MATH 141 (Section 3 & 4)
Prof. Meade

Exam 3
October 23, 2009

University of South Carolina
Fall 2009

Name: Key

Section: 003 / 004 (circle one)

Instructions:

1. There are a total of 7 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	30	
2	10	
3	10	
4	10	
5	10	
6	10	
7	20	
Extra Credit	5	
Total	100	

Enjoy Homecoming!

1. (30 points) Find the derivative of each function. Simplify your answer when possible.

(a) $y = x^{10} - \frac{1}{x^2} + \sqrt[5]{x} + e^3 = x^{10} - x^{-2} + x^{1/5} + e^3$

$y' = 10x^9 + 2x^{-3} + \frac{1}{5}x^{-4/5}$

(b) $g(t) = t^2 \tan(t)$

$g'(t) = t^2 \sec^2(t) + 2t\tan(t)$ (Product Rule)

(c) $y = \frac{1 + \sin(x)}{x + \cos(x)}$

$\frac{dy}{dx} = \frac{(x+\cos(x))(\cos(x)) - (1+\sin(x))(1-\sin(x))}{(x+\cos(x))^2}$ (Quotient Rule)

$= \frac{x\cos(x) + \underbrace{\boxed{\cos^2(x) - (1-\sin^2(x))}}_{=0}}{(x+\cos(x))^2} = \frac{x\cos(x)}{(x+\cos(x))^2}$

(d) $y = \arctan(e^{\sec(\theta)})$

$\frac{dy}{d\theta} = \frac{1}{1+(e^{\sec\theta})^2}\left(\frac{d}{d\theta}e^{\sec\theta}\right)$ (Chain Rule)

$= \frac{1}{1+e^{2\sec\theta}} e^{\sec\theta}\frac{d}{d\theta}(\sec\theta)$ (Chain Rule)

$= \frac{e^{\sec\theta}}{1+e^{2\sec\theta}} \sec\theta\tan\theta$

(e) $H(z) = \ln\left(\sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}\right) = \ln\left(\left(\dfrac{a^2-z^2}{a^2+z^2}\right)^{1/2}\right) = \dfrac{1}{2}\ln\left(\dfrac{a^2-z^2}{a^2+z^2}\right)$

HINT: Your final answer should be a rational function.

$$= \frac{1}{2}\left(\ln(a^2-z^2) - \ln(a^2+z^2)\right)$$

$$H'(z) = \frac{1}{2}\left(\frac{1}{a^2-z^2}\cdot\frac{d}{dz}(a^2-z^2) - \frac{1}{a^2+z^2}\cdot\frac{d}{dz}(a^2+z^2)\right)$$

$$= \frac{1}{2}\left(\frac{-2z}{a^2-z^2} - \frac{2z}{a^2+z^2}\right)$$

$$= -z\left(\frac{1}{a^2-z^2} + \frac{1}{a^2+z^2}\right)$$

$$= -z\left(\frac{(a^2+\cancel{z^2}) + (a^2-\cancel{z^2})}{(a^2-z^2)(a^2+z^2)}\right)$$

$$= \frac{-2az}{a^4-z^4}.$$

2. (10 points) Find the second derivative of $f(x) = \dfrac{x^2}{x^2+3}$.

$$f'(x) = \frac{(x^2+3)2x - x^2(2x)}{(x^2+3)^2} = \frac{6x}{(x^2+3)^2}$$

$$f''(z) = \frac{(x^2+3)^2\cdot 6 - 6x\cdot 2(x^2+3)(2x)}{\left((x^2+3)^2\right)^2}$$

$$= \frac{6\cancel{(x^2+3)}\left((x^2+3) - x\cdot 2\cdot 2x\right)}{(x^2+3)^{\cancel{4}\,3}}$$

$$= \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{18(1-x^2)}{(x^2+3)^3}.$$

3. (10 points) Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

Use implicit differentiation to find the slope of the tangent lines to the ellipse:

$$2x + 4y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{2x}{4y} = -\frac{x}{2y}.$$

The points on the ellipse that have tangent lines with slope 1 satisfy both: $x^2+2y^2=1$ & $-\frac{x}{2y}=1 \implies x=-2y$

so $(-2y)^2+2y^2=1.$
$4y^2+2y^2=1$
$6y^2=1$
$y^2=1/6$
$y=\pm 1/\sqrt{6}.$

When $y=+1/\sqrt{6}$, $x=-2y=-2/\sqrt{6}$.
& when $y=-1/\sqrt{6}$, $x=-2y=+2/\sqrt{6}$.

