Socle degrees of Frobenius powers
Lecture 3 — February 1, 2006
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Today's agenda:

I. Demonstrate the Magic machine for turning basis vectors into socle elements.

II. Outline the proof of (1) $\implies$ (2).

III. Get to work on some of the steps of (1) $\implies$ (2).

I. Demonstrate the Magic machine for turning basis vectors into socle elements.

Let $P = k[x_1, \dots, x_n]$ be a polynomial ring, I be an ideal of P with $\dim_k P/I$ finite, $\mathbb{F}$ be a resolution of P/I by free P-modules, and $\mathbb{G}$ be the (Koszul complex) resolution of $k = P/(x_1, \dots, x_n)$ by free P-modules. The isomorphism

$$(*) \qquad H_n(\mathbb{F} \otimes k) \cong H_n(\operatorname{Tot}(\mathbb{F} \otimes \mathbb{G})) \cong H_n(P/I \otimes \mathbb{G})$$

provides a method for converting the basis elements of $\mathbb{F}_n$ into socle elements of P/I. I illustrate with an example. Let $I = (x^2, xy, y^2)$. In this case, $\mathbb{F}$ is

$$0 \to \underbrace{P(-3)^2}_{F_2} \xrightarrow{f_2 = \begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}} \underbrace{P(-2)^3}_{F_1} \xrightarrow{f_1 = [\, y^2 \quad -xy \quad x^2 \,]} \underbrace{P}_{F_0},$$

$\mathbb{G}$ is

$$0 \to \underbrace{P(-2)}_{G_2} \xrightarrow{g_2 = \begin{bmatrix} y \\ -x \end{bmatrix}} \underbrace{P(-1)^2}_{G_1} \xrightarrow{g_1 = [\, x \quad y \,]} \underbrace{P}_{G_0},$$

and $\mathbb{F} \otimes \mathbb{G}$ is

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_2 \otimes G_2 & \xrightarrow{f_2 \otimes 1} & F_1 \otimes G_2 & \xrightarrow{f_1 \otimes 1} & F_0 \otimes G_2 \\
 & & {\scriptstyle 1\otimes g_2}\downarrow & & {\scriptstyle 1\otimes g_2}\downarrow & & {\scriptstyle 1\otimes g_2}\downarrow \\
0 & \longrightarrow & F_2 \otimes G_1 & \xrightarrow{f_2 \otimes 1} & F_1 \otimes G_1 & \xrightarrow{f_1 \otimes 1} & F_0 \otimes G_1 \\
 & & {\scriptstyle 1\otimes g_1}\downarrow & & {\scriptstyle 1\otimes g_1}\downarrow & & {\scriptstyle 1\otimes g_1}\downarrow \\
0 & \longrightarrow & F_2 \otimes G_0 & \xrightarrow{f_2 \otimes 1} & F_1 \otimes G_0 & \xrightarrow{f_1 \otimes 1} & F_0 \otimes G_0.
\end{array}
$$

Start with $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes 1$ in $F_2 \otimes G_0$ in the lower left hand corner. We see that this element represents an element of the homology of $H_2(\mathbb{F} \otimes k)$. One can extend this element to get an element of the homology of $H_2(\operatorname{Tot}(\mathbb{F} \otimes \mathbb{G}))$:

$$
\begin{array}{ccccc}
 & & & & 1 \otimes y \\
 & & & & \downarrow \\
 & & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \longrightarrow & 1 \otimes \begin{bmatrix} y^2 \\ -xy \end{bmatrix} \\
 & & \downarrow & & \\
\begin{bmatrix} 1 \\ 0 \end{bmatrix} & \longrightarrow & \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} & &
\end{array}
$$

The indicated element of $H_2(\operatorname{Tot}(\mathbb{F} \otimes \mathbb{G}))$ gives rise to the element y of the socle of P/I. To answer the question that our freshman ask: "Yes, it always works like that." We can use the idea of the snaky game to prove both isomorphisms in (*).

II. Outline the proof of (1) $\implies$ (2).

Recall that our goal is the following result.

