Socle degrees of Frobenius powers
Lecture 4 — February 8, 2006
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My typed notes have gotten ahead of what I have actually said. So, most of the typed notes for today were already typed last week. The new material concerns the proof of step 4.

The hypothesis. Let k be a field of positive characteristic p, P be the polynomial ring $k[x_1,\dots,x_n]$, $C=(f_1,\dots,f_c)$ be generated by a homogeneous regular sequence in P, R be the ring P/C, and I be a homogeneous ideal of P with P/I a Gorenstein ring and P/I finite dimensional as a k-vector space. Assume that the socles of R/IR and $R/I^{[p]}R$ have the same dimension, and that

$$D_i = pd_i-(p-1)a(R),$$

for all i, where the socle degrees of R/IR are $\{d_i\}$, the socle degrees of $R/I^{[p]}R$ are $\{D_i\}$, and $a(R)$ is $\sum|f_i|-\sum|x_i|$.

The goal. Prove that $\operatorname{pd}_R R/IR<\infty$.

Last week I outlined an 8 step program to reach this goal. Today I will march through the 8 steps.

Step 1. $\operatorname{Tor}_c^P(P/I,P/C)=\frac{I:C}{I}(-\sum|f_i|)$.

Proof of Step 1. Let $\mathbb{G}$ be the Koszul complex which resolves P/C. The end of $\mathbb{G}$ is

$$0\to P(-\sum_{i=1}^{c}|f_i|)\xrightarrow{\begin{bmatrix}f_1\\ \vdots\\ f_c\end{bmatrix}}\begin{matrix}P(-\sum\limits_{\substack{i=1\\ i\neq 1}}^{c}|f_i|)\\ \oplus\\ P(-\sum\limits_{\substack{i=1\\ i\neq 2}}^{c}|f_i|)\\ \oplus\\ \vdots\\ \oplus\\ P(-\sum\limits_{\substack{i=1\\ i\neq c}}^{c}|f_i|)\end{matrix}\to\dots\,.$$

We may compute $\operatorname{Tor}_c(P/I,P/C)$ by tensoring the above resolution with P/I (that is setting $I=0$) and then computing homology. So, $\operatorname{Tor}_c(P/I,P/C)$ is the kernel of

$$\frac{P}{I}(-\sum_{i=1}^{c}|f_i|) \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_c \end{bmatrix}} \begin{matrix} \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq 1}}^{c}|f_i|) \\ \bigoplus \\ \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq 2}}^{c}|f_i|) \\ \bigoplus \\ \vdots \\ \bigoplus \\ \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq c}}^{c}|f_i|), \end{matrix}$$

which is $\frac{I:C}{I}(-\sum\limits_{i=1}^{c}|f_i|)$, as claimed.

Step 2. We can connect the generator degrees of $\frac{I:C}{I}$ to the socle degrees of P/I.

Proof of Step 2. We will use two statements about Gorenstein duality. Assume that P/I is a finite dimensional vector space and is a Gorenstein ring. Let N be the socle degree of P/I. Let M be a finitely generated P/I-module. Then

A. $\operatorname{Hom}_{P/I}(\operatorname{Hom}_{P/I}(M, P/I), P/I) = M$, and

B. $\dim_k \operatorname{Hom}_{P/I}(M, P/I)_d = \dim_k M_{N-d}$ for all d.

Of course, the point is that $\operatorname{Hom}_{P/I}(\_, P/I)$ exactly turns P/I modules upside down!

Anyhow, I claim that if $\{\delta_i\}$ are the generator degrees of $\frac{I:C}{I}$, then $\delta_i = N - d_i$.

