MATH 141 (Section 11 & 12)
Prof. Meade

Exam 3 – Practice
October 26, 2007

University of South Carolina
Fall 2007

Name: ______________________
Section: 011 / 012 (circle one)

Instructions:

1. There are a total of 5 problems on 5 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	40	
2	30	
3	10	
4	10	
5	10	
Total	100	

Study Smart!

1. (40 points) Find $\dfrac{dy}{dx}$ in each of the following. Indicate each time you use Implicit Differentiation (ID) or Logarithmic Differentiation (LD).

(a) $y = \sin(\tan(3x))$

$$\frac{dy}{dx} = \cos(\tan(3x)) \cdot \frac{d}{dx}\tan(3x)$$
$$= \cos(\tan(3x)) \cdot \sec^2(3x) \cdot \frac{d}{dx}(3x)$$
$$= \cos(\tan(3x))\ \sec^2(3x) \cdot 3$$

ID

(b) $x^3 + y^3 = 3xy^2$

$$\frac{d}{dx}\left(x^3 + y(x)^3\right) = \frac{d}{dx}\left(3xy^2\right)$$
$$3x^2 + 3y^2\frac{dy}{dx} = 3y^2 + 3x\left(2y\frac{dy}{dx}\right)$$
$$3x^2 - 3y^2 = \left(6xy - 3y^2\right)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3x^2 - 3y^2}{6xy - 3y^2} = \frac{3(x^2 - y^2)}{3(2xy - y^2)} = \frac{x^2 - y^2}{2xy - y^2}$$

(c) $y = \ln(\sin^2(x))$

$$\frac{dy}{dx} = \frac{1}{\sin^2(x)} \cdot 2\sin(x)\cos(x) = \frac{2\cancel{\sin(x)}\cos(x)}{\sin^2(x)} = 2\frac{\cos(x)}{\sin(x)}$$

LD

(\$ID)

(d) $y = \dfrac{(x^2-8)^{1/3}\sqrt{x^2+1}}{x^6 - 7x + 5}$

$$\ln y = \ln\left(\frac{(x^2-8)^{1/3}\sqrt{x^2+1}}{x^6-7x+5}\right) = \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^2+1) - \ln(x^6-7x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\frac{1}{x^2-8}(2x) + \frac{1}{2}\frac{1}{x^2+1}(2x) - \frac{1}{x^6-7x+5}(6x^5-7)$$
$$\frac{dy}{dx} = y\left(\frac{2x}{3(x^2-8)} + \frac{x}{x^2+1} - \frac{6x^5-7}{x^6-7x+5}\right)$$

(e) $y = e^{7x} + \arcsin(2x)$

$$\frac{dy}{dx} = e^{7x} \cdot 7 + \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = 7e^{7x} + \frac{2}{\sqrt{1-4x^2}}$$

2. (30 points) Evaluate each of the following limits. Identify each indeterminate form that you encounter and indicate each time you use l'Hôpital's Rule.

(a) $\lim_{x\to 0} \frac{e^x - 1}{\sin(x)} \underset{l'H(\frac{0}{0})}{=} \lim_{x\to 0} \frac{e^x}{\cos(x)} = \frac{1}{1} = 1$

(b) $\lim_{x\to 0^+} \frac{1-\ln(x)}{e^{1/x}} \underset{l'H(\frac{\infty}{\infty})}{=} \lim_{x\to 0^+} \frac{-1/x}{e^{1/x}(-1/x^2)} = \lim_{x\to 0^+} \frac{x}{e^{1/x}} = 0$

(c) $\lim_{x\to\infty} xe^{-x}$ $\quad \infty\cdot 0$ $= \lim_{x\to\infty} \frac{x}{e^x} \underset{l'H(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

(d) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$ $\quad \infty - \infty$ $= \lim_{x\to 0} \frac{e^x - 1 - x}{x(e^x-1)} \underset{l'H(\frac{0}{0})}{=} \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + xe^x}$

