Algebra and Logic Seminar Notes

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Compatibility of Equations with the Real Line: Variations on a Theme of Walter Taylor

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LECTURE 0

The compatibility problem for equations with the real line

Consider $\mathbb{S}^1$, the unit circle with its usual topology. It is easy to see that the equation

$$x + y \approx y + x$$

is compatible with $\mathbb{S}^1$ in the sense that there is a way to interpret the two-place operation symbol $+$ as a two-place continuous operation on circle so that the equation becomes true for all x and y in $\mathbb{S}^1$.

In general, we say a set Σ of equations is **compatible** with a topological space $\mathbb{T}$ just in case continuous operations on $\mathbb{T}$, of the appropriate ranks, can be assigned to the operation symbols occurring in Σ so that all the equations in Σ become true. In other words, we also say that there is a topological algebra on $\mathbb{T}$ which is a model of Σ.

Since $\mathbb{S}^1$ can be turned into a topological Abelian group, we see that the set of equations axiomatizing Abelian groups is compatible with $\mathbb{S}^1$. By adopting the point of view that each equation is a constraint on some unknown continuous operations, we can construe this topological Abelian group as a solution to the system of equations axiomatizing Abelian groups. This is the point of view adopted by mathematicians working in the field of functional equations. From that perspective it would be more natural to say that Σ is solvable over $\mathbb{T}$ than to say, as we will, that Σ is compatible with $\mathbb{T}$.

A **signature** is a system of operation symbols, each assigned some finite rank. Thus, we could think of a signature as a function which assigns finite ranks to operation symbols. When investigating groups one would typically use a signature supplied with one two-place operation symbol (for the product), one one-place operation symbol (for the formation of inverses), and perhaps a zero-place operation symbol (a constant symbol to name the identity element). Ring theory would use a different signature. It is customary to consider sets of equations of a particular fixed signature.

Each equation is just a finite string of symbols. So a finite set of equations is suitable as an input to an algorithm (alias: computer program). This gives rise to the following general problem.

The Equational Compatibility Problem for $\mathbb{T}$ in Signature ρ

> Let $\mathbb{T}$ be a topological space. Is there an algorithm for determining of arbitary finite sets of equations of signature ρ whether they are compatible with $\mathbb{T}$?

Solutions to the compatibility problem are known only for a few topological spaces. For the spheres $\mathbb{S}^n$ the answer is known to be affirmative for all n other than 1, 3, and 7. Walter Taylor (2000), using some deep results of algebraic topology, showed that the sets of equations compatible with these spheres had to be trivial in a certain specific sense which can be recognized algorithmically.

Taylor (2006) showed that the equational compatibility problem for the real line has a negative answer. He used the signature of rings expanded by three one-place operation symbols and a countably infinite list of constant symbols. The purpose of these lectures is follow in Taylor's footsteps to a variation on his result.

The topological algebra that places a central role in our enterprise is

$$\mathbf{R} = \langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$$

where $+, \cdot, -, 1$ and $|\ |$ are the usual operations on the real numbers (we take $-$ to be the operation of forming negatives, rather than the operation of subtraction) and where $\sin^*(x) = \sin(\frac{\pi}{2}x)$ for all real numbers x. We reserve τ to denote the signature of this algebra. The operation symbols of this signature will be

$$\mathbf{+}, \mathbf{\cdot}, \mathbf{-}, \mathbf{1}, |\ |, \text{ and } \mathbf{sin}^*$$

Here is the theorem we aim to prove:

Main Theorem. *There is no algorithm for determining of an equation of signature τ whether it is compatible with the real line.*

The chief ways that this theorem differs from Taylor's is that no additional constant symbols are needed and that even limiting our attention to single equations as opposed to finite sets of equations still resists an algorithmic solution.

Our approach has three stages.

A set Δ of equations of signature ρ **determines** the topological algebra $\mathbf{T}$ over the space $\mathbb{T}$ provided $\mathbf{T}$ is, up to maps that are isomorphisms (simultaneously in the algebraic and topological senses), the unqiue topological algebra of signature ρ over $\mathbb{T}$ which is a model of Δ.

The Finite Determination Theorem. *The topological algebra $\mathbf{R}$ is determined by a finite set of equations. This set includes the equational axioms of rings with unit.*

This theorem is a minor variation of work by Walter Taylor (2006).

Now we turn to the second stage of our approach. The **equational theory** of an algebra is the set of all equations (in the signature of the algebra) which are true in the algebra. An equational theory is said to be **undecidable** provided there is no algorithm for determining whether an arbitrary equation belongs to the theory.

The Equational Undecidability Theorem. *The equational theory of $\mathbf{R}$ is undecidable.*

This theorem is a refinement theorem proved by Daniel Richardson (1968). His reasoning relied on the negative solution to Hilbert's Tenth Problem for Exponential Diophantine Equations established by Davis, Putnam, and Robinson (1961). The completion of the resolution of Hilbert's Tenth Problem (Matiyasevich, 1970) led to stronger forms of Richardson's work.

Now let Δ be the finite set mentioned in the Finite Determination Theorem and let $s \approx t$ be any equation of signature τ. It is clear that $s \approx t$ is true in $\mathbf{R}$ if and only if $\Delta \cup \{s \approx t\}$ is compatible with the real line. Therefore any algorithm for determining compatibility with $\mathbb{R}$ would lead to an algorithm for deciding the equational theory of $\mathbf{R}$. This establishes the Main Theorem altered to deal with finite sets of equations in place of single equations.

The third stage in our approach is to apply the following theorem.

The One Equation Collapsing Theorem. *Any finite set of equations which includes the equational axioms for rings with unit is logically equivalent to a single equation. Moreover, there is an algorithm which, upon input of such a finite set of equations, will output the single equation.*

Two sets of equations are said to be **logically equivalent** if they have exactly the same models. Notice that this theorem applies to any signature which includes the signature of rings with unit. This theorem was announced by Alfred Tarski in 1966, see (Tarski, 1968). Tarski actually framed a somewhat stronger theorem than the one displayed above. Independently, Grätzer and McKenzie (1967) announced a similar result obtained by different reasoning. A proof of Tarski's result can be found in (McNulty, 2004), while a proof of the result a George Grätzer and Ralph McKenzie is included in (Grätzer and Padmanabhan, 1978).

An application of the one equation collapse to $\Delta \cup \{s \approx t\}$ completes the proof of the Main Theorem.

LECTURE 1

Finite determination of $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$

Let Δ_0 be the following set of equations (here we regard $\mathbf{0}$ as an abbreviation for $\mathbf{-1+1}$):

$$\begin{array}{cc} x + (y + z) \approx (x + y) + z & x \cdot (y \cdot z) \approx (x \cdot y) \cdot z \\ x + y \approx y + x & x \cdot y \approx y \cdot x \\ x + \mathbf{0} \approx x & x \cdot \mathbf{1} \approx x \\ -x + x \approx \mathbf{0} & x \cdot (y + z) \approx x \cdot y + x \cdot z \end{array}$$

The set Δ_0 is just the familiar axioms for commutative rings with unit.

Let Δ_1 be the set consisting of:

$$|x^2| \approx x^2 \qquad |-x^2| \approx x^2$$

and let Δ_2 be the set consisting of:

$$\mathbf{sin}^*(x + y) - \mathbf{sin}^*(x - y) \approx 2\,\mathbf{sin}^*\,y\,\mathbf{sin}^*(x + \mathbf{1}) \qquad \mathbf{sin}^*\,\mathbf{1} \approx \mathbf{1}$$
$$\mathbf{sin}^*(\mathbf{sin}^*(x) + \mathbf{1}) \approx |\,\mathbf{sin}^*(\mathbf{sin}^*(x) + \mathbf{1})|$$

THEOREM 0. Δ_0 *determines* $\langle \mathbb{R}, +, \cdot, -, 1 \rangle$.

PROOF. Let $\mathbf{R}^\circ = \langle \mathbb{R}, \hat{+}, \hat{\cdot}, \hat{-}, \hat{1} \rangle$ be a commutative ring with unit on $\mathbb{R}$ such that all the fundamental operations are contiuous. We have to show that this algebra is isomorphic to the reals with the standard operations by a map that is also a homeomorphism.