Theorem. *Let k be a field of positive characteristic p, P be the polynomial ring $k[x_1, \dots, x_n]$, C be the homogeneous complete intersection ideal $C = (f_1, \dots, f_c)$ in*

P and R be P/C. Let I be a homogeneous ideal in P with P/I a finite dimensional vector space over k. Suppose that the socle degrees of R/IR are $d_1 \le \cdots \le d_\ell$ and that the socle degrees of $R/I^{[p]}R$ are $D_1 \le \cdots \le D_L$. Then the following statements are equivalent:
(1) $L = \ell$ and $D_i = pd_i - (p-1)a(R)$ for all i, and
(2) The ring R/IR has finite projective dimension as an R-module.

Remark. In the present context $a(R)$ is $\sum |f_i| - \sum |x_i|$.

We outline the proof of (1) $\implies$ (2) **under the additional hypothesis that P/I is a Gorenstein ring.** Our original proof did have this additional hypothesis. The proof is more direct and the result is better without the additional hypothesis; however, without the additional hypothesis one must make many calculations which involve "the canonical module". The canonical module of a Gorenstein ring is itself. If I make this additional hypothesis, then I can hide the fact that we are making canonical module calculations.

Assume (1). The following steps will yield (2).

Step 1. $\operatorname{Tor}_c(P/I, P/C) = \frac{I:C}{I}(-\sum |f_i|)$. (Actually, you know how to prove this already.)

Step 2. We can connect the generator degrees of $\frac{I:C}{I}$ to the socle degrees of P/I. (This uses Gorenstein duality.)

Step 3. Suppose the generators of $\operatorname{Tor}_c(P/I, P/C)$ have degrees $\{\gamma_i\}$ and the generators of $\operatorname{Tor}_c(P/I^{[p]}, P/C)$ have degrees $\{\Gamma_i\}$. Then $\Gamma_i = p\gamma_i$. (This is a straightforward calculation.)

Step 4. Use the generators of $\operatorname{Tor}_c(P/I, P/C)$ to produce the generators of $\operatorname{Tor}_c(P/I^{[p]}, P/C)$. (This is a delicate linear independence argument.)

Step 5. Drag the answer to Step 4 through the double complex machine to learn that

$$I^{[p]} : C = (I : C)^{[p]}(f_1 \cdots f_c)^{p-1} + I^{[p]}.$$

(I will leave this step out of these lectures.)

Step 6. Prove that the conclusion to step 5 implies

$$I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C.$$

(This step is very similar to step 4.)

Step 7. Prove that the conclusion of Step 6 implies $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$. (This is a grubby calculation. It is the one piece of the proof in this direction that I included in the notes for last week. I copied this calculation into the present notes. I might not bother to write them on the board.)

Step 8. We are finished by the Theorem of Avramov and Claudia Miller (see the last seminar talk given by John Olmo last semester.): If $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$, then $\operatorname{pd}_R(R/IR) < \infty$. (There is nothing for us to do here!)

III. Get to work on some of the steps of (1) $\implies$ (2).

Proof of Step 1. Let $\mathbb{G}$ be the Koszul complex which resolves P/C. The end of $\mathbb{G}$ is

$$0 \to P(-\sum_{i=1}^{c} |f_i|) \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_c \end{bmatrix}} \begin{matrix} P(-\sum\limits_{\substack{i=1\\ i\neq 1}}^{c} |f_i|) \\ \bigoplus \\ P(-\sum\limits_{\substack{i=1\\ i\neq 2}}^{c} |f_i|) \\ \bigoplus \\ \vdots \\ \bigoplus \\ P(-\sum\limits_{\substack{i=1\\ i\neq c}}^{c} |f_i|) \end{matrix} \to \dots .$$

We may compute $\operatorname{Tor}_c(P/I, P/C)$ by tensoring the above resolution with P/I (that is setting $I = 0$) and then computing homology. So, $\operatorname{Tor}_c(P/I, P/C)$ is the kernel of

$$\frac{P}{I}(-\sum_{i=1}^{c} |f_i|) \xrightarrow{\begin{bmatrix} f_1 \\ \vdots \\ f_c \end{bmatrix}} \begin{matrix} \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq 1}}^{c} |f_i|) \\ \bigoplus \\ \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq 2}}^{c} |f_i|) \\ \bigoplus \\ \vdots \\ \bigoplus \\ \frac{P}{I}(-\sum\limits_{\substack{i=1\\ i\neq c}}^{c} |f_i|), \end{matrix}$$

which is $\frac{I:C}{I}(-\sum\limits_{i=1}^{c} |f_i|)$, as claimed.

