MATH 141 (Section 11 & 12)
Prof. Meade

Exam 2
October 3, 2007

University of South Carolina
Fall 2007

Name: Key
Section: 011 / 012 (circle one)

Instructions:

1. There are a total of 7 problems (including the Extra Credit problem) on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	24	
2	6	
3	20	
4	24	
5	6	
6	12	
7	8	
Extra Credit	8	
Total	100	

Beat Kentucky!

1. (24 points) Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sin(3x)}{\tan(3x)} = \lim_{x\to 0} \frac{\sin(3x)}{\sin(3x)/\cos(3x)}$

$= \lim_{x\to 0} \frac{\cancel{\sin(3x)}\cos(3x)}{\cancel{\sin(3x)}}$

$= \lim_{x\to 0} \cos(3x) = \cos(0) = 1.$

(b) $\lim_{x\to 2} \frac{x^3 - x^2}{x - 1} = \frac{8-4}{2-1} = \frac{4}{1} = 4$

(c) $\lim_{x\to\infty} \frac{x^3 - x^2 + 10}{4x - x^2} = \lim_{x\to\infty} \frac{x^3\left(1 - \frac{1}{x} + \frac{10}{x^3}\right)}{x^2\left(\frac{4}{x} - 1\right)}$

$= \left(\lim_{x\to\infty} \frac{x^3}{x^2}\right) \cdot \left(\lim_{x\to\infty} \frac{1 - \frac{1}{x} + \frac{10}{x^3}}{\frac{4}{x} - 1}\right)$

$= \left(\lim_{x\to\infty} x\right) \cdot (-1)$

$= \infty \cdot (-1)$

$= -\infty$

2. (6 points) Find all points where $f(x) = \left|4 - \dfrac{8}{x^4 + x}\right|$ is *not* continuous.

The absolute value function is continuous for all inputs.
The only possible locations of discontinuities are where

$x^4 + x = 0$

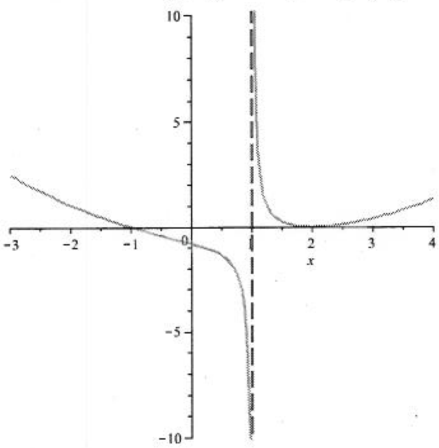
$x(x^3+1) = 0$ so $x = 0$ or $x^3 + 1 = 0$

$x^3 = -1$

$x = -1$

There are 2 discontinuities, at $x=0$ and at $x=-1$.

3. (20 points) The figure below shows the graph of $y = f'(x)$ for an unspecified function f.



(a) For what values of x does the curve $y = f(x)$ have a horizontal tangent line?

$x = -1$ and $x = 2$

(b) Over what intervals does the curve $y = f(x)$ have tangent lines with positive slope?

$(-\infty, -1) \cup (1, 2) \cup (2, \infty)$

(c) Over what intervals does the curve $y = f(x)$ have tangent lines with negative slope?

$(-1, 1)$

(d) At what points does the curve $y = f(x)$ not have a tangent line?

$x = 1$.

4. (24 points) Find the derivative of each of the following functions.

(a) $f(x) = x^3 + 5\sqrt{x} - \frac{3}{x^8} = x^3 + 5x^{1/2} - 3x^{-8}$

$$f'(x) = 3x^2 + 5\left(\tfrac{1}{2}x^{-1/2}\right) - 3\left(-8x^{-9}\right)$$
$$= 3x^2 + \tfrac{5}{2}x^{-1/2} + 24x^{-9}$$
$$= 3x^2 + \frac{5}{2\sqrt{x}} + \frac{24}{x^9}$$

(b) $f(x) = (1 + \sec(x))(x^2 - \cos(x))$

$$f'(x) = (1+\sec(x))'(x^2 - \cos(x)) + (1+\sec(x))(x^2-\cos(x))'$$
$$= \sec(x)\tan(x)(x^2 - \cos(x)) + (1+\sec(x))(2x - (-\sin(x)))$$
$$= \sec(x)\tan(x)(x^2 - \cos(x)) + (1+\sec(x))(2x + \sin(x)).$$

(c) $f(x) = \frac{3x+1}{2x^2+1}$

$$f'(x) = \frac{(2x^2+1)(3x+1)' - (3x+1)(2x^2+1)'}{(2x^2+1)^2}$$
$$= \frac{(2x^2+1)(3) - (3x+1)(4x)}{(2x^2+1)^2}$$
$$= \frac{6x^2+3 - (12x^2+4x)}{(2x^2+1)^2}$$
$$= \frac{-6x^2-4x+3}{(2x^2+1)^2}$$

5. (6 points) The differentiable functions f and g have $f(1) = 1$, $g(1) = 2$, $f'(1) = 3$, and $g'(1) = -1$. Evaluate $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=1}$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=1} = \frac{g(1)f'(1) - f(1)g'(1)}{g(1)^2}$$

$$= \frac{2(3) - (1)(-1)}{2^2} = \frac{6+1}{4} = \underline{\underline{\frac{7}{4}}}$$

6. (12 points) The formula for the volume of a cube of side x inches is $V = x^3$ cubic inches.

(a) What is the average rate at which the volume of a cube as the sides increase from 2 inches to 4 inches?

$$r_{ave} = \frac{V(4) - V(2)}{4-2} = \frac{4^3 - 2^3}{4-2} = \frac{64-8}{2} = \frac{56}{2} = \underline{\underline{28}}$$

(b) What is the instantaneous rate at which the volume of a cube changes when $x = 5$ inches?

$$V' = 3x^2$$

$$r_{inst} = V'(5) = 3(5)^2 = 3(25) = \underline{\underline{75}}.$$

7. (8 points)

(a) State the definition of the derivative of a function $f(x)$ in terms of a limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use the definition of the derivative to find $f'(x)$ for $f(x) = \dfrac{x}{x-1}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\frac{x+h}{(x+h)-1} - \frac{x}{x-1}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + \cancel{hx} - \cancel{x} - h - (\cancel{x^2} + \cancel{hx} - \cancel{x})}{h(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{\cancel{h}(x+h-1)(x-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)(x-1)} = \frac{-1}{(x-1)^2}$$

Extra Credit (8 points) Suppose that a function f is differentiable at $x = 1$ and $\lim_{h \to 0} \dfrac{f(1+h)}{h} = 5$.
Find $f(1)$ and $f'(1)$.

For $\lim_{h \to 0} \frac{f(1+h)}{h} = 5$ to be possible, this limit must be indeterminate of form $\frac{0}{0}$.
Thus, $\lim_{h \to 0} f(1+h) = 0$. Because f is differentiable at $x=1$, it's also continuous there.
So $f(1) = \lim_{h \to 0} f(1+h) = 0$. Next, $\lim_{h \to 0} \frac{f(1+h)}{h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = f'(1)$

So that $f'(1) = \lim_{h \to 0} \frac{f(1+h)}{h} = 5$.

$f(1) = 0$ and $f'(1) = 5$.