In the typical axiomatic presentation, tracing back to Eudoxus and Euclid in ancient times and Weierstrass, Dedekind, and Cantor in the 19th century, the real line is seen as a complete ordered field, this field being unique up to isomorphism which preserves the order relation as well as the field operations. See the text (Hewitt and Stromberg, 1965) for an exposition of this approach. Since the topology on the real line is determined by the order relation, such isomorphisms are also homeomorphisms. We will take advantage of this standard approach, but it falls short in the present situation for three reasons:

- In a field 0 and 1 must be distinct and any nonzero element must have a multiplicative inverse. These facts cannot be expressed with equations.
- Most of the salient facts about the order relation (or the set of "positive" numbers) are not equational either, although they can be expressed by elementary sentences.
- The completeness principle (i.e. the Least Upper Bound Axiom) is not expressible by even a set of elementary sentences.

To remedy these shortfalls, we mostly appeal to the continuity of the basic operations of $\mathbf{R}^\circ$.

To simplify notation put $\hat{0} = \hat{-}\hat{1}\hat{+}\hat{1}$. In the presence of the axioms of commutative rings, we see that $\hat{0} \neq \hat{1}$ since $\mathbb{R}$ has more than one element. This takes care of part of the first shortfall.

Here $<$ will always stand for the usual ordering of $\mathbb{R}$. So either $\hat{0} < \hat{1}$ or $\hat{1} < \hat{0}$. We will assume $\hat{0} < \hat{1}$ and later see that this assumption is harmless. Our ambition is to prove that $\langle \mathbb{R}, \hat{+}, \hat{\cdot}, \hat{-}, \hat{1}, < \rangle$ is a complete ordered field. Since we already know that $<$ is a complete ordering of $\mathbb{R}$, it remains to verify the following:

(1) Every element of $\mathbb{R}$ different from $\hat{0}$ has a $\hat{\cdot}$-multiplicative inverse. (So $\mathbf{R}^\circ$ is a field).
(2) If $a > \hat{0}$, then $\hat{0} > \hat{-}a$.
(3) If $a > \hat{0}$ and $b > \hat{0}$, then $a \hat{+} b > \hat{0}$.
(4) If $a > \hat{0}$ and $b > \hat{0}$, then $a \hat{\cdot} b > \hat{0}$.

It is (1) above which presents the most difficulty.

Claim 0. *If $a > c$, then $a \hat{+} b > c \hat{+} b$.*

Proof. Let $\sigma_b(x) = x \hat{+} b$ for all real x. We suppose $b \neq \hat{0}$, as that case is too easy. The function σ_b is continuous and $\sigma_{\hat{-}b}$ is its continuous inverse (using the properties of commutative rings). So σ_b is either strictly increasing or strictly decreasing. Every continuous strictly decreasing function on $\mathbb{R}$ must have a fixed point. σ_b has a fixed point if and only if $b = \hat{0}$, according to commutative ring theory. So σ_b is strictly increasing and the claim is established. □

Now to establish (3) above just note

$$a > \hat{0} \text{ and } b > \hat{0} \Rightarrow a \hat{+} b > \hat{0} \hat{+} b > \hat{0} \hat{+} \hat{0} = \hat{0}$$

To establish (2) above notice

$$\begin{aligned} a > \hat{0} &\Rightarrow a \hat{+} (\hat{-}a) > \hat{0} \hat{+} (\hat{-}a) \\ &\Rightarrow \hat{0} > \hat{-}a \end{aligned}$$

For each natural number n we define $\hat{n}$ by the following recursive condition

$$\widehat{n+1} = \hat{n} \hat{+} \hat{1} \text{ for all natural numbers } n$$

where the base of the recursion, namely $\hat{0}$, as well as $\hat{1}$, are already in hand. It is easy to establish by induction that $\hat{2} \hat{\cdot} \hat{n} = \widehat{2n}$.

Claim 1. *If $b > \hat{0}$, then $\langle \hat{n} \hat{\cdot} b \mid n$ is a natural number$\rangle$ is a strictly increasing sequence with no upper bound.*

Proof. Let $b_n = \hat{n} \hat{\cdot} b$. Since $b > \hat{0}$ and $b_{n+1} = \widehat{n+1} \hat{\cdot} b = (\hat{n} \hat{+} \hat{1}) \hat{\cdot} b = \hat{n} \hat{\cdot} b \hat{+} \hat{1} \hat{\cdot} b = \hat{n} \hat{\cdot} b \hat{+} b = b_n \hat{+} b$, we see that $b_{n+1} = b_n \hat{+} b > b_n$. So the sequence is strictly increasing by Claim 0. Suppose it had an upper bound. Then it would have a least upper bound L and $\lim_{n\to\infty} b_n = L$. Now the subsequence of terms with even indices must also have the same limit $L = \lim_{n\to\infty} b_{2n}$. Plainly $\hat{0} < b < L$. But now consider, with the help of the continuity of $\hat{\cdot}$:

$$\begin{aligned}
\hat{2}\hat{\cdot}L &= \hat{2}\hat{\cdot}\lim_{n\to\infty} b_n \\
&= \lim_{n\to\infty} (\hat{2}\hat{\cdot}b_n) \\
&= \lim_{n\to\infty} (\hat{2}\hat{\cdot}\hat{n}\hat{\cdot}b) \\
&= \lim_{n\to\infty} (\widehat{2n}\hat{\cdot}b) \\
&= \lim_{n\to\infty} b_{2n} \\
&= L.
\end{aligned}$$

This means that $L\hat{+}L = \hat{2}\hat{\cdot}L = L$. It follows from the axioms of rings that $L = \hat{0}$, a contradiction. Thus the sequence $\langle b_n \mid n \text{ is a natural number}\rangle$ is a strictly increasing unbounded sequence. □

Claim 2. *The ring $\mathbf{R}^\circ$ is a field.*

Proof. It is enough to show that if $b > \hat{0}$ then b is invertible. Now the map $\mu_b : \mathbb{R} \to \mathbb{R}$ defined so that $\mu_b(x) = x\hat{\cdot}b$ for all real x is a continuous function. Let n be large enough so that $b_n > \hat{1}$. Then $\mu_b(\hat{n}) = b_n > \hat{1} > \hat{0} = \mu_b(\hat{0})$. By the Intermediate Value Theorem, this means there is $c \in \mathbb{R}$ so that $\hat{1} = \mu_b(c) = c\hat{\cdot}b$. So b is invertible as desired. □

Claim 3. *If $a > \hat{0}$ and $b > \hat{0}$, then $a\hat{\cdot}b > \hat{0}$.*

Proof. Since μ_b is continuous the image of the interval $(\hat{0}, \infty)$ must be an interval as well. Because $\mathbf{R}^\circ$ is a field, it is an integral domain and so $\hat{0}$ is not in the image. On the other hand $\mu_b(\hat{1}) = b > \hat{0}$. But then the image of $(\hat{0}, \infty)$ must be included in $(\hat{0}, \infty)$, and the claim is established. □

There were two cases at the beginning of the proof of the theorem, namely $\hat{0} < \hat{1}$ and $\hat{1} < \hat{0}$. The argument above demonstrates the theorem in the first case. The argument in the other case is entirely similar. In items (2) through (4) one should reverse the inequalities. One could also note that $\mu_{\widehat{-1}}$ is a bicontinuous automorphism of $\mathbf{R}^\circ$. □

Theorem 1. *$\Delta_0 \cup \Delta_1$ determines $\langle \mathbb{R}, +, \cdot, -, 1, |\ |\rangle$.*

This theorem is too easy to prove.

The Finite Determination Theorem. *$\Delta_0 \cup \Delta_1 \cup \Delta_2$ determines $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^*\rangle$.*

Proof. Let f be a continuous function on $\mathbb{R}$ so that

$$(\star) \qquad \begin{aligned}
f(x+y) - f(x-y) &= 2f(y)f(x+1) \\
f(f(x)+1) &= |f(f(x)+1)| \\
f(1) &= 1
\end{aligned}$$

We have to prove that $f(x) = \sin(\frac{\pi}{2}x)$.

First, we note that

$$f(0) = 0 \text{ and } f(x+2) = -f(x).$$

The equation on the left can be deduced that substituting 0 for both x and y in ($\star$), while the equation on the right follows from($\star$) by substituting 1 for y and $x+1$ for x.

