Accumulation Functions

Objective The purpose of this lab assignment is to reinforce the understanding of definite integrals and the Fundamental Theorem of Calculus through the use of accumulation function. This investigation will provide experience working with definite integrals that cannot be evaluated in terms of elementary functions.

Background An *accumulation function* is a function that gives the "area" under the the graph of a function $y = f(t)$ from a fixed value a to a variable value x. That is, $F(x) = \int_a^x f(t)\, dt$.

If the integrand is the speed of an object, the corresponding accumulation function is the distance traveled since a specified initial time. (Recall that speed = |velocity|.)

Observe that an accumulation function is a function — the variable is the upper limit of integration. If f is continuous on an interval containing a then the Fundamental Theorem of Calculus tells us that the accumulation function is differentiable (and hence continuous) on this interval and that the derivative is $F'(x) = f(x)$.

Discussion Enter, and execute, the following Maple commands in a Maple worksheet.

Example 1: Continuous Integrand

```
> restart;                                          # clear Maple's memory
> with( plots );                                    # load package
> f := t -> 2/sqrt(t) + sin(t) + 1;                 # integrand (positive)
> F := unapply( Int( f(t), t=1..x ), x );           # accumulation fn
> plot( [ f(t), F(t) ], t=1..10 );                  # visualize f and F
> dF := D(F);                                       # derivative of accum fn
> plot( [ f(t), F(t), dF(t) ], t=1..10 );           # visualize F'(x) = f(x)
>       style=[line,line,point] );
```

Example 2: Discontinuous Integrand

```
> f := t -> if frac(t/2)<1/2                        # discontinuous & periodic
>              then 3
>              else -1
>           end if;
> f(t);                                             # ERR: cannot evaluate boolean
> P1:=plot( 'f'(t), t=0..10, discont=true ):        # plot f (unevaluated)
> F :=unapply( int( ''f''(t), t=0..x ), x );        # accumulation fn
> P2:=plot( 'F'(x), x=0..10, color=blue ):          # plot F (unevaluated)
> display( [ P1, P2 ] );                            # visualize f and F
```

Example 3: Speed *vs.* Velocity

```
> v := t -> piecewise( t< 2, t^2,                   # velocity: continuous
>                      t<20, 4-(t-2)/2,
>                      t<35, -5+(t-20)^2/25,
>                            4 );
> s := t -> unapply( abs(v(t)), t );                # speed: cont & non-neg
> plot( ['v'(t),'s'(t)], t=0..40,                   # compare s and v
>       style=[line,point] );                       #
> V := unapply( Int( v(t), t=0..x ), x );           # accum fn for v
> S := unapply( Int( s(t), t=0..x ), x );           # accum fn for s
> plot( [V(t),S(t)], t=0..40,                       # compare s and v
```

Notes

(1) Observe the use of an `if` ... `then` ... `end if;` statement to define the integrand in Example 2. If a function like this is evaluated with a symbolic name, e.g., `x` or `t`, Maple cannot determine which value to return. In these cases Maple responds with an error message saying Maple "cannot determine if this expression is true or false: ...". The resolution to this problem is to use single quotes, '*expr*', to delay evaluation of *expr*.

(2) Each evaluation of an expression removes one level of single quotes. Thus, in the definition of `F` in Example 2, it is necessary to delay evaluation of `f` in `int` and in `unapply`, hence the two levels of single quotes.

(3) In Example 3, note that the two accumulation functions are parallel whenever the velocity is positive. The accumulation function for the speed is distance traveled function; the accumulation function for the velocity is the position function.

Questions

(1) The (natural) logarithm function is the foremost example of an accumulation function. The integrand is the reciprocal function $f(t) = \frac{1}{t}$, $t \neq 0$, and the definition of the logarithm function is

$$L(x) = \int_1^x \frac{1}{t}\, dt.$$

 (a) Find $L(1)$, $\lim_{x\to 0^+} L(x)$, and $\lim_{x\to\infty} L(x)$. Prepare a well-labeled graph of $y = f(x)$ and $y = L(x)$ that supports these findings.

 (b) Find $L'(x)$. What is the corresponding integral formula? Explain why this result does not contradict the Power Rule.

 (c) Use results from Calculus to explain why there is exactly one number c with the property that $L(c) = 1$. Your explanation should provide a reason why you know there is at least one number with this property and a separate reason why there is at most one number with this property. (Note: This is one definition of Euler's constant, e.)

(2) An example of another accumulation function that arises in numerous applications is the *sine integral* function. This function is defined to be an accumulation function for the damped sine function $f(t) = \frac{\sin t}{t}$:

$$S(x) = \int_0^x \frac{\sin t}{t}\, dt.$$

 (a) Find $S(0)$, $\lim_{x\to\infty} S(x)$, and $\lim_{x\to-\infty} S(x)$. Prepare a well-labeled graph of $y = f(x)$ and $y = L(x)$ that supports these findings.

 (b) Find the first and second derivatives of the sine integral.

 (c) Give at least one difference in the behavior of the integrands for $L(x)$ and $S(x)$ at $t = 0$.

(3) Consider the accumulation function defined by $P(x) = \int_0^x f(t)\, dt$ where the integrand is the function whose Maple implementation is

```
> f := t -> abs( 2*frac(2*t)-1)-1/2;
```

 (a) Prepare a plot of the integrand and accumulation function that shows that the accumulation function is periodic with period $\frac{1}{2}$, i.e., $P(x + \frac{1}{2}) = P(x)$.

 (b) Give an example of an integrand $g(t)$ that is periodic that does not have a periodic accumulation function $G(x) = \int_0^x g(t)\, dt$.

 (c) The periodicity of P is *not* an immediate consequence of the fact the integrand is periodic with period $\frac{1}{2}$. What property of the integrand is essential to making the accumulation function periodic?