MATH 141 (Section 1 & 2)
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University of South Carolina
Fall 2004

Exam 3
October 29, 2004

Name: Key
Section: 001 002 (circle one)

Instructions:

1. There are a total of 6 problems on 6 pages. Check that your copy of the exam has all of the problems.
2. Calculators may not be used for any portion of this exam.
3. You must show all of your work to receive credit for a correct answer.
4. Your answers must be written legibly in the space provided. You may use the back of a page for additional space; please indicate clearly when you do so.

Problem	Points	Score
1	20	
2	30	
3	15	
4	15	
5	10	
6	10	
Total	100	

Happy Halloween!

1. (20 points)

(a) Rewrite $\ln\left(\frac{x^2 \sin^3 x}{\sqrt{x^2+1}}\right)$ in terms of $r = \ln x$, $s = \ln \sin x$, $t = \ln(x+1)$, and $v = \ln(x^2+1)$.

$$\ln\left(\frac{x^2 \sin^3 x}{\sqrt{x^2+1}}\right) = \ln(x^2) + \ln(\sin^3 x) - \ln\left(\sqrt{x^2+1}\right)$$
$$= 2\ln x + 3\ln(\sin x) - \tfrac{1}{2}\ln(x^2+1)$$
$$= 2r + 3s - \tfrac{1}{2}v$$

(b) Solve for x: $\ln\left(\frac{1}{x}\right) + \ln(2x^3) = \ln 8$.

$$\ln\left(\tfrac{1}{x}\right) + \ln(2x^3) = \ln 8$$
$$-\ln x + \ln 2 + 3\ln x = \ln 8$$
$$\ln 2 + 2\ln x = \ln 8$$
$$2\ln x = \ln 8 - \ln 2 = \ln\left(\tfrac{8}{2}\right)$$
$$\ln x = \tfrac{1}{2}\ln(4) = \ln(4^{1/2}) = \ln 2 \quad \text{so} \quad x = e^{\ln 2} = 2.$$

; or: $\ln 8 = \ln\left(\frac{1}{x}\right) + \ln(2x^3)$

$= \ln\left(\frac{2x^3}{x}\right) = \ln(2x^2)$

so $2x^2 = e^{\ln 8} = 8$. Now, $x^2 = 4$

so $x = \pm 2$. But, $\ln\left(\frac{1}{x}\right)$ requires $x >$

so the only solution is $x = 2$.

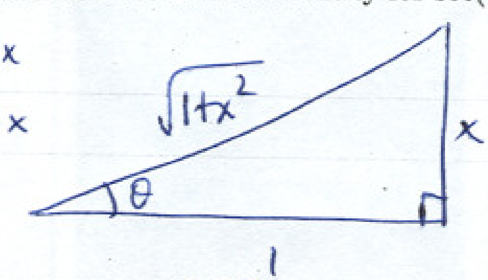
(c) Find the exact value of $\sin^{-1}\left(\frac{1}{2}\right)$.

because $\sin \pi/6 = 1/2$ we know $\sin^{-1}\left(\frac{1}{2}\right) = \pi/6$.

(d) Use the "triangle method" to find an identity for $\sec(\tan^{-1}(x))$.

let $\theta = \tan^{-1} x$

then $\tan\theta = x$



$$\sec(\tan^{-1} x) = \sec\theta = \frac{\sqrt{1+x^2}}{1} = \sqrt{1+x^2}.$$

2. (30 points) Find the derivative of each of the following functions.

(a) $f(x) = \ln\left(\frac{x}{1+x^2}\right) = \ln x - \ln(1+x^2)$

$$f'(x) = \frac{1}{x} - \frac{1}{1+x^2}\cdot 2x = \frac{1}{x} - \frac{2x}{1+x^2}$$

it's ok to stop here

$$= \frac{1+x^2 - 2x(x)}{x(1+x^2)}$$

$$= \frac{1-x^2}{x(1+x^2)}$$

or: $f'(x) = \frac{1}{\frac{x}{1+x^2}} \cdot \frac{(1+x^2)(1) - x(2x)}{(1+x^2)^2}$

$$= \frac{\cancel{1+x^2}}{x} \cdot \frac{1+x^2-2x^2}{(1+x^2)^{\cancel{2}}}$$

$$= \frac{1-x^2}{x}$$

(b) $f(x) = e^{\sin(x)}$

$$f'(x) = e^{\sin x} \frac{d}{dx}\sin x$$

$$= \cos x \; e^{\sin x}$$

(c) $f(x) = 3^{-x}$

$$f'(x) = 3^{-x}\ln 3 \cdot \frac{d}{dx}(-x)$$

$$= -(\ln 3)\,3^{-x}$$

(d) $f(x) = \sin^{-1}(x) + \cos^{-1}(x)$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0.$$

(e) $f(x) = \tan^{-1}\left(x^2\right)$

$$f'(x) = \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2) = \frac{2x}{1+x^4}.$$

3. (15 points) Let $f(x) = \sqrt{x+4} + 1$ for $x \geq -4$.

(a) [8 points] Find a formula for $f^{-1}(x)$.