Proof. In this argument, "Hom" means "$\operatorname{Hom}_{P/I}$" and "$\otimes$" means "$\otimes_{P/I}$". Use Nakayama's Lemma to see that the generator degrees of $\frac{I:C}{I}$ are equal to the degrees of $\frac{I:C}{I+\mathfrak{m}(I:C)}$. Recall that R/IR is the same as $P/(I+C)$; and therefore, the socle of R/IR is equal to

$$\frac{(I+C):\mathfrak{m}}{I+C} = \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \tfrac{P}{I+C})$$

and by **A**, this is equal to

$$\operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\operatorname{Hom}(\tfrac{P}{I+C}, \tfrac{P}{I}), \tfrac{P}{I})) = \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\tfrac{I:(I+C)}{I}, \tfrac{P}{I}))$$

$$= \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\tfrac{I:C}{I}, \tfrac{P}{I})).$$

Now use the Adjoint Isomorphism Theorem, which says

$$\mathrm{Hom}(A\otimes B, C) = \mathrm{Hom}(A, \mathrm{Hom}(B, C)),$$

to see that the socle of R/IR is equal to

$$\mathrm{Hom}(\tfrac{P}{\mathfrak{m}} \otimes \tfrac{I:C}{I}, \tfrac{P}{I}) = \mathrm{Hom}(\tfrac{I:C}{I+\mathfrak{m}(I:C)}, \tfrac{P}{I}).$$

Finally, we use **B** to complete the proof. $\square$

Step 3. Suppose the generators of $\mathrm{Tor}_c(P/I, P/C)$ have degrees $\{\gamma_i\}$ and the generators of $\mathrm{Tor}_c(P/I^{[p]}, P/C)$ have degrees $\{\Gamma_i\}$. Then $\Gamma_i = p\gamma_i$.

Proof of Step 3. We have

- the socle degrees of R/IR are $\{d_i\}$,
- the socle degrees of $R/I^{[p]}R$ are $\{D_i\}$,
- $D_i = pd_i - (p-1)(\sum |f_j| - \sum |x_j|)$,
- the generator degrees of $\mathrm{Tor}_c(P/I, P/C)$ are $\{\gamma_i\}$,
- the generator degrees of $\mathrm{Tor}_c(P/I^{[p]}, P/C)$ are $\{\Gamma_i\}$,
- $\mathrm{Tor}_c(P/I, P/C) = \frac{I:C}{C}(-\sum |f_i|)$,
- $\mathrm{Tor}_c(P/I^{[p]}, P/C) = \frac{I^{[p]}:C}{C}(-\sum |f_i|)$,
- the generator degrees of $\frac{I:C}{C}$ are $\{N - d_i\}$, and
- the generator degrees of $\frac{I^{[p]}:C}{C}$ are $\{\boxed{pN-(p-1)(-\sum |x_j|)} - D_i\}$.

(Recall that the (1) $\Leftarrow$ (2) part of the proof tells us that if the socle degree of P/I is N, then the socle degree of $P/I^{[p]}$ is $pN-(p-1)a(P)$. This explains the formula inside the box.)

Our job is to "Do the Math." I find it convenient to let $\{\delta_i\}$ be the generator degrees of $\frac{I:C}{C}$ and $\{\Delta_i\}$ be the generator degrees of $\frac{I^{[p]}:C}{C}$. We have

$$\Gamma_i = \Delta_i + \sum |f_j| = pN - (p-1)(-\sum |x_j|) - D_i + \sum |f_j|$$

$$= pN - (p-1)(-\sum |x_j|) - \Big(pd_i - (p-1)(\sum |f_j| - \sum |x_j|)\Big) + \sum |f_j|$$

$$= pN - pd_i + (p-1)\sum |f_j| + \sum |f_j| = pN - pd_i + p\sum |f_j|$$

$$= p((N - d_i) + \sum |f_j|) = p(\delta_i + \sum |f_j|) = p\gamma_i,$$

as claimed. $\square$

Step 4. Use the generators of $\operatorname{Tor}_c(P/I, P/C)$ to produce the generators of $\operatorname{Tor}_c(P/I^{[p]}, P/C)$.