This leads us to the 2 points:
$\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$ & $\left(\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right)$.

4. (10 points) The radius of a sphere is decreasing at a rate of 4 mm/s. How fast is the volume changing when the diameter is 80 mm? $\frac{dr}{dt} = -4$ mm/s.

$(r = 40\text{mm})$ NOTE: A sphere with *radius* r has volume $V = \frac{4}{3}\pi r^3$.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

when $r = 40$ mm & $\frac{dr}{dt} = -4$ mm/s : $\frac{dV}{dt} = 4\pi(40)^2\cdot(-4)$
$= -25{,}600\pi \text{ mm}^3/\text{s}.$

The volume is decreasing at the rate of $25{,}600\pi$ mm^3/s.

Note: $1000 \text{ mm}^3 = 1 \text{ cm}^3 = 1 \text{ ml}$
so this rate is actually
25.6π ml/s.

5. (10 points) Find the absolute maximum and absolute minimum values of $f(x) = (x^2 - 4)^3$ on the interval $[-1, 2]$.

This is a continuous function on a closed interval, so there will be an absolute maximum and an absolute minimum. These will occur at an endpoint or at a critical number.

$f'(x) = 3(x^2-4)^2(2x) = 0 \iff x = 0$ or $x^2 = 4$ $(x = \pm 2)$.

	x	$f(x)$
endpoints	-1	$(-3)^3 = -27$
	2	$0^3 = 0$
critical numbers	0	$(-4)^3 = -64$
	~~-2~~	(not in the interval)

The absolute maximum value is 0
The absolute minimum value is -64.

6. (10 points) Suppose that f is a differentiable function with $f(2) = 12$ and $3 \leq f'(x) \leq 5$ for all values of x. Show that $30 \leq f(8) \leq 42$. (Be sure to show your work.)

HINT: Use the Mean Value Theorem on the interval $[2, 8]$.

By the MVT : $\dfrac{f(8)-f(2)}{8-2} = f'(c)$ for some number c in $(2,8)$

But we know $3 \leq f'(c) \leq 5$ so :

$$3 \leq \frac{f(8)-f(2)}{8-2} \leq 5$$

$$3 \leq \frac{f(8)-12}{6} \leq 5.$$

$$18 \leq (f(8)-12) \leq 30.$$

$$30 \leq f(8) \leq 42$$

7. (20 points) Let $f(x) = 3x^4 - 8x^3 + 6x^2$.

(a) Find the intervals of increase and decrease.

(b) Find the intervals of concavity.

(c) Find the local maximum and local minimum values.

(d) Find the inflection points

Clearly label each of your answers.

$$f'(x) = 12x^3 - 24x^2 + 12x = 12x(x^2 - 2x + 1) = 12x(x-1)^2.$$

$$f''(x) = 36x^2 - 48x + 12 = 12(3x^2 - 4x + 1) = 12(3x-1)(x-1)$$

critical #'s: $x = 0$ & $x = 1$.

possible infl. pts: $x = \frac{1}{3}$ & $x = 1$.

$f'(x)$	−	0	+	0	+
x		0		1	

$f''(x)$	+		+	0	−	0	+
x		0		$1/3$		1	

f is increasing on $(0, \infty)$ & decreasing on $(-\infty, 0)$

the graph of f is concave upward on $(-\infty, 1/3)$ & $(1, \infty)$ & concave downward on $(\frac{1}{3}, 1)$.

f has a local minimum at $x = 0$ (& has no local maximum)

f has inflection points at $(\frac{1}{3}, \frac{11}{27})$ & $(1, 1)$

Extra Credit (5 points) Sketch the graph of the function in the previous problem. Be sure to clearly label all extrema and inflection points.

HINT: $f\left(\frac{1}{3}\right) = \frac{11}{27} \approx 0.4$