$y = \sqrt{x+4} + 1$

$y - 1 = \sqrt{x+4}$

$(y-1)^2 = x+4$

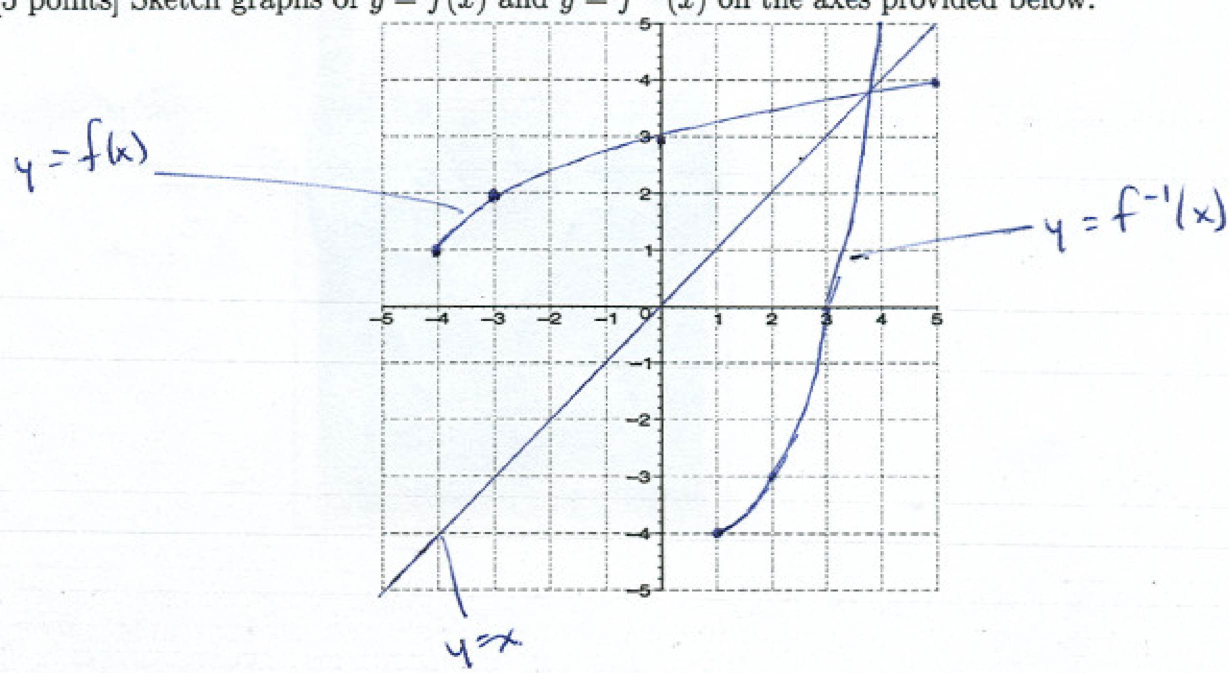
$(y-1)^2 - 4 = x.$ so $f^{-1}(x) = (x-1)^2 - 4.$

(b) [2 points] What is the domain of f^{-1}?

Note that the range of f is $y \geq 1$.

Thus, the domain of f^{-1} is $x \geq 1$.

(c) [5 points] Sketch graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the axes provided below.



4. (15 points) Evaluate the following limits. Identify each time l'Hôpital's Rule is applied, including the type of indeterminate form.

(a) $\lim_{x\to 2} \frac{x^3-2x-4}{x^2-x-2}$ $\overset{l'H: \frac{0}{0}}{=}$ $\lim_{x\to 2} \frac{3x^2-2}{2x-1} = \frac{10}{3}$

$\lim_{x\to 2} x^3-2x-4=0$

$\lim_{x\to 2} x^2-x-2=0$

(b) $\lim_{x\to 0^+} \frac{\cos x}{\ln x}$ $= 0$

$\lim_{x\to 0^+} \cos x = 1$

This is not an indeterminate form!

$\lim_{x\to 0^+} \ln x = -\infty$

(c) $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x$ $= \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{3}{x}\right)^x\right)} = \lim_{x\to\infty} e^{x\ln\left(1+\frac{3}{x}\right)} = e^{\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)}$.

$\lim_{x\to\infty} 1+\frac{3}{x} = 1$

$\lim_{x\to\infty} x = \infty$

This problem is indeterminate with form 1^∞.

$\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right) = \lim_{x\to\infty} \frac{\ln\left(1+\frac{3}{x}\right)}{x^{-1}}$

$l'H: \infty\cdot 0$

$= \lim_{x\to\infty} \frac{\frac{1}{1+\frac{3}{x}}\left(\cancel{-}3x^{\cancel{-2}}\right)}{\cancel{-}x^{\cancel{-2}}}$

$l'H: \frac{0}{0}$

$= \lim_{x\to\infty} \frac{3}{1+\frac{3}{x}}$

$= 3$

Thus, $\lim_{x\to\infty} \left(1+\frac{3}{x}\right)^x = e^{\lim_{x\to\infty} x\ln\left(1+\frac{3}{x}\right)} = e^3$.

5. (10 points) A function f that is continuous for all real numbers has the following sign chart for its first and second derivatives.

interval	sign of $f'(x)$	sign of $f''(x)$
$x < -1$	-	+
$-1 < x < 1$	-	-
$1 < x < 2$	+	-
$2 < x < 4$	+	+
$4 < x$	-	+

(a) On what intervals is f increasing?

$1 < x < 2$ & $2 < x < 4$

(b) On what intervals is f decreasing?

$x < -1$, $-1 < x < 1$, & $x > 4$

(c) On what intervals is f concave up?

$x < -1$, $2 < x < 4$, & $x > 4$

(d) On what intervals is f concave down?

$-1 < x < 1$, & $1 < x < 2$

(e) Find the x-coordinate of all inflection points. — points where $f''(x)$ changes sign

$x = -1$ & $x = 2$.

6. (10 points) A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 3 cm?

Remember to provide appropriate units for your final answer.

$V = \frac{4}{3}\pi r^3$

$\frac{dr}{dt} = -15$ cm/min

& $r = 3$ cm.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\frac{dV}{dt} = 4\pi (3)^2 (-15) = -540\pi$ cm^3/min.