MATH 142 (Section 502)
Prof. Meade

Exam 4
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University of South Carolina
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Name: Key
Section 502

Instructions:

1. There are a total of 5 problems on 7 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	24	
2	12	
3	26	
4	18	
5	20	
Total	100	

Happy Thanksgiving!

1. (24 points) [8 points each] Determine if each series converges absolutely, converges conditionally, or diverges.

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{k^3}{e^k}$

Abs. Ratio Test

$$\rho = \lim_{k\to\infty} \left| \frac{(-1)^{k+1} \frac{(k+1)^3}{e^{k+1}}}{(-1)^k \frac{k^3}{e^k}} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^3}{k^3} \frac{e^k}{e^{k+1}} \right|$$

$$= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^3 \cdot \frac{1}{e} = \frac{1}{e} < 1$$

so this series converges absolutely

(b) $\sum_{k=1}^{\infty} \sin\left(\frac{k\pi}{2}\right)$

$\lim_{k\to\infty} \sin\left(\frac{k\pi}{2}\right)$ does not exist because these terms follow the pattern $1, 0, -1, 0, \ldots$

~~and~~ $\therefore$ this series diverges.

(c) $\sum_{k=1}^{\infty}(-1)^{k+1} \frac{k+1}{k^2+1}$

these terms look like $\frac{(-1)^{k+1}}{k}$ for k large.

Limit Comparison with $\sum_{k=1}^{\infty} \frac{1}{k}$ (which diverges).

$$\lim_{k\to\infty} \frac{\frac{k+1}{k^2+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k(k+1)}{k^2+1} = \lim_{k\to\infty} \frac{k^2+k}{k^2+1} = 1.$$

so $\sum_{k=1}^{\infty} \frac{k+1}{k^2+1}$ diverges.

Alt. Series Test: $\lim_{k\to\infty} (-1)^{k+1} \frac{k+1}{k^2+1} = 0.$

is $|a_{k+1}| \overset{?}{<} |a_k|$? $\quad \frac{(k+1)+1}{(k+1)^2+1} \overset{?}{<} \frac{k+1}{k^2+1}$

$$(k^2+1)(k+2) \overset{?}{<} (k+1)(k^2+2k+2)$$

$$k^3+2k^2+k+2 \overset{?}{<} k^3+3k^2+4k+2$$

$$0 < k^2+3k \leftarrow \text{this is true!}$$

so this series converges conditionally

2. (12 points) Consider the convergent series

$$\frac{1}{1\cdot 2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \cdots.$$

(a) Write this series in summation notation.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\cdot 2^k}$$

(b) How many terms are needed to approximate the sum of this series to two decimal place accuracy? (Do not approximate the sum!)

HINT: $0.5 = 1/2$, $0.05 = 1/20$, $0.005 = 1/200$, $0.0005 = 1/2000$, etc.

Two decimal place accuracy after n terms where the $(n+1)^{st}$ term is smaller than 0.005.

That is: $|a_{n+1}| < 0.005$

$$\frac{1}{(n+1)2^{n+1}} < \frac{1}{200}$$

or $$(n+1)\,2^{n+1} > 200.$$

n	$(n+1)2^{n+1}$
1	$2\cdot 2^2 = 8$
2	$3\cdot 2^3 = 24$
3	$4\cdot 2^4 = 64$
4	$5\cdot 2^5 = 160$
5	$6\cdot 2^6 = 384 > 200$

so 5 terms are needed.

3. (26 points) [6 points each] Let $f(x) = \cos(2x)$.

(a) Find the Taylor polynomials of orders $n = 0$, 1, 2, 3, and 4 about $x = \pi/2$.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$	$\frac{1}{n!}f^{(n)}(\pi/2)$
0	$\cos(2x)$	-1	-1
1	$-2\sin(2x)$	0	0
2	$-4\cos(2x)$	4	$4/2 = 2$
3	$8\sin(2x)$	0	0
4	$16\cos(2x)$	-16	$-16/24 = \frac{-2}{3}$.

$P_0(x) = -1$

$P_1(x) = -1 + 0(x-\pi/2) = -1$

$P_2(x) = -1 + 0(x-\pi/2) + 2(x-\pi/2)^2$

$P_3(x) = -1 + 0(x-\pi/2) + 2(x-\pi/2)^2 + 0(x-\pi/2)^3$

$P_4(x) = -1 + 0(x-\pi/2) + 2(x-\pi/2)^2 + 0(x-\pi/2)^3 - \frac{2}{3}(x-\pi/2)^4$.

EXTRA CREDIT Find the Taylor series for $f(x)$ about $x = \pi/2$.