Now in ($\star$) substitute $\frac{h}{2}$ for y and kh for x, and reason as follows:

$$\begin{aligned}
2f\left(\frac{h}{2}\right) f(kh+1) &= f\left(kh+\frac{h}{2}\right) - f\left(kh-\frac{h}{2}\right) \\
f(kh+1) &= \frac{1}{2}\left[f\left(\frac{(2k+1)h}{2}\right) - f\left(\frac{(2k-1)h}{2}\right)\right]\frac{1}{f(\frac{h}{2})} \\
f(kh+1)h &= \left[f\left(\frac{(2k+1)h}{2}\right) - f\left(\frac{(2k-1)h}{2}\right)\right]\frac{\frac{h}{2}}{f\left(\frac{h}{2}\right)} \\
\sum_{k=1}^{n} f(kh+1)h &= \left[\sum_{k=1}^{n}\left[f\left(\frac{(2k+1)h}{2}\right) - f\left(\frac{(2k-1)h}{2}\right)\right]\right]\frac{\frac{h}{2}}{f(\frac{h}{2})} \\
\sum_{k=1}^{n} f(kh+1)h &= \left[f\left(\frac{(2n+1)h}{2}\right) - f\left(\frac{h}{2}\right)\right]\frac{\frac{h}{2}}{f(\frac{h}{2})}
\end{aligned}$$

Now in the last equation put $h = \frac{x}{n}$. This gives

$$\sum_{k=1}^{n} f\left(\frac{kx}{n}+1\right)\frac{x}{n} = \left[f\left(x+\frac{x}{2n}\right) - f\left(\frac{x}{2n}\right)\right]\frac{\frac{x}{2n}}{f(0+\frac{x}{2n}) - f(0)}$$

The sum on the left side is a Riemann sum associated with the integral $\int_0^x f(t+1)\,dt$. Since f is continuous this integral must exist. So by letting $n \to \infty$ in the last equation we see that

$$\int_0^x f(t+1)\,dt = \frac{f(x)}{f'(0)}.$$

According to the Fundamental Theorem of Calculus the left side is differentiable. So the right side must be as well and we find

$$f'(0)f(x+1) = f'(x).$$

In turn, this gives

$$f'(0)f'(x+1) = f''(x) \text{ as well as } f'(0)f(x+2) = f'(x+1)$$

Combining these two equations with $f(x+2) = -f(x)$ we find the elementary differential equation

$$f''(x) + (f'(0))^2 f(x) = 0.$$

All the solutions to this equation have the form

$$f(x) = A\sin(f'(0)x) + B\cos(f'(0)x).$$

Since $f(0) = 0$ we see that $B = 0$. It follows that $f'(x) = Af'(0)\cos(f'(0)x)$. This implies that $f'(0) = Af'(0)$. But $f(1) = 1$ now entails that $f'(0) \neq 0$. Therefore $A = 1$. This means

$$f(x) = \sin(f'(0)x).$$

It remains to show that $f'(0) = \frac{\pi}{2}$. Since $f(1) = 1$ we have that $\sin(f'(0)) = 1$. This means that $f'(0) = \frac{\pi}{2} + 2m\pi$ for some integer m. We will argue that $|f'(0)| \leq \frac{\pi}{2}$, which suffices.

Now we know that $f'(0)f(x+1) = f'(x) = f'(0)\cos(f'(0)x)$ and that $f'(0) \neq 0$. Thus $f(x+1) = \cos(f'(0)x)$. Now that last equation in Δ_2 asserts that $f(f(x)+1)$ is not negative. This means

$$\cos(f'(0)\sin(f'(0)x)) \geq 0.$$

Since $\sin(f'(0)x)$ takes on all values between -1 and 1, we see that $\cos y \geq 0$ whenever $|y| \leq |f'(0)|$. It follows that $|f'(0)| \leq \frac{\pi}{2}$, as desired. □

The Provenance of These Results. The theorems in this lecture belong to the theory of functional equations, an old and widely applicable branch of mathematics. *Æquationes Mathematicæ* is a journal devoted to this branch. The text of J. Aczél (1966) gives an historical perspective on this field. Given the extent of the literature about functional equations, it is difficult to know the origins of the results given in this lecture, dealing as they do with such familiar functions. My own very cursory glance through the literature of functional equations gives the impression that addition, subtraction, multiplication, and division of real numbers are usually assumed as given. Thus Theorem 0 above likely originated with Walter Taylor (2006). On the other hand, the investigation of systems of equations characterizing the sine and cosine functions were undertaken by d'Alembert is the middle of the 18th century (d'Alembert, 1747a; 1747b; 1750; 1769), and can be found in Cauchy's 1821 text *Cours d'analyse de l'École Polytechnique*, see (Cauchy, 1992). The first two equations in Δ_2 are variants of familiar equations and the details of proof of the Finite Determination Theorem that use Riemann sums, telescoping series, and appeal to the integrability of continuous function, the Fundamental Theorem of Calculus, and the solutions to differential equations were suggested by reading a paper of Vaughan (1955). The last equation in Δ_2 and its use in the proof of the Finite Determination Theorem to demonstrate $f'(0) = \frac{\pi}{2}$ are again ideas of Walter Taylor.

LECTURE 2

The Equational Theory of $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$ is Undecidable

Our aim is to prove that there is no algorithm for determining of an arbitrary equation whether it is true in $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$. Our argument will show that any algorithm for deciding this equational theory will allow us to devise another algorithm for solving Hilbert's Tenth Problem. So some words about Hilbert's Tenth Problem are in order.

A **Diophantine equation** is simply an equation where each side is a polynomial in some finite number of variables and whose coefficients are integers. The equation $2x^2y + 2xy^2 \approx 4xyz^5$ is a Diophantine equation. Hilbert asked for a method which would determine which Diophantine equations have solutions in the integers.

A set A of integers is called **Diophantine** provided there are polynomials $p(x, y_0, \ldots, y_{n-1})$ and $q(x, y_0, \ldots, y_{n-1})$ with integer coefficients such that

$$A = \{k \mid p(k, y_0, \ldots, y_{n-1} \approx q(k, y_0, \ldots, y_{n-1}) \text{ has a solution in the integers}\}.$$

This notion can be extended to give meaning to an m-ary relation or a k-ary function on the integers being Diophantine.

It was conjectured by Davis (1950) that the Diophantine sets were exactly the sets for which there are algorithms for listing their elements (in no particular order and with repetitions allowed in the list). While this seemed very doubtful at the time, Davis, Putnam, and Robinson (1961) were able to show that it was true, provided the operations allowed in building the "polynomials" p and q included the exponential function 2^x. Not only did they prove that the exponential Diophantine sets were exactly the listable sets, but they also proved that if any function that exhibits roughly exponential growth is Diophantine in the ordinary sense then the Diophantine sets and the listable sets coincide. Matiyasevich (1970) demonstrated that a function with exponential growth related to the Fibonacci sequence was in fact Diophantine. In this way Davis's Conjecture was verified. Since it had been established by Turing (1936) that there are listable sets which are not decidable, Hilbert's Tenth Problem is seen to have negative solution in a strong sense. Let H be a set of integers whose characteristic function is not algorithmic (that is membership in H cannot be determined algorithmically) but which is nevertheless listable. There is a particular natural number n and specific polynomials $p(x, y_0, \ldots, y_{n-1})$ and $q(x, y_0 \ldots, y_{n-1}$ such that

$$H = \{k \mid p(k, y_0, \ldots, y_{n-1}) \approx q(k, y_0, \ldots, y_{n-1}) \text{ has a solution in the integers}\}.$$

The smallest known value for n is 9. This was discovered by Matiyasevich. A detailed exposition can be found in (Jones, 1982). Since p and q are fixed, one can also bound the degrees. In particular, this means that there is no algorithm for determining of an arbitrary Diophantine equation in no more than 9 variables, whether it is solvable in the integers.

Expand the signature τ by adding 9 constant symbols, $\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_8$. Call this expanded signature τ'. For arbitrary polynomials $s(y_0, \ldots, y_8)$ and $t(y_0, \ldots, y_8)$ let

$$\Delta_{s,t} = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \{\mathbf{sin}^*(\mathbf{2c}_i) \approx \mathbf{0} \mid i < 9\} \cup \{s(\mathbf{c}_0, \ldots, \mathbf{c}_8) \approx t(\mathbf{c}_0, \ldots, \mathbf{c}_8)\}.$$

Since $\Delta_0 \cup \Delta_1 \cup \Delta_2$ determines $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$ it is evident that $\Delta_{s,t}$ is compatible with the real line if and only if $s(y_0, \ldots, y_8) \approx t(y_0, \ldots, y_8)$ has a solution in the integers. This gives us Taylor's Theorem, see (Taylor, 2006):

Taylor's Compatibility Theorem. *There is no algorithm for determining whether finite sets of equations are compatible with the real line.*

Indeed, we see a sharper result. Namely that the compatibility problem for the real line in the signature τ' has a negative solution. Our intention in the rest of this lecture is to see that we can get rid of those nine constant symbols, and along the way prove that the equational theory of $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$ is undecidable. That last result is interesting in its own right.

