

Graphical Understanding of Limits

Objective

The purpose of this lab is to help you develop your intuitive understanding of limits. This will be accomplished by determining limits using Maple plots of functions that would otherwise be difficult to visualize or manipulate.

Background

Limits are the foundation for all of calculus. As such it is essential that you develop a solid understanding of this concept. The basic picture you need to have in mind is that limits do *not* depend on the value of the function at the limit point. In fact, a limit can exist even if the function is not defined at the limit point.

In this lab you will use your skills to plot a Maple function and use this as an aid in evaluating limits. You will also learn to use Maple to define and plot *piecewise-defined* functions.

New Maple commands introduced in this lab include `limit`, for evaluating one- and two-sided limits, and `piecewise`, for entering piecewise-defined functions.

Discussion

The fundamental idea behind limits is that the value of a function at the limit point is not important. In some cases, a function may not be defined at the limit point (Example 1). In other cases, the value of the function may be defined but different from the limit (Example 2). When the function is defined at the limit point and this value agrees with the corresponding limit the function is said to be *continuous* at this limit point (Example 3). Example 4 is a very common limit.

Example 1: $\lim_{x \rightarrow 1} \frac{x^3 - 5x^2 + 8x - 4}{x - 1}$

```
> restart; # clear Maple's memory
> f := x -> (x^3-5*x^2+8*x-4)/(x-1); # define function
> f(1); # function has a singularity at x = 1
> plot( f(x), x=-4..4 ); # plot function on [-4,4]
> f(1.1); # value at a point near x = 1
> f(1.01); # value at a point closer to x = 1
> f(1.0001); # ...
> limit( f(x), x=1 ); # limiting value
```

Example 2: $\lim_{x \rightarrow 1} g(x)$ where $g(x) = \begin{cases} 1 & x < 1 \\ 2 & x = 1 \\ 3 & x > 1 \end{cases}$

```
> g := x -> piecewise(x<1,1,x=1,2,x>1,3); # piecewise-defined function
> g(1); # function is defined at x = 1
> plot( g(x), x=0..2, y=0..3, discontinuous=true # plot function on [0,2]
);
> limit( g(x), x=1 ); # two-sided limit
> limit( g(x), x=1, left ); # one-sided limit from left
> limit( g(x), x=1, right ); # one-sided limit from right
```

Example 3: $\lim_{x \rightarrow 2} |x^2 - x - 2|$

```
> h := x -> abs(x^2-x-2);           # define function
> h(2);                             # value of function at x = 2
> plot( h(x), x=-2..3 );           # plot of function
> limit( h(x), x=2 );              # agrees with limit at x = 2
>                                     # ==> h(x) continuous at x = 2
```

Example 4: $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

```
> s := x -> sin(x)/x;             # define function
> plot( s(x), x=-5*Pi..5*Pi );    # limit at x = 0 is clear
> s(0);                             # value of function at x = 0
> limit( s(x), x=0 );              # consistent with plot?
```

Note

- (1) A Maple plot is really a collection of points connected in sequence. As a result, when Maple plots a discontinuous function it normally connects the points. To force Maple to look for discontinuities and to create a reasonable plot, include the optional argument `discont=true` in the `plot` command.

Questions

- (1) Let

$$f(x) = \frac{x^3 - 7x^2 + 15x - 9}{x^4 - 6x^3 + 5x^2 + 24x - 36}.$$

For what values of x is the graph of $y = f(x)$ discontinuous? For each discontinuity, c , of f , is it possible to define $f(c)$ so that f is continuous at c ? (Explain.)

- (2) Evaluate each of the following limits:

$$\lim_{x \rightarrow 0} \frac{\sin(3x)}{x} \qquad \lim_{x \rightarrow 0} \frac{\sin(x)}{4x} \qquad \lim_{x \rightarrow 0} \frac{\sin(ax)}{\sin(bx)}.$$

- (3) Let $F(x) = \frac{\sin x}{\sqrt{1 - \cos x}}$. Evaluate

$$\lim_{x \rightarrow 0} F(x) \qquad \lim_{x \rightarrow 0} |F(x)|.$$

Use one-sided limits to explain why only one of these limits exists.

- (4) One of $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$ and $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right)$ exists and the other does not exist. Which limit exists? What is the value of this limit? What property of limits is used to evaluate this limit?
- (5) Define g to be the piecewise-defined function

$$g(x) = \begin{cases} 2 - x^2 & x < 1 \\ \sqrt{x-1} - ax + 2 & x \geq 1 \end{cases}.$$

For what value(s) of a is g continuous for all real numbers? *Hint:* Look at one-sided limits.