MATH 141 (Section 5 & 6)
Prof. Meade

Exam 4
November 25, 2009

University of South Carolina
Fall 2009

Name: Key
Section: 005 / 006 (circle one)

Instructions:

1. There are a total of 8 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	27	
2	18	
3	10	
4	9	
5	10	
6	8	
7	9	
8	9	
Total	100	

Happy Thanksgiving! Beat Clemson!!

1. (27 points) Evaluate each definite integral.

(a) $\int_0^1 (\sqrt{t} - t^2)\, dt$ $= \int_0^1 t^{1/2} - t^2 \, dt$

$= \left(\frac{t^{3/2}}{3/2} - \frac{t^3}{3} \right) \Big|_0^1$

$= \left(\frac{2}{3} t^{3/2} - \frac{1}{3} t^3 \right) \Big|_0^1$

$= \left(\frac{2}{3} - \frac{1}{3} \right) - (0 - 0)$

$= \frac{1}{3}.$

(b) $\int_2^4 (3-t)^9 \, dt$ $= \int_1^{-1} u^9 (-du) = - \int_1^{-1} u^9 du$

let $u = 3-t$

then $\frac{du}{dt} = -1$ so $dt = -du$

$t = 2 \Rightarrow u = 3-2 = 1$

$t = 4 \Rightarrow u = 3-4 = -1$

$= -\frac{1}{10} u^{10} \Big|_1^{-1}$

$= -\frac{1}{10} (-1)^{10} - \left(-\frac{1}{10} (1)^{10} \right)$

$= -\frac{1}{10} + \frac{1}{10}$

$= 0$

(c) $\int_{-\pi/2}^{\pi/2} \cos(x) e^{\sin(x)} \, dx$ $= \int_{-1}^{1} e^u du = e^u \Big|_{-1}^{1} = e^1 - e^{-1}.$

let $u = \sin x$

then $\frac{du}{dx} = \cos x$ ~~dx~~

so $du = \cos x \, dx$

$x = -\pi/2 \Rightarrow u = \sin(-\pi/2) = -1$

$x = \pi/2 \Rightarrow u = \sin(\pi/2) = 1$

2. (18 points) Evaluate each indefinite integral.

(a) $\int (2x - e^x)\, dx$ $= x^2 - e^x + C$

(b) $\int y\sqrt{y+2}\, dy$ $= \int y \cdot u^{1/2}\, du = \int (u-2)\, u^{1/2}\, du$

let $u = y+2 \Rightarrow y = u-2$

then $\frac{du}{dy} = 1$

so $dy = du$

$= \int u^{3/2} - 2u^{1/2}\, du$

$= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C$

$= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$

$= \frac{2}{5}(y+2)^{5/2} - \frac{4}{3}(y+2)^{3/2} + C$

(c) $\int \frac{x+1}{x^2+2x+5}\, dx$ $= \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\int \frac{1}{u}\, du$

let $u = x^2 + 2x + 5$

then $\frac{du}{dx} = 2x + 2$

$= 2(x+1)$

so $(x+1)dx = \frac{1}{2}du$

$= \frac{1}{2}\ln u + C$

$= \frac{1}{2}\ln(x^2 + 2x + 5) + C.$

3. (10 points) Find the function f with $f'(t) = 2\cos(t) + \sec^2(t)$ for $-\pi/2 < t < \pi/2$ and $f(\pi/3) = 4\sqrt{3}$.

$$f(t) = 2\sin t + \tan t + C$$

$$4\sqrt{3} = f\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{\pi}{3}\right) + \tan\frac{\pi}{3} + C$$

$$= 2\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}/2}{1/2}\right) + C$$

$$= \sqrt{3} + \sqrt{3} + C$$

$$= 2\sqrt{3} + C$$

So $C = 4\sqrt{3} - 2\sqrt{3}$
$= 2\sqrt{3}.$

$$f(t) = 2\sin(t) + \tan(t) + 2\sqrt{3}.$$

4. (9 points) If $\int_1^5 f(x)\,dx = 12$, $\int_4^5 f(x)\,dx = 4$ and $\int_2^4 f(x)\,dx = -2$, find $\int_1^2 f(x)\,dx$.

