Algebra Problems Set Twenty One

Due 28 April 2015

Problem 78.
Let E be the splitting field over $\mathbb{Q}$ of $x^4 - 2$. Determined all the fields intermediate between E and $\mathbb{Q}$. Draw a diagram of the lattice of intermediate fields.

Problem 79.
Prove that the field of real numbers has only one ordering that makes it into an ordered field. In contrast, prove that $\mathbb{Q}[\sqrt{2}]$ has exactly two such orderings.

Problem 80.
Let R be a real closed field and let $f(x) \in R[x]$. Suppose that $a < b$ in R and that $f(a)f(b) < 0$. Prove that there is $c \in R$ with $a < c < b$ such that c is a root of $f(x)$.

Problem 81.
Let R be a real closed field, let $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + x^n = f(x) \in R[x]$, and put $M = |a_0| + |a_1| + \cdots + |a_{n-1}| + 1$. Prove that every root of $f(x)$ which belongs to R belongs to the interval $[-M, M]$.