The idea for how to proceed was put forward by Richardson (1968). At that time, the solution to Hilbert's Tenth Problem was not known, so Richardson based his work on the earlier result of Davis, Putnam, and Robinson concerning the solution of exponential Diophantine equations. Just a few years after Richardson's paper appeared, Matiyasevich (1970) completed the work begun by Davis, Putnam, and Robinson to solve Hilbert's Tenth Problem. We can take advantage of that.

Given ten polynomials $s = s(x_0, \ldots, x_8), k_0(x_0, \ldots, x_8), \ldots, k_8(x_0, \ldots, x_8)$ with integer coefficient and no other variables apart from the nine listed we associate a term $F_s(x_0, \ldots, x_8)$ defined as follows:

$$F_s(x_0, \ldots, x_8) = 10^2 \left[s^2(x_0, \ldots, x_8) + \sum_{i<9} (\mathbf{sin}^*\, 2x_i)^2 k_i^2(x_0, \ldots, x_8) \right] - \mathbf{1}.$$

The Richardson Inequality Lemma. *There is an algorithm which upon input of any polynomial $s(x_0, \ldots, x_8)$ with integer coefficients, will produce polynomials $k_0(x_0, \ldots, x_8), \ldots, k_8(x_0, \ldots, x_8)$, also with integer coefficients, such that these polynomials, when evaluated at any 9-tuple of reals, always exceed 1, and such that the following statements are equivalent:*

(i) *there is a solution of $s(x_0, \ldots, x_8) \approx \mathbf{0}$ in the integers.*

(ii) *there is a 9-tuple of real numbers to which the term function denoted by $F_s(x_0, \ldots, x_8)$ assigns a negative value.*

Proof. Here is how to construct the polynomials k_i. Let $f_i = \frac{\partial s^2}{\partial x_i}$. So $f_i(x_0, \ldots, x_8)$ is again a polynomial. We obtain $k_i(x_0, \ldots, x_8)$ from $f_i(x_0, \ldots, x_8)$ replacing each coefficient c by $|c|+2$ and by replacing each occurrence of x_i by $x_i^2 + 2$. Indeed, let $f(x_0, \ldots, x_8)$ be any polynomial. We take f° to be the polynomial obtained from f by the replacements just described. So if f were $3x_0x_2^3 - 4x_1$ then the corresponding f° would be the result of simplifying $(3+2)(x_0^2+2)(x_2^2+2)^3 + (4+2)(x_1^2+2)$. Here is the claim that motivates this choice of k_i as f_i°.

Claim 4. *For any reals $a_0, a_1, \ldots, a_8$ and any numbers δ_i (for $i < 9$) so that $|\delta_i| < 1$ we have that $f^\circ(a_0, \ldots, a_8) > f(a_0 + \delta_0, \ldots, a_8 + \delta_8)$ and $f^\circ(a_0, \ldots, a_8) > 1$ for any polynomial f.*

Proof. We prove this induction on the complexity of the polynomials f.

The base step of the induction has two cases: f is a constant c and f is the variable x_j. In the first case we note that $c < |c| + 2$ and in the second case we see that $a_j + \delta_j \leq |a_j + \delta_i| \leq |a_j| + |\delta_j| < |a_j| + 1 \leq a_j^2 + 2$ and the base step of the induction is secure.

For the inductive step there are also two cases to consider, namely $f(x_0, \ldots, x_8) = g(x_0, \ldots, x_8) + h(x_0, \ldots, x_8)$ and $f(x_0, \ldots, x_8) = g(x_0, \ldots, x_8)h(x_0, \ldots, x_8)$, where the inductive hypothesis tells us that

$$g(a_0 + \delta_0, \ldots, a_8 + \delta_8) < g^\circ(a_0, \ldots, a_8) > 1$$
$$h(a_0 + \delta_0, \ldots, a_8 + \delta_8) < h^\circ(a_0, \ldots, a_8) > 1$$

Since

$$(g(x_0, \ldots, x_8) + h(x_0, \ldots, x_8))^\circ = g^\circ(x_0, \ldots, x_8) + h^\circ(x_0, \ldots, x_8)$$

and

$$(g(x_0, \ldots, x_8)h(x_0, \ldots, x_8))^\circ = g^\circ(x_0, \ldots, x_8)h^\circ(x_0, \ldots, x_8),$$

the desired conclusions follow. □

Now let us turn to the proof of Richardson's Inequality Lemma. It is easy to see that (i) implies (ii). For the converse, suppose the real numbers $a_0, \ldots, a_8$ have the property that $F_s(a_0, \ldots, a_8) < 0$. We use $\lfloor a \rceil$ to denote the integer closest to the real number a, chosing the larger integer in case a falls halfway between two integers. We note that because of the continuity of F_s we can suppose without loss of generality that no a_i falls halfway between two integers.

We will demonstrate that $s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) < 1$. Since $s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil)$ must be a nonegative integer, it will follow that $s(x_0, \ldots, x_8) \approx \mathbf{0}$ is solvable in the integers. This will verify that (ii) implies (i).

We know that $F_s(a_0, \ldots, a_8) < 0$. This entails that

$$s^2(a_0, \ldots, a_8) + \sum_{i<9} (\sin^* 2a_i)^2 k_i^2(a_0, \ldots, a_8) < \frac{1}{10^2}.$$

Hence $s^2(a_0, \ldots, a_8) < \frac{1}{10}$ and $|\sin^* a_i| k_i(a_0, \ldots, a_8) < \frac{1}{10}$ for all $i < 9$.

Now according to the Mean Value Theorem in Several Variables there is a point $(c_0, \ldots, c_8)$ on the line segment joining $(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil)$ and $(a_0, \ldots, a_8)$ so that

$$s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) - s^2(a_0, \ldots, a_8) = \nabla s^2(c_0, \ldots, c_8) \cdot ((\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) - (a_0, \ldots, a_8))$$
$$s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) - s^2(a_0, \ldots, a_8) = \sum_{i<9} \frac{\partial s^2(x_0, \ldots, x_8)}{\partial x_i} \Big|_{(c_0, \ldots, c_8)} (\lfloor a_i \rceil - a_i)$$

which gives, by the claim and the Triangle Inequality,

$$s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) < s^2(a_0, \ldots, a_8) + \sum_{i<9} k_i(a_0, \ldots, a_8) |\lfloor a_i \rceil - a_i|$$

The graph of $y = |\lfloor x \rceil - x|$ is periodic with period 1 and it looks like a sawtooth with the bottom points on the X-axis at the integers and the top points with Y-coordinate 1 and X-coordinates halfway between successive integers. The graph of $y = |\sin^* 2x|$ is also periodic with period 1 and it looks like a row of 'sine" humps with the bottom points on the X-axis at integer coordinates and high points with Y-coordinate 1 and X-coordinates a values halfway between successive integers. Evidently, $|\lfloor x \rceil - x| \leq |\sin^* 2x|$ of all x. So we arrive at

$$s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) < s^2(a_0, \ldots, a_8) + \sum_{i<9} |\sin^* 2a_i| k_i(a_0, \ldots, a_8).$$

There are 10 nonnegative terms on the right side of this inequality and we know each is bounded above by $\frac{1}{10}$. Finally, this yields $s^2(\lfloor a_0 \rceil, \ldots, \lfloor a_8 \rceil) < 1$ just as we desire.

□

Richardson (1968) framed this Inequality Lemma in a slightly different way. In the first place his version applied to a wider class of functions, where ours just concerns polynomials (in 9 variables). On the other hand Richardson needed k_i^4 and 10^4 were we used k_i^2 and 10^2, these were refinements in Richardson's method that where advanced by Caviness (1970).

The Equational Undecidability Theorem. *The equational theory of* $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$ *is undecidable.*

Proof. Recall that we use $\mathbf{R}$ to denote $\langle \mathbb{R}, +, \cdot, -, 1, |\ |, \sin^* \rangle$.

