral(s) in either (b) or (c).
   If you do not find that the area is 5/6, please see me before you begin problems 2 and 3.

$$A_1 = \int_0^1 x^2 dx + \int_1^2 2-x\,dx$$

$$= \frac{1}{3}x^3\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2$$

$$= \frac{1}{3} + \left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{3} + \frac{1}{2}$$

$$= \frac{5}{6}$$

$$A_2 = \int_0^1 2 - y - y^{1/2}\,dy$$

$$= 2y - \frac{1}{2}y^2 - \frac{2}{3}y^{3/2}\Big|_0^1$$

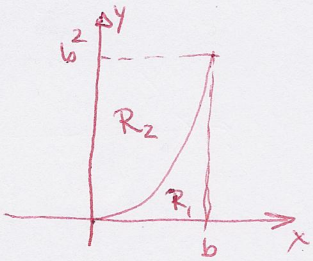
$$= 2 - \frac{1}{2} - \frac{2}{3}$$

$$= \frac{12-3-4}{6}$$

$$= \frac{5}{6}.$$

4. (20 points) Let $\mathcal{R}_1$ be the region bounded by $y = x^2$, $y = 0$, and $x = b$, where $b > 0$. Let $\mathcal{R}_2$ be the region bounded by $y = x^2$, $x = 0$, and $y = b^2$.

(a) Is there a value of b such that $\mathcal{R}_1$ and $\mathcal{R}_2$ have the same area?



$$A_1 = \int_0^b x^2\,dx = \frac{1}{3}x^3\Big|_0^b = \frac{b^3}{3}.$$

$$A_2 = \int_0^b (b^2 - x^2)\,dx = \left(b^2x - \frac{x^3}{3}\right)\Big|_0^b = b^3 - \frac{b^3}{3} = \frac{2b^3}{3}.$$

$\frac{b^3}{3} \neq \frac{2b^3}{3}$ for all $b > 0$ No.

(A_2 is always twice A_1).

(b) Is there a value of b such that $\mathcal{R}_1$ sweeps out the same volume when rotated about the x-axis and the y-axis?

$$V_1 = \int_0^b \pi (x^2)^2\,dx = \pi \int_0^b x^4\,dx = \pi \frac{x^5}{5}\Big|_0^b = \pi \frac{b^5}{5}.$$

$$V_2 = \int_0^b 2\pi x (x^2)\,dx = 2\pi \int_0^b x^3\,dx = 2\pi \frac{x^4}{4}\Big|_0^b = \pi \frac{b^4}{2}$$

$V_1 = V_2$: $\pi \frac{b^5}{5} = \pi \frac{b^4}{2}$

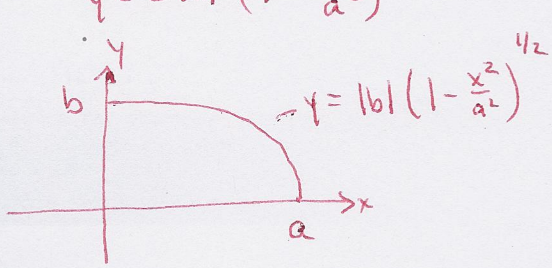
$b = \frac{5}{2}$.   Yes.

5. (10 points) Set up, but do not evaluate, an integral for the length of the part of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the first quadrant. Do not evaluate this integral.

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm |b| \left(1 - \frac{x^2}{a^2}\right)^{1/2}$$



$$y' = \frac{1}{2}|b|\left(1 - \frac{x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right).$$

$$= -\frac{|b|}{a^2}\frac{x}{\sqrt{1 - \frac{x^2}{a^2}}}$$

$$L = \int_0^a \left(1 + (y')^2\right)^{1/2} dx = \int_0^a \left(1 + \frac{b^2}{a^4}\frac{x^2}{1 - x^2/a^2}\right)^{1/2} dx$$

$$= \int_0^a \left(1 + \frac{b^2x^2}{a^4 - a^2x^2}\right)^{1/2} dx$$