Volume of a Solid of Revolution

Objective   This lab investigates volumes of solids of revolution. The `VolumeOfRevolution` command can be used to visualize the region in a 3-D plot, set up a definite integral for the volume of the solid, or compute a numeric approximation to the volume of the solid. The same command is used for both the method of washers and the method of shells. A maplet interface to this command is also presented.

Background   The general formulae for the volume of a solid of revolution by the method of washers and the method of shells can be summarized as

$$V = \int_a^b \pi\left(r_{out}(x)^2 - r_{in}(x)^2\right)\ dx \text{ and } V = \int_a^b 2\pi x\left(h_{top}(x) - h_{bot}(x)\right)\ dx.$$

The three-dimensional plots produced by the `VolumeOfRevolution` command can be rotated in real time. To do this, first click the left mouse button on the plot. Then, while holding down the left mouse button, drag the plot until it rotates to the view you desire. (It's difficult to describe this process; just try it!) The icons in the context bar for a 3D plot can be used for some modifications to the plot. A full description of these features is provided in the online help: `?worksheet,plotinterface,style3`

In Example 1 the region in the first quadrant bounded above by the graph of $y = x^2 + 1$ on $[0, 2]$ is revolved around the x-axis ($y = 0$) to form a 3-D solid. Example 2 revolves the same region around the y-axis ($x = 0$). Example 3 illustrates the usage of the `VolumeOfRevolution` command for a solid with nontrivial top and bottom curves. Example 4 describes the usage of the `VolumeOfRevolution` maplet to produce the same results as obtained in Example 1.

In each of these examples the first step is to visualize the 2-D region *and* the axis. In many cases it is easiest to do this manually. Maple commands producing this visual aid are provided for Examples 1 – 3. (In the interest of space, these commands are omitted from the listing of Examples 2 and 3.)

Discussion   Enter, and execute, the following Maple commands in a Maple worksheet.

Example 1: Method of Washers – Axis: $y = 0$ (Horizontal)

```
> restart;                                      # clear Maple's memory
> with( plots );                                # load package
> with( Student[Calculus1] );                   # load package
> R := x -> x^2 + 1;                            # define outer radius
> Pr:=plot( [ R(x),                             # outer radius
>               [[0,R(0)],[0,0],[2,0],[2,R(2)]] # outline of region
>           ], x=0..2, color=blue ):
> Pa:=plot( 0, x=-1..3, thickness=2 ):          # horizontal axis
> display( [Pr,Pa], axes=framed );              # display region and axis
> VolumeOfRevolution( R(x), x=0..2,             # 3-D plot (rotate it!)
>         axis=horizontal, output=plot );
> q1:=VolumeOfRevolution( R(x), x=0..2,         # volume as definite integral
>         axis=horizontal, output=integral );
> q2:=value( q1 );                              # exact volume
> evalf( q2 );                                  # floating-point approximation
```

Example 2: Method of Shells – Axis: $x = 0$ (Vertical)

```
> VolumeOfRevolution( R(x), x=0..2,          # 3-D plot (rotate it!)
>          axis=vertical, output=plot );
> q3 := VolumeOfRevolution( R(x), x=0..2,    # vol as definite integral
>          axis=vertical, output=integral );
> q4 := value( q3 );                          # exact volume
> evalf( q4 );                                # approximation
```

Example 3: Volume Between Two Surfaces

```
> top := x -> 2 + sin(x);                     # define outer radius
> bot := x -> 1;                              # define inner radius
> VolumeOfRevolution( top(x), bot(x),         # 3-D plot (rotate it!)
>          x=0..2*Pi, axis=horizontal, output=plot );
> q5 := VolumeOfRevolution( top(x), bot(x),   # vol as definite integral
>          x=0..2*Pi, axis=horizontal, output=integral );
> q6 := value( q5 );                          # exact volume
> evalf( q6 );                                # approximation
```

Example 4: The `VolumeOfRevolution` maplet
- From your browser, launch the `VolumeOfRevolution` maplet.
- In the **Function** field, enter `x^2+1`.
- Set **a** $= 0$ and **b** $= 2$.
- Check that the **Horizontal axis** box is checked.
- To see the solid, click the **Plot** button.
- To see the definite integral and its value (exact and approximate), click the **Volume** button.

Notes

(1) The `VolumeOfRevolution` maplet complains if the problem contains parameters. Fortunately, the `VolumeOfRevolution` *command* is capable of working with problems with parameters (except that a plot cannot be created).

(2) The `VolumeOfRevolution` command (and maplet) use the Method of Washers when the axis is horizontal and the Method of Shells when the axis is vertical.

Questions

(1) Let R denote the region in the first quadrant bounded above by the graph of $y = -x^2 + 4x$ for $x \in [0, 3]$. Find a definite integral for the volume of the solid generated by revolving R about the x-axis. What is the value of this integral?

(2) Find a definite integral for the volume of the solid generated by revolving R about the y-axis. What is the value of this integral?

(3) Let S denote the region in the first quadrant bounded by the graphs of $y = -x^2 + 4x + 2$ and $y = x + 2$ for $x \in [0, 3]$. Find a definite integral for the volume of the solid generated by revolving S about the x-axis. What is the value of this integral? *Write the integrand of the definite integral as π times the difference of two squares.*

Bonus Explain, in one or two complete sentences, why the volume of the region in the first quadrant bounded by the graphs of $y = -x^2 + 4x$ and $y = x$ for $x \in [0, 3]$ revolved about the line $y = -2$ is the same as the volume found in Question (3).

Challenge Let S_k be the solid formed when one arch of $y = \sin(x)$, $0 \le x \le \pi$, is revolved about the line $y = k$. Determine k, $0 \le k \le 1$, so that S_k has maximum and minimum volume. *Hint: Obtain a formula for the volume as a function of k. Graph this function.*