

Definite Integrals and Riemann Sums

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Overview

This lab will help to develop your understanding of the definite integral as defined via Riemann sums and as computed via the Fundamental Theorem of Calculus.

Maple Essentials

- The *Riemann Sums* tutor can be started from the Tools menu:

Tools → **Tutors** → **Calculus - Single Variable** → **Riemann Sums ...**

Note: *The Riemann Sums and Approximate Integrals tutors are identical.*

- New Maple commands introduced in this lab include:

Command	Description
<code>ApproximateInt</code>	visual, numeric, or symbolic representation of a Riemann sum; in the <code>Student[Calculus1]</code> package
<code>int</code>	used for definite and indefinite integrals

Note: *See the Additional Notes section for the syntax and usage of these commands.*

Preparation

Review the definition of area under a curve and approximations of area (last part of §6.4 in Anton) and the Fundamental Theorems of Calculus (§6.5 in Anton). In particular, you should be able to explain the symbols and meaning of the following two equations:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k^*) \Delta x$$

$$\int_a^b f(x) dx = F(b) - F(a) \quad \text{where } F \text{ is an antiderivative of } f$$

Activities

- Use the `Riemann Sums` tutor to approximate $\int_2^{10} \frac{1}{x} dx$ with the Riemann sum $\sum_{k=1}^4 f(x_k^*) \Delta x$

where:

- x_k^* is the left endpoint of each subinterval
 - x_k^* is the right endpoint of each subinterval
 - x_k^* is the midpoint of each subinterval
- Make a table of approximate values for $\int_2^{10} \frac{1}{x} dx$ using left, right, or midpoint approximations (your choice) with $n = 4, 8, 16, 32, 64,$ and 128 subintervals. What can you say about these numbers as the number of subintervals increases?

3. The following sequence of Maple commands reinforces the Fundamental Theorem of Calculus for the definite integral $\int_a^b \frac{1}{1+x^2} dx$ with general limits of integration:

```
> with( Student[Calculus1] );           # load package
> f := 1/(1+x^2);                       # enter the integrand
> q1 := ApproximateInt( f, a..b, partition = n, # Riemann sums with n subintervals
>           output=sum );
> limit( q1, n=infinity );              # exact area
> F := int( f, x );                     # antiderivative of f (F(x))
> eval( F, x=b ) - eval( F, x=a );     # area by Fund Thm of Calculus
> int( f, x=a..b );                     # definite integral
```

4. (a) Repeat *Activity 2* for the following definite integrals:

$$\int_0^{\pi/2} \cos(x) dx \quad \int_2^6 x^3 dx \quad \int_{-1}^3 e^{-x} dx \quad \int_0^4 \frac{x}{x+1} dx \quad \int_0^4 \frac{x}{x^3+1} dx$$

$$\int_0^{3\pi/2} \cos(x) dx \quad \int_0^5 \sqrt{x} dx \quad \int_{-1}^3 x e^{-x} dx \quad \int_0^4 \frac{x}{x^2+1} dx \quad \int_0^4 \frac{x}{x^4+1} dx$$

- (b) Repeat *Activity 3* for the *integrands* in the above examples.

Additional Notes

- The `ApproximateInt` and `int` commands can be used instead of the tutors. Execute `with(Student[Calculus1]);` before using the `ApproximateInt` command.
- The graph of the midpoint approximation to the area under the curve $y = f(x)$ on the interval $[a, b]$ with N subintervals is obtained with the Maple command:
`> ApproximateInt(f, a..b, partition=N, method=midpoint, output=plot);`
*In this plot, what do the numbers labeled by **Approximate Value** and **Area** represent?*
- The left endpoint and right endpoint approximations are obtained by replacing `method=midpoint` with `method=left` and `method=right`, respectively. (Another valid option is `method=random`.)
- An animation showing the convergence of the Riemann sum approximations to the exact area can be obtained by changing `output=plot` to `output=animation`.
`> ApproximateInt(f, a..b, partition=N, method=left, output=animation);`
- Similarly, `output=sum` instructs Maple to return the Riemann sum in sigma notation.
- The Maple syntax for the indefinite integral $\int f(x) dx$ is `int(f, x);`.
- The Maple syntax for the definite integral $\int_a^b f(x) dx$ is `int(f, x = a .. b);`.

Assignment

This is the last week of the semester, and you have already completed three projects and three quizzes. Congratulations! There is no assignment this week.