MATH 142 (Section H01)
Prof. Meade

Exam 1
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University of South Carolina
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Name: Key
Section H01

Instructions:

1. There are a total of 4 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1 (p. 3)	24	
1 (p. 4)	24	
2	28	
3	10	
4	14	
Total	100	

Good Luck!

There is no exam content on this page.

1. (48 points) Evaluate the following integrals.

(a) $\displaystyle\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

seeing the x^3dx (and that $x^8 = (x^4)^2$)
leads me to consider the substitution:

$u = x^4$

$du = 4x^3dx$

$$\int \frac{x^3}{\sqrt{1-x^8}}\,dx = \frac{1}{4}\int \frac{4x^3\,dx}{\sqrt{1-(x^4)^2}} = \frac{1}{4}\int \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{4}\arcsin(u) + C$$

$$= \frac{1}{4}\arcsin(x^4) + C.$$

(b) $\displaystyle\int x\arctan(x)\,dx$

The product of a power and an inverse trig fn.
suggest integration by parts with

$u = \arctan x \qquad dv = x\,dx$

$du = \dfrac{dx}{1+x^2} \qquad v = \frac{1}{2}x^2$

$$\int x\arctan x\,dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

now: $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$

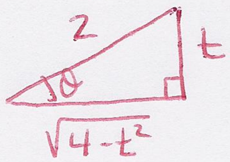
$$\begin{array}{r} 1 \\ 1+x^2 \overline{)\, x^2 } \\ \underline{-(x^2+1)} \\ -1 \end{array}$$

$$\int x\arctan x\,dx = \frac{1}{2}x^2\arctan x - \frac{1}{2}\int 1 - \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{2}x^2\arctan x - \frac{1}{2}(x - \arctan x) + C.$$

(c) $\int t^3\sqrt{4-t^2}\,dt$

Using a trig. subst: $t = 2\sin\theta$ so $dt = 2\cos\theta\,d\theta$ & $\sqrt{4-t^2} = 2\cos\theta$



$$\int t^3\sqrt{4-t^2}\,dt = \int (2\sin\theta)^3(2\cos\theta)\,2\cos\theta\,d\theta = 32\int \sin^3\theta\cos^2\theta\,d\theta$$

$$= 32\int \sin\theta\,(1-\cos^2\theta)\cos^2\theta\,d\theta$$

$u = \cos\theta$
$du = -\sin\theta\,d\theta$

$$= -32\int (1-u^2)u^2\,du$$

$$= -32\int u^2 - u^4\,du$$

$$= -32\left(\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right) + C$$

* You can also do this with the substitution $u = 4-t^2$.

$$= -32\left(\tfrac{1}{3}\cos^3\theta - \tfrac{1}{5}\cos^5\theta\right) + C$$

$$= -32\left(\tfrac{1}{3}\left(\tfrac{\sqrt{4-t^2}}{2}\right)^3 - \tfrac{1}{5}\left(\tfrac{\sqrt{4-t^2}}{2}\right)^5\right) + C$$

$$= -\tfrac{4}{3}(4-t^2)^{3/2} + \tfrac{1}{5}(4-t^2)^{5/2} + C$$

(d) $\int_2^3 \frac{ds}{s^3-s}$

Partial fractions: $\frac{1}{s^3-s} = \frac{1}{s(s^2-1)} = \frac{1}{s(s+1)(s-1)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-1}$

$$= \frac{A(s+1)(s-1) + Bs(s-1) + Cs(s+1)}{s(s+1)(s-1)}$$

so $1 = A(s+1)(s-1) + Bs(s-1) + Cs(s+1)$

$s=1$: $1 = C(1)(2) \Rightarrow C = 1/2$
$s=0$: $1 = A(1)(-1) \Rightarrow A = -1$
$s=-1$: $1 = B(-1)(-2) \Rightarrow B = 1/2$

so $\frac{1}{s^3-s} = \frac{-1}{s} + \frac{1/2}{s+1} + \frac{1/2}{s-1}$

$$\int_2^3 \frac{ds}{s^3-s} = \int_2^3 \frac{-1}{s} + \frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{1}{s-1}\,ds = \left(-\ln|s| + \tfrac{1}{2}\ln|s+1| + \tfrac{1}{2}\ln|s-1|\right)\Big|_2^3$$

$$= -\ln 3 + \tfrac{1}{2}\ln 4 + \tfrac{1}{2}\ln 2 - \left(-\ln 2 + \tfrac{1}{2}\ln 3 + \tfrac{1}{2}\ln 1\right)$$

$$= -\tfrac{3}{2}\ln 3 + \tfrac{1}{4}\ln 4 + \tfrac{3}{2}\ln 2$$

2. (28 points) For each of the following improper integrals, evaluate the integral or show that it diverges.