$\underset{l'H(\frac{0}{0})}{=} \lim_{x\to 0} \frac{e^x}{e^x + e^x + xe^x} = \lim_{x\to 0} \frac{e^x}{(2+x)e^x} = \lim_{x\to 0} \frac{1}{2+x} = \frac{1}{2}.$

(e) $\lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^x$ $\quad 1^\infty$ $\quad$ let $L = \lim_{x\to\infty} \left(1 - \frac{3}{x}\right)^x$

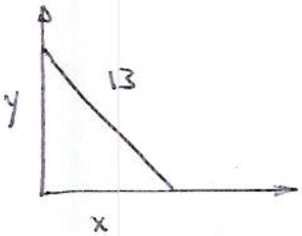
then $\ln(L) = \ln\left(\lim_{x\to\infty}\left(1-\frac{3}{x}\right)^x\right) = \lim_{x\to\infty} \ln\left(\left(1-\frac{3}{x}\right)^x\right) = \lim_{x\to\infty} x\ln\left(1-\frac{3}{x}\right)$

$= \lim_{x\to\infty} \frac{\ln(1-\frac{3}{x})}{x^{-1}} \underset{l'H(\frac{0}{0})}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{3}{x}}\cdot 3x^{-2}}{-x^{-2}}$

$= \lim_{x\to\infty} \frac{-3}{1-3/x} = -3.$

so $L = e^{\ln(L)} = e^{-3}.$

3. (10 points) A 13-ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 12 ft above the ground.



$$x^2+y^2=13^2$$

$$\frac{dy}{dt}=-2\text{ ft/s}.$$

$$y = 12\text{ ft}$$

$$\left(x=\sqrt{13^2-y^2} = \sqrt{13^2-12^2} = \sqrt{169-144} = \sqrt{25}=5\right)$$

Goal: Find $\frac{dx}{dt}$.

Using implicit differentiation:

$$\frac{d}{dt}\left(x(t)^2+y(t)^2\right) = \frac{d}{dt}(13^2)$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$2x\frac{dx}{dt}=-2y\frac{dy}{dt}$$

$$\frac{dx}{dt}=\frac{-2y}{2x}\frac{dy}{dt}=-\frac{y}{x}\frac{dy}{dt}=-\frac{12}{5}(-2)=\frac{24}{5}=4.8\text{ ft/s}.$$

The foot of the ladder is moving away from the wall at a rate of $\underline{\frac{24}{5}=4.8\text{ ft/s}}$.

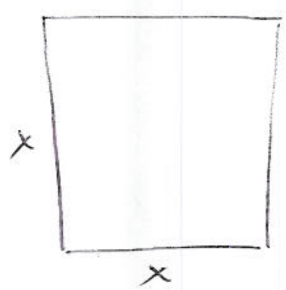
4. (10 points) Find the local linear approximation of $f(x) = \frac{1}{2+x}$ at $x_0 = 1$.

$= (2+x)^{-1}$

$$L(x) = f(1) + f'(1)(x-1)$$
$$= \frac{1}{3} - \frac{1}{9}(x-1)$$
$$= -\frac{x}{9} + \frac{1}{3} + \frac{1}{9}$$
$$= -\frac{x}{9} + \frac{4}{9}$$

$$f(1) = \frac{1}{2+1} = \frac{1}{3}$$
$$f'(x) = -(2+x)^{-2}(1)$$
$$f'(1) = -(2+1)^{-2} = -\frac{1}{9}.$$

5. (10 points) The side of a square is measured with a possible percentage error of $\pm 2\%$. Use differentials to approximate the percentage error in the area.



$$\left|\frac{dx}{x}\right| \le 0.02$$
$$A = x^2$$
$$dA = 2x\,dx$$
$$\frac{dA}{A} = \frac{2x\,dx}{A} = \frac{2x\,dx}{x^2} = 2\frac{dx}{x}$$
$$\left|\frac{dA}{A}\right| = \left|2\frac{dx}{x}\right| = 2\left|\frac{dx}{x}\right| \le 2(0.02) = 0.04$$

The area is computed with a possible percentage error of $\pm 4\%$.