Let s be any polynomial with integer coefficients and variables among $x_0, \ldots, x_8$. Observe that the equation $|F_s| \approx F_s$ is true in $\mathbf{R}$ if and only if $F_s < \mathbf{0}$ has no solutions in $\mathbf{R}$ if and only if $s \approx \mathbf{0}$ has no solutions in the integers, according to Richardson's Inequality Lemma. By the negative solution to Hilbert's Tenth Problem, we know there is no algorithm to settle whether $s \approx \mathbf{0}$ is solvable in the integers. So there can be no algorithm for determining which equations are true in $\mathbf{R}$. □

Actually, we have here a proof of something a bit stronger: there is no algorithm for determining which equations in only 9 variables are true in $\mathbf{R}$, that is the 9-variable equational theory of $\mathbf{R}$ is undecidable. Further methods suggested by Richardson lead to the even stronger conclusion that the 1-variable equational theory of $\mathbf{R}$ is undecidable. It should be said that Richardson did not actually obtain this result, because he based his construction not on the negative solution to Hilbert's Tenth Problem (unknown at the time), but on the forerunning result of Davis, Putnam, and Robinson. After the resolution of Hilbert's Tenth Problem, Wang (1974) provided a reworking of the method of Richardson, taking into account the simplifications suggested in (Caviness, 1970). Richardson (and after him Caviness and Wang) considered the usual sine function, not $\sin(\frac{\pi}{2}x)$ as we have done. Instead, Richard provided a constant symbol to denote π. Matiyasevich (1993) not only provides a detailed exposition of this reasoning of Richardson, Caviness, and Wang, but he also devised a way to avoid a constant to denote π. The adjustments to this line of reasoning to apply to $\sin(\frac{\pi}{2}x)$ instead of the sine function are provided in (McNulty, 2008).

One might wonder if the same result holds in the absence of the trigonometric function. The answer is no. It has been known since 1930 that not only the equational theory, but also the far richer elementary theory of $\langle \mathbb{R}, +, \cdot, -, 1, |\ | \rangle$ is decidable, see Tarski (1951).

Corollary 2. *The equational compatiblity problem for* $\mathbb{R}$ *in signature* τ *has a negative solution.*

LECTURE 3

Collapsing to One Equation

In this lecture we will prove the following theorem.

The One Equation Collapsing Theorem. *Any finite set of equations which includes the equational axioms for rings with unit is logically equivalent to a single equation. Moreover, there is an algorithm which, upon input of such a finite set of equations, will out put the single equation.*

Somewhat stronger versions of this result were announced independently in 1966 by Alfred Tarski (1968) and in 1967 by George Grätzer and Ralph McKenzie (1967). The respective proofs can be found in (McNulty, 2004) and in (Grätzer and Padmanabhan, 1978). The proof below follows Tarski's approach.

The establishment of the One Equation Collapsing Theorem completes the three stages in the proof of our Main Theorem, as described in Lecture 0.

It proves convenient to have a symbol for subtraction. We use $\dot{-}$. Formally, we understand $s \dot{-} t$ to be an abbreviation for $s + (-t)$. The equation below, which we denote by $\boldsymbol{\varepsilon}$, plays a key role.

$$(\boldsymbol{\varepsilon}) \qquad y \approx [(z \dot{-} z) \dot{-} (x \dot{-} y)] \dot{-} [(w \dot{-} w) \dot{-} x]$$

The equation $\boldsymbol{\varepsilon}$, easily seen to be a consequence of the axioms for rings, has itself many useful consequences.

The Cancellation Lemma. *Let $p, s,$ and t be any terms. The following cancellation laws hold:*

(a) $\boldsymbol{\varepsilon}, p \dot{-} s \approx p \dot{-} t \vdash s \approx t$.

(b) $\boldsymbol{\varepsilon}, s \dot{-} p \approx t \dot{-} p \vdash s \approx t$.

Proof. Here is the reasoning to establish (a.):

$$\begin{aligned} s &\approx [(z \dot{-} z) \dot{-} (p \dot{-} s)] \dot{-} [(w \dot{-} w) \dot{-} p] && \text{a substitution instance of } \boldsymbol{\varepsilon} \\ &\approx [(z \dot{-} z) \dot{-} (p \dot{-} t)] \dot{-} [(w \dot{-} w) \dot{-} p] && \text{since } p \dot{-} s \approx p \dot{-} t \\ &\approx t && \text{a substitution instance of } \boldsymbol{\varepsilon} \end{aligned}$$

To establish (b.) observe that the next two equations are substitution instances of $\boldsymbol{\varepsilon}$.

$$[(z \dot{-} z) \dot{-} (s \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} s] \approx p$$

$$p \approx [(z \dot{-} z) \dot{-} (t \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} t]$$

Consequently

$$[(z \dot{-} z) \dot{-} (s \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} s] \approx [(z \dot{-} z) \dot{-} (t \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} t].$$

But in view of $s \dot{-} p \approx t \dot{-} p$ we obtain

$$[(z \dot{-} z) \dot{-} (t \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} s] \approx [(z \dot{-} z) \dot{-} (t \dot{-} p)] \dot{-} [(w \dot{-} w) \dot{-} t].$$

Now two applications of the cancellation law (a.) give first

$$(w \dot{-} w) \dot{-} s \approx (w \dot{-} w) \dot{-} t$$

and then the desired result

$$s \approx t$$

□

The following equations are logical consequences of the equation $\boldsymbol{\varepsilon}$, but the proofs are left as challenges for the seminar.

$$(z \dot{-} z) \dot{-} [(z \dot{-} z) \dot{-} x] \approx x \qquad y \dot{-} y \approx z \dot{-} z \qquad x \dot{-} (z \dot{-} z) \approx x$$

We also note that the sets $\{\boldsymbol{\varepsilon}, s \approx t\}$ and $\{\boldsymbol{\varepsilon}, s \dot{-} t \approx z \dot{-} z\}$ are logically equivalent. We also leave this as a challenge to the seminar. In tackling these challenges it is important to use only the given equations, not other familiar properties of commutative rings with unit.

Now for any terms s and t we will let $\delta_{s,t}$ stand for the following equation

$$y \approx [(z \dot{-} z) \dot{-} (x \dot{-} y)] \dot{-} [(s \dot{-} t) \dot{-} x].$$

LEMMA 0. *Let s and t be any terms. The equation $\delta_{s,t}$ is logically equivalent to the set $\{\boldsymbol{\varepsilon}, s \approx t\}$ of equations.*

PROOF. We assume, without loss of generality, that the variables x, y, z, and w do not occur in $s \approx t$. Evidently, $\varepsilon,\ s \approx t \vdash \delta_{s,t}$, so it remains to establish $\delta_{s,t} \vdash \varepsilon$ and $\delta_{s,t} \vdash s \approx t$. These derivations are accomplished at once by an argument like the proofs left as challenges. We need three substitution instances of $\delta_{s,t}$ (rather than of ε). The first two are

$$z \approx [[(z \dot{-} z) \dot{-} (z \dot{-} z)] \dot{-} [(s \dot{-} t) \dot{-} z]] \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)]$$
$$z \approx [(z \dot{-} z) \dot{-} (z \dot{-} z)] \dot{-} [(s \dot{-} t) \dot{-} z]$$

The right side of the second equation occurs in the first equation, giving

$$z \approx z \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)]$$

Substitute $z \dot{-} z$ for z to obtain

$$(*) \qquad z \dot{-} z \approx (z \dot{-} z) \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)].$$

The third substitution instance of $\delta_{s,t}$ is

$$s \dot{-} t \approx [(z \dot{-} z) \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)]] \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)]$$

Applying $(*)$ twice to this equation we obtain first

$$s \dot{-} t \approx (z \dot{-} z) \dot{-} [(s \dot{-} t) \dot{-} (s \dot{-} t)]$$

and then

$$s \dot{-} t \approx z \dot{-} z.$$

But z does not occur in $s \dot{-} t$, so substituting w for z gives

$$s \dot{-} t \approx w \dot{-} w.$$

Now replace $s \dot{-} t$ in $\delta_{s,t}$ by $w \dot{-} w$ to obtain $\boldsymbol{\varepsilon}$. With $\boldsymbol{\varepsilon}$ in hand, the last equation displayed above yields $s \approx t$ by one of the challenges. $\square$

We are now prepared to prove the One Equation Collapsing Theorem.

