# Sequences

Douglas B. Meade

Department of Mathematics

## *Overview*

Sequences (and series) are the objects of interest for the next couple of weeks. The intent of this lab is to provide additional practice determining the convergence or divergence of a sequence of numbers. This will be accomplished using a drill maplet and explicit commands in a worksheet. In both cases, graphs of the terms of a sequence will be used to assist the understanding — but, remember that ultimately you will have to make these decisions based on the formula for the terms of the sequence.

## *Maple Essentials*

- A link to the *SequenceDrill* maplet will soon be added to the Maplets for Calculus webpage. A direct URL is
  `http://www.math.sc.edu/~meade/CalcMaplets/CalcUSC/SequenceDrill.maplet`
- New Maple commands introduced in this lab include:

| Command | Description |
|---|---|
| `!` | factorial, e.g., `5!` is 120 (try `100!`) |
| `->` | the *arrow operator* is used to define a function<br>For example, $a(n) = \sqrt{n^2+n+1} - n$ could be defined with<br>`> a := n -> sqrt(n^2+n+1)-n;`<br>Then, $a(10)$ is `a(10)` and $\lim_{n\to\infty} a_n$ is `limit( a(n), n=infinity );` |
| `seq` | create a *seq*uence of values<br>If `a` is a Maple function, the values of $a(1)$, $a(2)$, $a(3)$, and $a(4)$ can be obtained, simultaneously, using:<br>`> seq( a(n), n=1..4 );`<br>Points on the graph of a function could be obtained using:<br>`> seq( [n,a(n)], n=1..100 );`<br>**Note:** *Maple's* `seq` *command is similar to a* `for` *or* `do` *statement in other programming languages.* |
| `sum` (and `Sum`) | a finite or infinite sum (including infinite series)<br>The sum of the first $n$ positive integers is<br>`> sum( k^2, k=1..n );`<br>and the sum of a convergent geometric series is<br>`> sum( (3/4)^k, k=1..infinity );`<br>Observe that Maple automatically evaluates a `sum`. The `Sum` command is very similar, except that it is not automatically expanded and evaluated:<br>`> S := Sum( (3/4)^k, k=1..infinity );`<br>The `value` command forces evaluation of a `Sum`: `S = value(S);` |

## *Preparation*

Review the basic definitions of *convergence* and *divergence* of a sequence. In addition, review the basic qualitative properties of powers, logarithms, and exponentials. For example, any exponential grows faster (at $\infty$) than any polynomial; factorials grow faster than exponentials.

## *Assignment*

- The Activities cover a wide variety of sequences, including sequences with recursively-defined terms. The techniques introduced in the Activities should prove useful in checking your solutions to assigned problems from the text. Mastery Quiz 6 tests these skills.
- Hour Quiz 2 will be given the week immediately before Spring Break. The questions will focus on the lab content since Hour Quiz 1.

## *Activities*

1. Use the *SequenceDrill* maplet to determine the convergence or divergence of the following sequences. **Note:** *For additional examples, see the Exercises in your textbook or let the maplet generate random sequences.*

   (i) $\left\{\dfrac{n^2+3n-1}{n^2-5n-6}\right\}_{n=7}^{\infty}$ (ii) $\left\{n\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$ (iii) $\left\{\dfrac{n^3e^n}{\pi^n}\right\}_{n=1}^{\infty}$

   (iv) $\{(-1)^n\arctan(n)\}_{n=1}^{\infty}$ (v) $\left\{\dfrac{10^n}{n!}\right\}_{n=0}^{\infty}$ (vi ) $\left\{\dfrac{(-1)^n\sin(n)^2}{n}\right\}_{n=1}^{\infty}$

2. Determine if the following sequences converge or diverge, and the limit of any convergent sequence. (Let $a$ and $b$ be real numbers.)

   (i) $\left\{\left(\dfrac{n+a}{n+b}\right)^n\right\}_{n=1}^{\infty}$ (ii) $\left\{\sqrt{n^2+an+b}-n\right\}_{n=1}^{\infty}$ (iii) $\left\{an\sin\left(\dfrac{\pi}{n}\right)\right\}_{n=1}^{\infty}$

   (iv) $\left\{\dfrac{1}{k}\right\}_{k=1}^{\infty}$ (v) $\left\{\displaystyle\sum_{k=1}^{n}\dfrac{1}{1+(k/n)}\right\}_{k=1}^{\infty}$ (vi) $\left\{\displaystyle\sum_{k=1}^{n}\dfrac{k}{n^2}\right\}_{n=1}^{\infty}$

   (vii) $\left\{\dfrac{3+n^2\sin(n)}{2+n^2}\right\}_{n=1}^{\infty}$ (viii) $\left\{\dfrac{1}{n}\displaystyle\sum_{k=1}^{n}\dfrac{1}{1+(k/n)}\right\}_{n=1}^{\infty}$ (ix) $\left\{\displaystyle\sum_{k=1}^{n}\dfrac{k^2}{n^3}\right\}_{n=1}^{\infty}$

   **Note:** *The* SequenceDrill *maplet does not work well with sequences involving parameters. It's probably best to use explicit commands in a worksheet to help with these examples.*

3. A standard format for a recursively-defined sequence is $a_{n+1}=f(a_n)$, $n=2,3,\dots$ (with $a_1$ given explicitly). Under the assumptions that (i) $\{a_n\}$ converges to $L$ and (ii) $f$ is continuous function (at $L$), we find that $\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}a_n=L$ and

   $$\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}f(a_n)=f(\lim_{n\to\infty}a_n)=f(L).$$

   Thus, $L$ must be a solution to $L=f(L)$. While this equation might be difficult to solve by hand, Maple can be used to find a solution (exactly, numerically, or graphically).

   (a) Consider the sequence $\{a_n\}$ defined by $a_1=0$, $a_{n+1}=\sqrt{6+a_n}$, $n=1,\ 2,\ 3,\ \dots$. Assuming this sequence converges, find the (exact) value of the limit, $L$. Explain how a plot containing the graphs of $y=x$ and $y=\sqrt{6+x}$ confirms that this equation has a solution.

   (b) Consider the sequence $\{a_n\}$ defined by $a_1=1$, $a_{n+1}=\dfrac{1}{2}\left(a_n+\dfrac{r}{a_n}\right)$, $n=1,2,3,\dots$ where $r>0$. Assuming this sequence converges, show that $\lim\limits_{n\to\infty}r_n=\sqrt{a}$.