Proof of Step 4. Let $\mathbb{F}$ be a resolution of P/I by free P-modules. It follows that

$$\operatorname{Tor}_c(P/I, P/C) = H_c(\mathbb{F} \otimes P/C).$$

Kunz's Theorem guarantees that $\mathbb{F}^{[p]}$ is a resolution of $P/I^{[p]}$ by free P-modules; and therefore,

$$\operatorname{Tor}_c(P/I^{[p]}, P/C) = H_c(\mathbb{F}^{[p]} \otimes P/C).$$

A homology element of $H_c(\mathbb{F} \otimes P/C)$ is $[\bar{z}]$, where z is a column vector z in F_c with $d_c(z) \in CF_{c-1}$. The homology element $[\bar{z}]$ is non-zero if $z \notin \operatorname{im} d_{c+1} + CF_c$. We see that if $[\bar{z}]$ is a homology element of $H_c(\mathbb{F} \otimes P/C)$, then $[\bar{z}^{[p]}]$ is a homology element of $H_c(\mathbb{F}^{[p]} \otimes P/C)$. Furthermore, the degree of $[\bar{z}^{[p]}]$ is p times the degree of $[\bar{z}]$. We take a minimal generating set $[\bar{z}_1], \dots, [\bar{z}_\ell]$ for $H_c(\mathbb{F} \otimes P/C)$. We see that $[\bar{z}_1^{[p]}], \dots, [\bar{z}_\ell^{[p]}]$ are elements of $H_c(\mathbb{F}^{[p]} \otimes P/C)$ which have the correct degrees to be a minimal generating set. We can show that $[\bar{z}_1^{[p]}], \dots, [\bar{z}_\ell^{[p]}]$ are a minimal generating set by proving that they are linearly independent.

The argument goes by induction. A good way to convey the flavor of the argument, without overwhelming you with details, is to show the base case. (Only slight modifications are needed to do the inductive step.) Assume that $[\bar{z}]$ is non-zero element of $H_c(\mathbb{F} \otimes P/C)$ of least degree. We prove that $[\bar{z}^{[p]}]$ is non-zero element of $H_c(\mathbb{F}^{[p]} \otimes P/C)$. Suppose $[\bar{z}^{[p]}]$ is zero in $H_c(\mathbb{F}^{[p]} \otimes P/C)$. So

$$z^{[p]} \in \operatorname{im} d_{c+1}^{[p]} + CF_c^{[p]}.$$

I will prove that

$$(\bigstar) \quad z^{[p]} \in \operatorname{im} d_{c+1}^{[p]} + C^t F_c^{[p]} + C^{[p]} F_c^{[p]} \implies z^{[p]} \in \operatorname{im} d_{c+1}^{[p]} + C^{t+1} F_c^{[p]} + C^{[p]} F_c^{[p]},$$

for all t for which this makes sense. Once $(\bigstar)$ is established, then

$$z^{[p]} \in \operatorname{im} d_{c+1}^{[p]} + C^{[p]} F_c^{[p]}$$

because $C^t \subseteq C^{[p]}$ for $t \geq c(p-1)+1$. Now Kunz's Theorem (again!) tells us that $z \in \operatorname{im} d_{c+1} + CF_c$, and this is a contradiction because $[\bar{z}]$ is not zero in $H_c(\mathbb{F} \otimes P/C)$.

Now we prove (★). We are told that there are $y_\alpha \in F_c^{[p]}$ with

$$z^{[p]} - \sum f_1^{\alpha_1} \cdots f_c^{\alpha_c} y_\alpha \in \operatorname{im} d_{c+1}^{[p]} + C^{[p]} F_c^{[p]}$$

where the sum is taken over all c-tuples α with $\sum \alpha_i = t$ and $0 \le \alpha_i \le p-1$, for all i. Fix an α. Multiply by $f_1^{p-1-\alpha_1} \cdots f_c^{p-1-\alpha_c}$. Apply $d_c^{[p]}$. Observe that

$$(f_1 \cdots f_c)^{p-1} d_c^{[p]}(y_\alpha) \in C^{[p]} F_c^{[p]}.$$

But

$$C^{[p]} : (f_1 \cdots f_c)^{p-1} = C.$$

So

$$d_c^{[p]}(y_\alpha) \in C F_{c-1}^{[p]}.$$

In other words, $[\bar{y}_\alpha]$ is a homology element of $H_c(\mathbb{F}^{[p]} \otimes P/C)$. But $|y_\alpha| < |z|$; so, by hypotheis, $[\bar{y}_\alpha]$ is zero in $H_c(\mathbb{F}^{[p]} \otimes P/C)$; so,

$$y_\alpha \in \operatorname{im} d_{c+1}^{[p]} + C F_c.$$

Do this procedure for each α, to complete the proof of (★).