Proof. Let $\{p_0 \approx q_0, \dots, p_{m-1} \approx q_{m-1}\}$ be a finite set of equations which includes the equational axioms for rings with unit. We take $z_0, \dots, z_{m-1}$ and w be distinct variables that do not occur is any of these equations. Let

$$\Sigma = \{p_0 \cdot z_0 \approx q_0 \cdot z_0, \dots, p_{m-1} \cdot z_{m-1} \approx q_{m-1} \cdot z_{m-1}\} \cup \{x \cdot 1 \approx x, \boldsymbol{\varepsilon}\}$$

It is clear that Σ is logically equivalent to our original set of equations.

Here is another equation, which we will call γ and which is a consequence of the axioms for rings with unit.

$$(\gamma) \qquad (x \cdot 1) \dot{-} x \approx [y \cdot (w \dot{-} w)] \dot{-} [z \cdot (w \dot{-} w)]$$

We use p to denote $(x \cdot 1) \dot{-} x$ and q to denote $[y \cdot (w \dot{-} w)] \dot{-} [z \cdot (w \dot{-} w)]$. So γ is the equation $p \approx q$.

By substituting x for y, z, and w in γ we obtain

$$(x \cdot 1) \dot{-} x \approx [x \cdot (x \dot{-} x)] \dot{-} [x \cdot (x \dot{-} x)].$$

So we find that $x \cdot 1 \approx x$ follows from $\{\gamma, \boldsymbol{\varepsilon}\}$ with the help of the challenges made earlier. Another consequence is $y \cdot (w \dot{-} w) \approx z \cdot (w \dot{-} w)$.

It is convenient to let u_i denote $p_i \cdot z_i$ and r_i denote $q_i \cdot z_i$ for all $i < m$. Further, we use u_i^* to denote $p_i \cdot (w \dot{-} w)$ and r_i^* to denote $q_i \cdot (w \dot{-} w)$ for all $i < m$. Observe that $u_i^* \approx r_i^*$ is a logical cnsequence of $\{\gamma, \boldsymbol{\varepsilon}\}$ for all $i < m$.

Now let

$$s \text{ be the term } p \dot{-} \big([u_0 \dot{-} (\dots \dot{-} (u_{m-2} \dot{-} u_{m-1}) \dots)] \dot{-} [u_0^* \dot{-} (\dots \dot{-} (u_{m-2}^* \dot{-} u_{m-1}^*) \dots)]\big)$$

and

$$t \text{ be the term } q \dot{-} \big([r_0 \dot{-} (\dots \dot{-} (r_{m-2} \dot{-} r_{m-1}) \dots)] \dot{-} [r_0^* \dot{-} (\dots \dot{-} (r_{m-2}^* \dot{-} r_{m-1}^*) \dots)]\big).$$

The single equation that we have been seeking is $\delta_{s,t}$. By the Lemma, this equation is logically equivalent to $\{\boldsymbol{\varepsilon}, s \approx t\}$. So we see that at least $\delta_{s,t}$ is a consequence of Σ. It remains for us to show that $u_i \approx r_i$ for each $i < m$.

The equation $s \approx t$ written out in detail is

$$\begin{aligned} p \dot{-} \big([u_0 \dot{-} (\cdots \dot{-} (u_{m-2} \dot{-} u_{m-1}) \cdots)] \dot{-} [u_0^* \dot{-} (\cdots \dot{-} (u_{m-2}^* \dot{-} u_{m-1}^*) \cdots)]\big) \approx \\ q \dot{-} \big([r_0 \dot{-} (\cdots \dot{-} (r_{m-2} \dot{-} r_{m-1}) \cdots)] \dot{-} [r_0^* \dot{-} (\cdots \dot{-} (r_{m-2}^* \dot{-} r_{m-1}^*) \cdots)]\big). \end{aligned}$$

Substituting $w \dot{-} w$ for each z_i is this equation gives

$$\begin{aligned} p \dot{-} \big([u_0^* \dot{-} (\cdots \dot{-} (u_{m-2}^* \dot{-} u_{m-1}^*) \cdots)] \dot{-} [u_0^* \dot{-} (\cdots \dot{-} (u_{m-2}^* \dot{-} u_{m-1}^*) \cdots)]\big) \approx \\ q \dot{-} \big([r_0^* \dot{-} (\cdots \dot{-} (r_{m-2}^* \dot{-} r_{m-1}^*) \cdots)] \dot{-} [r_0^* \dot{-} (\cdots \dot{-} (r_{m-2}^* \dot{-} r_{m-1}^*) \cdots)]\big). \end{aligned}$$

In view of the challenges and the presence of $\boldsymbol{\varepsilon}$ we obtain

$$p \dot{-} (z \dot{-} z) \approx q \dot{-} (z \dot{-} z).$$

Applying the Cancellation Lemma we obtain $p \approx q$ and

$$\big([u_0 \dot- (\cdots \dot- (u_{m-2} \dot- u_{m-1}) \cdots)] \dot- [u_0^* \dot- (\cdots \dot- (u_{m-2}^* \dot- u_{m-1}^*) \cdots)]\big) \approx \big([r_0 \dot- (\cdots \dot- (r_{m-2} \dot- r_{m-1}) \cdots)] \dot- [r_0^* \dot- (\cdots \dot- (r_{m-2}^* \dot- r_{m-1}^*) \cdots)]\big)$$

by applying the Cancellation Lemma, this time to $s \approx t$. A third application of the Cancellation Lemma yields

$$(\star) \qquad u_0 \dot- (u_1 \dot- \cdots \dot- (u_{m-2} \dot- u_{m-1}) \cdots) \approx r_0 \dot- (r_1 \dot- \cdots \dot- (r_{m-2} \dot- r_{m-1}) \cdots).$$

Now substitution gives

$$u_0 \dot- (u_1^* \dot- \cdots \dot- (u_{m-2}^* \dot- u_{m-1}^*) \cdots) \approx r_0 \dot- (r_1^* \dot- \cdots \dot- (r_{m-2}^* \dot- r_{m-1}^*) \cdots)$$

but $\{\gamma, \boldsymbol{\varepsilon}\} \vdash u_i^* \approx r_i^*$ so we get

$$u_0 \dot- (u_1^* \dot- \cdots \dot- (u_{m-2}^* \dot- u_{m-1}^*) \cdots) \approx r_0 \dot- (u_1^* \dot- \cdots \dot- (u_{m-2}^* \dot- u_{m-1}^*) \cdots)$$

The Cancellation Lemma applied to the equation above gives

$$u_0 \approx r_0$$

and, from $(\star)$, we get

$$u_1 \dot- \cdots \dot- (u_{m-2} \dot- u_{m-1}) \cdots) \approx r_1 \dot- \cdots \dot- (r_{m-2} \dot- r_{m-1}) \cdots).$$

We can repeat this process to obtain $\delta_{s,t} \vdash u_i \approx r_i$ for all $i < m$, which is what was to be proved. □

Comments at the end. The second of the three stages of our proof of the Main Theorem departed from Walter Taylor's approach. While Taylor did not take on the task of collapsing to a single equation, this stage was a simple application of a theorem from the literature. A significant difference between the two approaches lies in our use of the absolute value as a basic operation. This came up in two places. In the first stage it was used to prove that $f'(0)$ was actually $\frac{\pi}{2}$. In the second stage, it was used to express the solvability of an inequality like $F < 0$ by the failure of an equation like $|F| \approx F$. This allowed us to avoid the nine additional constant symbols in our negative solution to the Compatibility Problem for the real line, that are present in Taylor's approach. (Actually, Richardson (1968) invented a method for squeezing these nine constants down to just one, see (McNulty, 2008) for an application appropriate here.) To get the correct value for $f'(0)$ in the first stage, Taylor introduced a new one-place operation symbol $\boldsymbol{\lambda}$ and he replaces our Δ_1 with the equation

$$\mathbf{sin}^*(\mathbf{sin}^*(x) + 1) = (\boldsymbol{\lambda}(x))^2.$$

While in Taylor's setting it is not necessary that the interpretation of $\boldsymbol{\lambda}$ on the real line be determined, by replacing Δ_1 by

$$\Delta_1' = \{\mathbf{sin}^*(\mathbf{sin}^*(x) + 1) = (\boldsymbol{\lambda}(x))^2, \boldsymbol{\lambda}(x + \mathbf{4}) \approx \boldsymbol{\lambda}(x), \boldsymbol{\lambda}(\mathbf{0}) \approx \mathbf{-1}, \boldsymbol{\lambda}(\mathbf{2}) \approx \mathbf{1}\}$$

we see that $\Delta_0 \cup \Delta_1' \cup \Delta_2$ determines the interpretation of $\boldsymbol{\lambda}$ as well. Taylor was even able to prove that the function determined in this way is a real-analytic function on the real line. By combining this with Richardson's method of getting by with one constant and with the One Equation Collapse, the following theorem emerges.