It's easiest to see the pattern in the $f^{(n)}(\pi/2)$ column. You have only even terms that are non-zero. They grow by factors of 4 and change signs. So

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}4^k}{(2k)!}(x-\pi/2)^{2k}$$

4. (18 points) [6 points each] Find the radius of convergence for each of these power series.

(a) $\sum_{k=1}^{\infty} \frac{x^k}{k}$

$$\rho = \lim_{k \to \infty} \left| \frac{x^{k+1}/k+1}{x^k/k} \right| = \lim_{k \to \infty} \left| \frac{k}{k+1} \frac{x^{k+1}}{x^k} \right|$$

$$= \lim_{k \to \infty} \frac{k}{k+1} |x| = |x| < 1$$

The radius of convergence is $\rho = 1$.

(b) $\sum_{k=1}^{\infty} \frac{(3x)^k}{k!}$

$$\rho = \lim_{k \to \infty} \left| \frac{(3x)^{k+1}/(k+1)!}{(3x)^k/k!} \right| = \lim_{k \to \infty} \left| \frac{k!}{(k+1)!} \frac{(3x)^{k+1}}{(3x)^k} \right|$$

$$= \lim_{k \to \infty} \left| \frac{3x}{k+1} \right| = 0$$

The radius of convergence is ∞.

(c) $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k (x+5)^k$

$$\rho = \lim_{k \to \infty} \left| \frac{\left(\frac{4}{3}\right)^{k+1} (x+5)^{k+1}}{\left(\frac{4}{3}\right)^k (x+5)^k} \right| = \lim_{k \to \infty} \left| \frac{4}{3} (x+5) \right|$$

$$\cancel{= \frac{4}{3} \lim_{k \to \infty}} \qquad = \frac{4}{3} |x+5| < 1$$

$$|x+5| < \frac{3}{4}.$$

The radius of convergence is $\rho = \frac{3}{4}$

5. (20 points) [5 points each] Consider the Maclaurin series

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k}$$

(a) Show that the interval of convergence for this series is $-1 < x \le 1$.

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+2} x^{k+1}/(k+1)}{(-1)^{k+1} x^k / k} \right| = \lim_{k\to\infty} \left| \frac{k}{k+1} \frac{x^{k+1}}{x^k} \right| = \lim_{k\to\infty} \frac{k}{k+1} |x| = |x| < 1.$$

Check endpoints:

$x = 1$: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} 1^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ is the alternating harmonic series: converges.

$x = -1$: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k} = -\sum_{k=1}^{\infty} \frac{1}{k}$ is the (negative of the) harmonic series, which diverges.

The interval of convergence is $-1 < x \le 1$.

(b) A Maclaurin series for $\ln(1-x)$ that is valid for $-1 \le x < 1$ is

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots . = -\sum_{k=1}^{\infty} \frac{x^k}{k}$$

Explain how this series, <u>and</u> its interval of convergence, are obtained directly from the Maclaurin series for $\ln(1+x)$.

Note that $\ln(1-x) = \ln(1+(-x))$

so $\ln(1-x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-x)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k x^k}{k} = -\sum_{k=1}^{\infty} \frac{x^k}{k}$.

This is valid for $-1 < -x \le 1$

which means $\underline{-1 \le x < 1}$.

(c) Use the given Maclaurin series for $\ln(1+x)$ and $\ln(1-x)$ to obtain the following Maclaurin series for $\ln\left(\frac{1+x}{1-x}\right)$:

$$\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right). = 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}$$

Extra Credit: How do you know this series converges for all $-1 < x < 1$?

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$
$$-\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \cdots\right)$$
$$= 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \cdots$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = 2\sum_{k=1}^{\infty} \frac{x^{2k-1}}{2k-1}.$$

This holds when x is valid for both $\ln(1+x)$ & for $\ln(1-x)$. That means $-1 < x \le 1$ and $-1 \le x < x$. This gives $-1 < x < 1$.

(d) Use one of the series in this problem to find a series that converges to $\ln(1.5)$.
Extra Credit: How do you know the series converges to $\ln(1.5)$?

You can use $x = 0.5$ in $\ln(1+x)$: $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}(0.5)^k}{k}$

or $x = -0.5$ in $\ln(1-x)$: $-\sum_{k=1}^{\infty} \frac{(-0.5)^k}{k} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(0.5)^k}{k}$

or $x = 1/5$ in $\ln\left(\frac{1+x}{1-x}\right)$: $2\sum_{k=1}^{\infty} \frac{(1/5)^{2k-1}}{2k-1}$.

Any of these works because the corresponding value of x is in the appropriate interval of convergence.