Step 5. Drag the answer to Step 4 through the double complex machine to learn that

$$I^{[p]} : C = (I : C)^{[p]} (f_1 \cdots f_c)^{p-1} + I^{[p]}.$$

(I will leave this step out of these lectures.)

Step 6. Prove that the conclusion to step 5 implies

$$I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]} C.$$

(This step is very similar to step 4.)

Step 7. Prove that the conclusion of Step 6 implies $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$.

Proof of Step 7. We want to prove that

(***) $$I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]} C$$

implies $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$. We show that $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$ by showing that

(**) $$\begin{gathered} R^{b_2} \xrightarrow{d_2} R^{b_1} \xrightarrow{d_1} R \to R/I \to 0 \quad \text{is exact} \\ \implies R^{b_2} \xrightarrow{d_2^{[p]}} R^{b_1} \xrightarrow{d_1^{[p]}} R \to R/I^{[p]} \to 0 \quad \text{is exact.} \end{gathered}$$

We show that (***) implies (**).

I make my calculation at the P-level. Let $a_1, \dots, a_{b_1}$ generate I in P; so,

$$d_1 = [\, a_1 \quad \dots \quad a_{b_1} \,]$$

and

$$d_1^{[p]} = [\, a_1^p \quad \dots \quad a_{b_1}^p \,] \,.$$

We think of d_2 as having two pieces:

$$d_2 = [\, d_2' \quad d_2'' \,]$$

where

$$P^{b_2{}'} \xrightarrow{d_2'} P^{b_1} \xrightarrow{d_1} P$$

is exact (and d_2'' is all of the extra columns that describe elements of I which are also in C.) Recall that Kunz's Theorem (ingredient (B) of the other direction) tells us that

$$P^{b_2{}'} \xrightarrow{(d_2')^{[p]}} P^{b_1} \xrightarrow{d_1^{[p]}} P$$

is exact.

Suppose v is in P^{b_1} with $d_1^{[p]}(v) \in C$. In other words,

$$d_1^{[p]}(v) \in I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C.$$

So, there exist $s_1, \dots, s_t \in I \cap C$; $\alpha_1, \dots, \alpha_t$ in P; and $c_1, \dots c_{b_1}$ in C so that

$$d_1^{[p]}(v) = \sum_{i=1}^{t} \alpha_i s_i^p + \sum_{i=1}^{b_1} a_i^p c_i.$$

Of course, there exists $v_i \in P^{b_1}$ with $d_1(v_i) = s_i$ (and therefore also $d_1^{[p]} v_1^{[p]} = s_i^p$). So,

$$d_1^{[p]}(v) = d_1^{[p]} \left(\sum_{i=1}^{t} \alpha_i v_i^{[p]} + \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix} \right) .$$

So,

$$v - \sum_{i=1}^{t} \alpha_i v_i^{[p]} - \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix}$$

is killed by $d_1^{[p]}$; hence is in the image of $(d_2')^{[p]}$. Finally, $d_1(v_i) = s_i \in I \cap C$, so $v_i = d_2''(w_i)$ for some w_i; hence, $v_i^{[p]} = (d_2'')^{[p]}(w_i^{[p]})$. Thus,

$$v \in \operatorname{im} d_2^{[p]} + CP^{b_1},$$

as desired.

Step 8. We are finished by the Theorem of Avramov and Claudia Miller (see the last seminar talk given by John Olmo last semester.): If $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$, then $\operatorname{pd}_R(R/IR) < \infty$. (There is nothing for us to do here!)