Theorem at the End. *In the signature of rings with unit, expanded by two one-place operation symbols* $\mathbf{sin}^*$ *and* $\boldsymbol{\lambda}$ *and one constant symbol, there is no algorithm for determining whether an arbitrary equation is compatible with the real line, in the real analytic sense.*

Appendix: From 9 to 1

The main objective of this series of lectures in the seminar was the negative solution to the compatibility problem over the real line for single equations of signature τ. To achieve this objective it was not necessary to develop Richardson's method for squeezing down to one variable from nine. On the other hand, this method has a number of interesting consequences, notably the Theorem at the End. So in this appendix I provide an adaption of Richardson's method suitable in our context.

Call a system $\langle e_0, e_1, e_2, \ldots \rangle$ of functions on $\mathbb{R}$ a system of **approximate decoding functions** if and only if given any natural number m, any real numbers $\chi_0, \ldots, \chi_{m-1}$, and any $\varepsilon > 0$, there is a real number η so that $|e_i(\eta) - \chi_i| < \varepsilon$ for all $i < m$.

Richardson's Approximate Decoding Lemma. *Let $h(x) = x \sin x$ and $g(x) = x \sin x^3$. Then*

$$\langle h(g^i(x)) \mid i \text{ is a natural number} \rangle = \langle h(x), h(g(x), h(g(g(x)))), \ldots \rangle$$

is a system of approximate decoding functions.

A very short argument, given by Wang (1974), is needed to deduce the negative solution Hilbert's Tenth Problem for one variable with respect to $\langle \mathbb{R}, +, \cdot, -, 1, \sin^* \rangle$ from the Richardson Inequality Lemma and Richardson's Approximate Decoding Lemma. What is needed to prove our Theorem at the End is a modification of Richardson's Approximate Decoding Lemma.

1. How to make a system of approximate decoding functions

Our Decoding Lemma. *Let $h(x) = x \sin^* x$ and $g(x) = x \sin^* x^3$. Then*

$$\langle e_i(x) \mid i \text{ is a natural number} \rangle := \langle h(x), h(g(x), h(g(g(x)))), \ldots \rangle$$

is a system of approximate decoding functions.

For each $i < n$ we let $\mathbf{e}_i(x)$ be the obvious term that denotes the corresponding decoding function e_i in the algebra $\mathbf{R}^*$.

Proof. The claim below is key to proving the Lemma. Our proof has been adapted from an argument of Matiyasevich (1993). Richardson's original argument can also be adapted in roughly the same way.

Claim. *Given any real numbers χ and ψ and any $\varepsilon > 0$, there is a real number η such that*

$$|h(\eta) - \chi| < \varepsilon$$
$$g(\eta) = \psi$$

Proof of the Claim. It is harmless to suppose $\varepsilon < 1$. Fix an integer k so large that $4k-1 > |\chi|$ and $4k-2 > |\psi|$ and

$$M\frac{6(4k-1)^2}{(2k+1)\pi+1} > \frac{4}{\varepsilon}$$

Consider the closed interval $[4k-1, 4k+1]$. Notice

$$\sin^*(4k-1) = \sin\frac{\pi}{2}(4k-1) = \sin(2\pi k - \frac{\pi}{2}) = \sin(-\frac{\pi}{2}) = -1$$
$$\sin^*(4k+1) = \sin\frac{\pi}{2}(4k+1) = \sin(2\pi k + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$$

So $\sin^*(x)$ maps $[4k-1, 4k+1]$ onto $[-1, 1]$. Hence, the image of the interval $[4k-1, 4k+1]$ under the map $h(x)$ must include the interval $[-4k+1, 4k+1]$. This means that there is η_0 in $[4k-1, 4k+1]$ so that $h(\eta_0) = \chi$, since $4k-1 > |\chi|$ entails that $-4k+1 < \chi < 4k+1$.

By continuity, h will map any sufficiently small interval about η_0 into the interval about χ of radius ε. Our argument depends on finding out how small sufficiently small must be. After a bit of reverse engineering, we take

$$\delta = \frac{\varepsilon}{(2k+1)\pi+1}.$$

Let ν be any element of $[\eta_0 - \delta, \eta_0 + \delta]$. According to the Mean Value Theorem, we pick $\hat\eta$ in $[\eta_0 - \delta, \eta_0 + \delta]$ so that $|h(\nu) - h(\eta_0)| \le |h'(\hat\eta)|\delta$.

Now just observe

$$\begin{aligned}
|h(\nu) - \chi| &= |h(\nu) - h(\eta_0)| \\
&\le |h'(\hat\eta)|\delta \\
&= \left|\sin(\frac{\pi}{2}\hat\eta) + \frac{\pi}{2}\hat\eta\cos(\frac{\pi}{2}\hat\eta)\right|\delta \\
&\le \left(1 + \frac{\pi}{2}(\eta_0+\delta)\right)\delta \\
&\le \left(1 + \frac{\pi}{2}\left(4k+1+\frac{\varepsilon}{(2k+1)\pi+1}\right)\right)\frac{\varepsilon}{(2k+1)\pi+1} \\
&\le \left(1 + 2k\pi + \frac{\pi}{2} + \frac{\pi}{(4k+2)\pi+2}\right)\frac{\varepsilon}{(2k+1)\pi+1} \\
&< \left(1 + 2k\pi + \frac{\pi}{2} + \frac{\pi}{2}\right)\frac{\varepsilon}{(2k+1)\pi+1} \\
&= (1 + 2k\pi + \pi)\frac{\varepsilon}{(2k+1)\pi+1} \\
&= ((2k+1)\pi+1)\frac{\varepsilon}{(2k+1)\pi+1} = \varepsilon
\end{aligned}$$

This means

$$|h(\nu) - \chi| < \varepsilon$$

for any ν in $[\eta_0 - \delta, \eta_0 + \delta]$.

So it remains to find η in $[\eta_0 - \delta, \eta_0 + \delta]$ so that $g(\eta) = \psi$. Now the cubing function maps $[\eta_0 - \delta, \eta_0 + \delta]$ onto $[(\eta_0 - \delta)^3, (\eta_0 + \delta)^3]$. Also, observe

$$\begin{aligned}
(\eta_0 + \delta)^3 - (\eta_0 - \delta)^3 &= 6\eta_0^2\delta + 2\delta^3 \\
&\geq 6(4k-1)^2\delta \\
&= 6(4k-1)^2 \frac{\varepsilon}{(2k+1)\pi + 1} \\
&= \frac{6(4k-1)^2}{(2k+1)\pi + 1}\varepsilon \\
&> \frac{4}{\varepsilon}\varepsilon = 4
\end{aligned}$$

Therefore, as ν ranges over $[\eta_0 - \delta, \eta_0 + \delta]$ we find that ν^3 ranges over an interval of length at least 4. In turn, this means that $\frac{\pi}{2}\nu^3$ ranges over an interval of length at least 2π. Consequently, as ν ranges over $[\eta_0 - \delta, \eta_0 + \delta]$ we conclude that $\sin^*(\nu^3) = \sin(\frac{\pi}{2}\nu^3)$ takes on all values between -1 and 1.

Then $g(\nu) = \nu \sin^*(\nu^3)$ has to take on all values between $\delta - \eta_0$ and $\eta_0 - \delta$. Recall that $4k - 1 \leq \eta_0$. Since we have $|\psi| < 4k - 2$, we know that ψ will lie between $\delta - \eta_0$ and $\eta_0 - \delta$ and we can pick η in $[\eta_0 - \delta, \eta_0 + \delta]$ so that $g(\eta) = \psi$ and $|h(\eta) - \chi| < \varepsilon$, as desired. This completes the proof of the Claim. □

Our whole line of reasoning in support of the Lemma can now be concluded by a straightforward induction on the natural number m to the effect that for all reals $\chi_0, \dots, \chi_{m-1}$ and every $\varepsilon > 0$, there is η_m so that

$$|e_i(\eta_m) - \chi_i| < \varepsilon \text{ for all } i < m.$$

The base step holds vacuously.