(a) $\int_0^\infty \frac{t\,dt}{(t^2+1)^2}$ $= \lim_{A\to\infty} \int_0^A \frac{t\,dt}{(t^2+1)^2} = \lim_{A\to\infty} \frac{1}{2}\int_1^A u^{-2}\,du$

$u = t^2+1$

$du = 2t\,dt$

$= \lim_{A\to\infty} \frac{1}{2}\left(-u^{-1}\right)\Big|_1^A$

$= -\frac{1}{2}\lim_{A\to\infty}\left(\frac{1}{A}-1\right)$

$= \frac{1}{2}$

(b) $\int_0^{\pi/4} \frac{\sec^2(\theta)}{1-\tan(\theta)}\,d\theta$ $= \lim_{A\to \pi/4^-}\int_0^A \frac{\sec^2\theta}{1-\tan\theta}\,d\theta$

$u = 1-\tan\theta$

$du = -\sec^2\theta\,d\theta$

Note that $\tan \pi/4 = 1$ so there is a singularity at $\theta = \pi/4$.

$= \lim_{A\to \pi/4^-} -\int_{\theta=0}^{\theta=A} \frac{du}{u}$

$= \lim_{A\to \pi/4^-} -\ln|u|\Big|_{\theta=0}^{\theta=A}$

$= \lim_{A\to \pi/4^-} -\ln|1-\tan\theta|\Big|_0^A$

$= \lim_{A\to \pi/4^-} -\underbrace{\ln|\underbrace{1-\tan A}_{\to 0}|}_{\to -\infty} + \underbrace{\ln|1|}_{=0} = +\infty$

so this integral diverges.

3. (10 points) Write the form of the partial fraction decomposition of $\frac{3x^2 - x + 1}{x^4 + x^3 + x^2}$.

$x^4+x^3+x^2 = x^2(x^2+x+1)$

note that x^2+x+1 does not factor, because $b^2-4ac = 1^2-4(1)(1) = -3 < 0$.

so the form for

$$\frac{3x^2+x+1}{x^4+x^3+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+x+1}$$

4. (14 points) Use the table of integrals provided on this page to evaluate $\int \frac{\sqrt{2-x^2}}{x}\,dx$. Indicate each time you use an entry from the table.

This is entry 3, with $a^2=2$ and $u=x$.
$(a=\sqrt{2})$

so $\int \frac{\sqrt{2-x^2}}{x}\,dx = \sqrt{2-x^2} - \sqrt{2}\ln\left|\frac{\sqrt{2}+\sqrt{2-x^2}}{x}\right| + C.$

1. $\int \frac{\sqrt{u^2-a^2}}{u}\,du = \sqrt{u^2-a^2} - a\arccos\left(\frac{a}{|u|}\right) + C$
2. $\int \frac{\sqrt{u^2-a^2}}{u^2}\,du = \frac{-1}{u}\sqrt{u^2-a^2} + \ln\left|u+\sqrt{u^2-a^2}\right| + C$
3. $\int \frac{\sqrt{a^2-u^2}}{u}\,du = \sqrt{a^2-u^2} - a\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right| + C$
4. $\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \frac{-1}{u}\sqrt{a^2-u^2} - \arcsin\left(\frac{u}{a}\right) + C$
5. $\int \frac{\sqrt{2au-u^2}}{u}\,du = \sqrt{2au-u^2} + a\arccos\left(\frac{a-u}{a}\right) + C$
6. $\int \frac{\sqrt{2au-u^2}}{u^2}\,du = \frac{-2}{u}\sqrt{2au-u^2} - \arccos\left(\frac{a-u}{a}\right) + C$