$$\int_1^2 f(x)dx = \int_1^5 f(x)dx - \int_2^5 f(x)dx$$

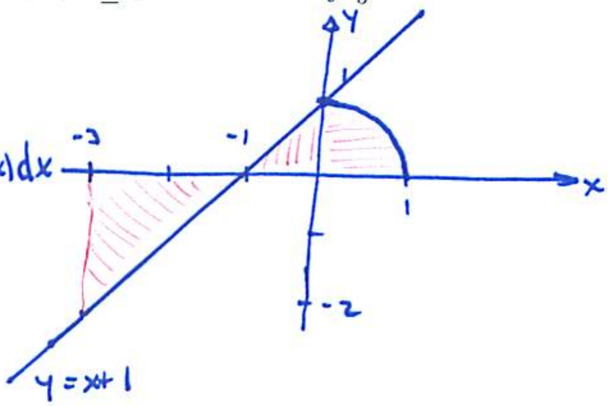
$$= \int_1^5 f(x)dx - \left(\int_2^4 f(x)dx + \int_4^5 f(x)dx\right)$$

$$= 12 - (-2+4)$$

$$= 12 - 2$$

$$= 10.$$

5. (10 points) Let $f(x) = \begin{cases} x+1 & \text{if } -3 \le x \le 0 \\ \sqrt{1-x^2} & \text{if } 0 < x \le 1 \end{cases}$. Evaluate $\int_{-3}^{1} f(x)\, dx$ by interpreting the integral as a difference of areas.



$$\int_{-3}^{1} f(x)dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= -\left(\tfrac{1}{2}\cdot 2\cdot 2\right) + \left(\tfrac{1}{2}\cdot 1\cdot 1\right) + \tfrac{1}{4}\pi(1)^2$$

$$= -2 + \tfrac{1}{2} + \tfrac{\pi}{4}$$

$$= \tfrac{\pi}{4} - \tfrac{3}{2}.$$

6. (8 points)

(a) Write $\int_0^3 (2-x^2)\, dx$ as the limit of an appropriate summation.

$\Delta x = \frac{3-0}{n} = \frac{3}{n}$

$x_i = \frac{3i}{n}$

$$\int_0^3 (2-x^2)dx = \lim_{n\to\infty} \sum_{i=1}^{n} (2-x_i^2)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2-\left(\tfrac{3i}{n}\right)^2\right)\cdot \tfrac{3}{n}$$

(b) Evaluate the answer in (a). (Work that involves the Fundamental Theorem of Calculus earns no credit.)

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(2-\left(\tfrac{3i}{n}\right)^2\right)\tfrac{3}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{6}{n} - \tfrac{9i^2}{n^2}\cdot\tfrac{3}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\tfrac{6}{n} - \tfrac{27}{n^3}i^2\right)$$

$$= \lim_{n\to\infty} \cancel{\sum_{i=1}^{n}}\ \tfrac{6}{n}\cdot n - \tfrac{27}{n^3}\cdot \tfrac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \left(6 - \tfrac{9(n+1)(2n+1)}{2n^2}\right)$$

$$= \lim_{n\to\infty} 6 - \tfrac{9}{2}\,\tfrac{2n^2+3n+1}{n^2} = 6 - \tfrac{9}{2}\cdot 2 = 6-9 = -3.$$

7. (9 points) A storage tank initially contains 5000 liters of water. Water flows into the top of the tank at a rate of $r(t) = 200 - 4t$ liters per minute, where $0 \le t \le 50$ (in minutes). Find the amount of water in the tank after the first 10 minutes.

$V = 5000 -$ total amt. drained in 10 min.

$= 5000 - \int_0^{10} 200 - 4t \, dt$

$= 5000 - \left(200t - 2t^2\right)\Big|_0^{10}$

$= 5000 - ((2000 - 200) - 0)$

$= 5000 - 1800$

$= 3200$ liters.

8. (9 points) Suppose $g(x) = \displaystyle\int_{-1}^{x^3} \frac{u^3}{u^2+1}\, du$.

(a) Find $g'(x) = \dfrac{(x^3)^3}{(x^3)^2+1} \cdot \dfrac{d}{dx}(x^3) = \dfrac{x^9}{x^6+1} \cdot 3x^2 = \dfrac{3x^{11}}{x^6+1}.$

by the FTC

(b) Find $g(-1) = \displaystyle\int_{-1}^{(-1)^3} \frac{u^3}{u^2+1}\, du = \int_{-1}^{-1} \frac{u^3}{u^2+1}\, du = 0$

same limits (interval length is 0)

(c) Explain why $g(1) = 0$. The integrand $f(u) = \dfrac{u^3}{u^2+1}$ is an odd function:

$f(-u) = \dfrac{(-u)^3}{(-u)^2+1} = \dfrac{-u^3}{u^2+1} = -f(u)$

limits symmetric about the origin

so $g(1) = \displaystyle\int_{-1}^{1} \frac{u^3}{u^2+1}\, du = 0$

HINT: No part of this problem requires any integration, just properties of definite integrals and the FTC.