Limits, Infinity, and Asymptotes

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Overview

Asymptotes for functions are sometimes easy to identify from a graph. The actual definitions of asymptotes is in terms of limits:

Asymptote	Equation	Definition
Horizontal	$y = L$	$\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$
Vertical	$x = a$	$\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$

This lab is designed to provide experience finding asymptotes. Some limits will be evaluated by inspection. You will learn several ways to use Maple to help evaluate more difficult limits.

Maple Essentials

- The *Rational Functions* tutor is started from the Maple 9.5 user interface under the Tools menu:

  Tools → Tutors → Precalculus → Rational Functions ...

- The *LimitCheck* maplet is available from USC at the URL:

  `http://www.math.sc.edu/~meade/141L-F04/maplets/CalcUSC/LimitCheck.maplet`

  This maplet can also be run via MapleNet at the URL:

  `http://maplenet.math.sc.edu/maplenet/141L-F04/CalculusI/LimitCheck.html`

Preparation

To prepare for this lab, review the definitions of horizontal and vertical asymptotes. Also review basic facts about rational functions.

Activities

Your task is to identify all horizontal and vertical asymptotes for as many of the functions on the back of this page as possible. For rational functions the *Rational Functions* tutor can be used to obtain a graph of the function and its asymptotes. This is a start, but you still need to identify the exact equations for the asymptotes.

The equations of the asymptotes for the graph of a function are found by evaluating appropriate limits. All horizontal asymptotes are found by evaluating $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Vertical asymptotes exist when one of the one-sided limits or the two-sided limit of $f(x)$ at $x = a$ is ∞ or $-\infty$. The challenge here is to be able to identify appropriate values of a.

When possible, evaluate limits by inspection. For more difficult limits, use the *LimitCheck* maplet or enter a `limit` command in your Maple worksheet (see Notes on reverse).

Assignment

Think about the following questions:

- What property of a rational function determines whether it has a horizontal asymptote?
- Does every hole in the domain of a function lead to a vertical asymptote?
- Can the graph of a function cross the graph of its horizontal asymptotes? Its vertical asymptotes?
- How many horizontal asymptotes can a graph have? (Explain.)

1. $f(x) = \dfrac{3x^2 + 2x - 1}{x + 2}$ [default function in *Rational Functions* tutor]
2. $f(x) = \dfrac{3x^2 + 2x - 1}{x^2 + 2}$
3. $f(x) = \dfrac{3x^2 + 2x - 1}{x^2 - 4}$
4. $f(x) = \dfrac{2x + 3}{x + 4}$ [#25(a), p. 120, Anton]
5. $f(x) = \dfrac{x^2 + 1}{x + 1}$ [#25(c), p. 120, Anton]
6. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$
7. $f(x) = \dfrac{x^2 - 1}{5x^2 + 1}$ [#26(a), p. 121, Anton]
8. $f(x) = \left(1 + \dfrac{3}{x}\right)^x$ [#25(b), p. 120, Anton]
9. $f(x) = \dfrac{\sin x}{x}$ [#26(c), p. 121, Anton]
10. $f(x) = \dfrac{x - \sin x}{x^3}$ [#34, p. 121, Anton]
11. $f(x) = \dfrac{\sqrt{x^2 + 4} - 2}{x}$ [#40, p. 130, Anton]
12. $f(t) = \dfrac{t^3 + 3t^2 - 12t + 4}{t^3 - 4t}$ [#13, p. 130, Anton]
13. $f(x) = \dfrac{\sqrt{x^2 + 1} + 2x}{x}$

Notes:

- The *Rational Functions* tutor does not provide any control over the viewing window. However, when you click the **Close** button the plot is returned to the Maple worksheet. From here the viewing window can be modified via the **Axes** entry in the context menu produced when the right mouse button is pressed when the cursor is over the graph.
- If you need help evaluating some of these limits, you should consider using the `LimitCheck` maplet or using Maple's palettes to create a `limit` command. Once you are comfortable with the Maple syntax you might want to type the `limit` command.
- The Maple syntax for limits is:

Limit	Maple Command
$\lim_{x \to a} f(x)$	`limit( f(x), x=a );`
$\lim_{x \to a^-} f(x)$	`limit( f(x), x=a, left );`
$\lim_{x \to a^+} f(x)$	`limit( f(x), x=a, right );`

  The constant ∞ is represented in Maple as `infinity`. (Your TA will show you how to use the **Expression** and **Symbol** palettes to avoid typing so much.)