Here is the inductive step. Suppose reals $\chi_0, \dots, \chi_m$ are given along with $\varepsilon > 0$. The inductive hypothesis applied to the system $\chi_1, \dots, \chi_m$ yields η_m so that

$$|e_i(\eta_m) - \chi_i| < \varepsilon \text{ for all } i \text{ with } 1 \leq i \leq m.$$

Use the Claim to obtain η_{m+1} so that $|h(\eta_{m+1}) - \chi_0| < \varepsilon$ and $g(\eta_{m+1}) = \eta_m$.

It follows that

$$\begin{aligned}
|e_0(\eta_{m+1}) - \chi_0| &= |h(\eta_{m+1}) - \chi_0| < \varepsilon \\
|e_1(\eta_{m+1}) - \chi_1| &= |e_0(g(\eta_{m+1})) - \chi_1| = |e_0(\eta_m) - \chi_1| < \varepsilon \\
|e_2(\eta_{m+1}) - \chi_2| &= |e_1(g(\eta_{m+1})) - \chi_2| = |e_1(\eta_m) - \chi_2| < \varepsilon \\
&\vdots \\
|e_m(\eta_{m+1}) - \chi_m| &= |e_{m-1}(g(\eta_{m+1})) - \chi_m| = |e_{m-1}(\eta_m) - \chi_m| < \varepsilon,
\end{aligned}$$

which is exactly what we need. We have devised a system of approximate decoding functions from just $\cdot$ and $\sin^*$. This completes the proof of the Lemma. □

2. Just one unknown: Hilbert's Tenth Problem for $\mathbf{R}^*$

Let $\mathbf{R}^*$ be the algebra $\langle \mathbb{R}, +, \cdot, -, 1, \sin^* \rangle$.

A Negative Solution to Hilbert's Tenth Problem in One Unknown for $\mathbf{R}^*$. *There is no algorithm for determining whether an equation of signature τ^* in one variable is solvable in $\mathbf{R}^*$.*

We repeat here Wang's proof, with one small change (as Wang allowed the rational number $\frac{1}{2}$).

With each polynomial $p(x_0, \ldots, x_8)$ with integer coefficients we will associate a term $G_p(x)$ in the signature of $\mathbf{R}^*$ so that the following are equivalent

(i) $p(x_0, \ldots, x_8) \approx \mathbf{0}$ has a solution in the integers
(ii) $G_p(x) < \mathbf{0}$ has a solution in the real numbers
(iii) $G_p(x) \approx \mathbf{0}$ has a solution in the real numbers.

Moreover, there will be an algorithm which upon input of $p(x_0, \ldots, x_8)$ will output the associated $G_p(x)$. In this way Hilbert's 10^{th} Problem in one variable for $\mathbf{R}^*$ will be reduced to Hilbert's 10^{th} Problem in n variables for the ring of integers, and our Theorem will be established.

First, let $H_p(x_0, \ldots, x_8)$ be the term

$$2(10^2)\left[p^2(x_0, \ldots, x_8) + \sum_{i<9} (\mathbf{sin}^*\, 2x_i)^2 k_i^2(x_0, \ldots, x_8)\right] \mathbf{-1},$$

where the terms $k_i(x_0, \ldots, x_8)$ come from the Richardson Inequality Lemma.

Notice that $H_p(x_0, \ldots, x_8) = 2F_p(x_0, \ldots, x_8) \mathbf{+1}$, where $F_p(x_0, \ldots, x_8)$ also comes from the Richardson Inequality Lamma.

Finally, put

$$G_p(x) = H_p(\mathbf{e}_0(x), \mathbf{e}_1(x), \ldots, \mathbf{e}_8(x)) = 2F_p(\mathbf{e}_0(x), \ldots, \mathbf{e}_8(x)) \mathbf{+1}\,.$$

Now it follows from our Lemma by the continuity of H_p that

$$H_p(b_0, \ldots, b_8) < 0 \text{ for some } b_0, \ldots, b_8 \in \mathbb{R}$$

if and only if

$$G_p(\eta) = H_p(e_0(\eta), \ldots, e_8(\eta)) < 0 \text{ for some } \eta \in \mathbb{R}.$$

A similar equivalence prevails with the inequalities going in the other direction.

First, we argue that (i) imples (ii) implies (iii). Suppose that $a_0, \ldots, a_8$ are integers so that $p(a_0, \ldots, a_8) = 0$. Then $F_p(a_0, \ldots, a_8) = -1$, which implies that $H_p(a_0, \ldots, a_8) = -1$ as well. Consequently, $G_p(\mu)$ is negative for some real number μ. On the other hand it is easy to see that $H_p(\frac{\pi}{4}, \ldots, \frac{\pi}{4})$ must be positive. So $G_p(\nu)$ must be positive for some real number ν, by our Lemma and the continuity of H_p. By the Intermediate Value Theorem there must be a real number η so that $G_p(\eta) = 0$.

To see that (iii) implies (i), suppose that $G_p(\eta) = 0$. Then there are real numbers $\beta_0, \ldots, \beta_8$ so that $0 = 2F_p(\beta_0, \ldots, \beta_8) + 1$. This means that $F_p(\beta_0, \ldots, \beta_8)$ is negative. By the Richardson Inequality Lemma, we conclude that $p(x_0, \ldots, x_8) \approx \mathbf{0}$ has a solution in the integers.

That's all that needed to be proved.

3. Proofs at the End

Theorem at the End. *In the signature of rings with unit, expanded by two one-place operation symbols* $\mathbf{sin}^*$ *and* $\boldsymbol{\lambda}$ *and one constant symbol, there is no algorithm for determining whether an arbitrary equation is compatible with the real line, in the real analytic sense.*

Proof. Let τ' be the signature with operation symbols

$$+, \cdot, -, \mathbf{1}, \mathbf{sin}^*, \boldsymbol{\lambda}, \mathbf{c}$$

where $\mathbf{c}$ is a new constant symbol. We reduce Hilbert's 10^{th} Problem in one variable for $\mathbf{R}^*$ to the equational compatibility problem for $\mathbb{R}$ and τ'. We just proved above that the former problem is algorithmically unsolvable, hence the equational compatibility problem must also be algorithmically unsolvable.

Let $\Delta = \Delta_0 \cup \Delta_1' \cup \Delta_2$ and let Γ be any finite set of equations of the signature of $\mathbf{R}^*$ so that x is the only variable to occur in Γ. Let Γ' result from Γ by substituting the constant symbol $\mathbf{c}$ for the variable x. We contend that

$$\Gamma \text{ has a solution in } \mathbf{R}^* \text{ if and only if } \Delta \cup \Gamma' \text{ is compatible with } \mathbb{R}.$$

The left-to-right implication is immediate from the natural interpretations for the operation symbols. The right-to-left implication is immediate since Δ determines $\mathbf{R}^*$ according to the modified Finite Determination Theorem.

Now, we complete the proof by appealing to the One Equation Collapse. □

The One-Variable Equational Undecidability Theorem. *The one-variable equational theory of* $\langle \mathbb{R}, +, \cdot, -, 1, \sin^*, |\ \ | \rangle$ *is algorithmically undecidable.*

Proof. In Section 2 we noted that there is an algorithm which given any polynomial p with integer coefficients will produce a term $G_p(x)$ in one variable so that

$$\begin{gathered} p(x_0, \dots, x_{n-1}) \approx \mathbf{0} \text{ has a solution in the integers} \\ \text{if and only if} \\ G_p(x) < \mathbf{0} \text{ has a solution in the real numbers} \\ \text{if and only if} \\ G_p(x) \approx \mathbf{0} \text{ has a solution in the real numbers.} \end{gathered}$$

It is our contention that the one-variable equation $|G_p(x)| \approx G_p(x)$ fails in $\mathbf{R}$ if and only if $p(x_0, \dots, x_{n-1}) \approx \mathbf{0}$ has a solution in the integers.

This contention holds since $p(x_0, \dots, x_{n-1}) \approx \mathbf{0}$ has a solution in the integers if and only if $G_p(x) < \mathbf{0}$ has a solution in the reals if and only if the equation $|G_p(x)| \approx G_p(x)$ fails in $\mathbf{R}$.

So Hilbert's 10^{th} Problem over the integers reduces to the algorithmic decision problem for the one-variable equational theory of $\mathbf{R}$. Hence that latter problem must be algorithmically undecidable. □

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