Algebra Problems Set Twelve and a Half

Due 31 March 2015

Problem 4. Prove that every algebraically closed field of prime characteristic is infinite.

Problem 5. Let $\mathbf{R}$ be a commutative ring and let I be a finitely generated nontrivial ideal of $\mathbf{R}$. Prove that $\mathbf{R}$ has an ideal M such that each of the following properties holds:

i. I is not a subset of M, and
ii. for all ideals J of $\mathbf{R}$, if $M \subseteq J$ and $M \neq J$, then $I \subseteq J$.

Problem 6. Prove that there is a polynomial $f(x) \in \mathbb{R}[x]$ such that
(a) $f(x) - 1$ belongs to the ideal $(x^2 - 2x + 1)$;
(b) $f(x) - 2$ belongs to the ideal $(x + 1)$, and
(c) $f(x) - 3$ belongs to the ideal $(x^2 - 9)$.

Problem 7. Let the field $\mathbf{E}$ be an extension of the field $\mathbf{F}$ so that $[\mathbf{E} : \mathbf{F}]$ is finite. Let $f(x) \in \mathbf{F}[x]$ be irreducible and of degree p where p is a prime number. Prove that if $f(x)$ is not irreducible in $\mathbf{E}[x]$, then p divides $[\mathbf{E} : \mathbf{F}]$.