Proof of Step 2. We will use two statements about Gorenstein duality. These statements are not independent; indeed, either one could be used to prove the other.

Furthermore, maybe the real key statement is that if P/I is Gorenstein and a finite dimensional vector space, then P/I is an injective P/I-module (which means that the functor $\operatorname{Hom}_{P/I}(\_, P/I)$ is an exact functor.) At Bard College, Lars Christenson tossed off that "most of us think of this as the definition of Gorenstein".

Assume that P/I is a finite dimensional vector space and is a Gorenstein ring. Let N be the socle degree of P/I. Let M be a finitely generated P/I-module. Then

A. $\operatorname{Hom}_{P/I}(\operatorname{Hom}_{P/I}(M, P/I), P/I) = M$, and

B. $\dim_k \operatorname{Hom}_{P/I}(M, P/I)_d = \dim_k M_{N-d}$ for all d.

Of course, the point is that $\operatorname{Hom}_{P/I}(\_, P/I)$ exactly turns P/I modules upside down!

Anyhow, I claim that if $\{\delta_i\}$ are the generator degrees of $\frac{I:C}{I}$, then $\delta_i = N - d_i$.

Proof. In this argument, "Hom" means "$\operatorname{Hom}_{P/I}$" and "$\otimes$" means "$\otimes_{P/I}$". Use Nakayama's Lemma to see that the generator degrees of $\frac{I:C}{I}$ are equal to the degrees of $\frac{I:C}{I+\mathfrak{m}(I:C)}$. Recall that R/IR is the same as $P/(I+C)$; and therefore, the socle of R/IR is equal to

$$\frac{(I+C):\mathfrak{m}}{I+C} = \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \tfrac{P}{I+C})$$

and by **A**, this is equal to

$$\operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\operatorname{Hom}(\tfrac{P}{I+C}, \tfrac{P}{I}), \tfrac{P}{I})) = \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\tfrac{I:(I+C)}{I}, \tfrac{P}{I}))$$

$$= \operatorname{Hom}(\tfrac{P}{\mathfrak{m}}, \operatorname{Hom}(\tfrac{I:C}{I}, \tfrac{P}{I})).$$

Now use the Adjoint Isomorphism Theorem, which says

$$\operatorname{Hom}(A \otimes B, C) = \operatorname{Hom}(A, \operatorname{Hom}(B, C)),$$

to see that the socle of R/IR is equal to

$$\operatorname{Hom}(\tfrac{P}{\mathfrak{m}} \otimes \tfrac{I:C}{I}, \tfrac{P}{I}) = \operatorname{Hom}(\tfrac{I:C}{I+\mathfrak{m}(I:C)}, \tfrac{P}{I}).$$

Finally, we use **B** to complete the proof. $\square$

Proof of Step 3. We have

- the socle degrees of R/IR are $\{d_i\}$,
- the socle degrees of $R/I^{[p]}R$ are $\{D_i\}$,

- $D_i = pd_i - (p-1)(\sum |f_j| - \sum |x_j|)$,
- the generator degrees of $\operatorname{Tor}_c(P/I, P/C)$ are $\{\gamma_i\}$,
- the generator degrees of $\operatorname{Tor}_c(P/I^{[p]}, P/C)$ are $\{\Gamma_i\}$,
- $\operatorname{Tor}_c(P/I, P/C) = \frac{I:C}{C}(-\sum |f_i|)$,
- $\operatorname{Tor}_c(P/I^{[p]}, P/C) = \frac{I^{[p]}:C}{C}(-\sum |f_i|)$,
- the generator degrees of $\frac{I:C}{C}$ are $\{N - d_i\}$, and
- the generator degrees of $\frac{I^{[p]}:C}{C}$ are $\{\boxed{pN - (p-1)(-\sum |x_j|)} - D_i\}$.

(Recall that the (1) $\Leftarrow$ (2) part of the proof tells us that if the socle degree of P/I is N, then the socle degree of $P/I^{[p]}$ is $pN - (p-1)a(P)$. This explains the formula inside the box.)

Our job is to "Do the Math." I find it convenient to let $\{\delta_i\}$ be the generator degrees of $\frac{I:C}{C}$ and $\{\Delta_i\}$ be the generator degrees of $\frac{I^{[p]}:C}{C}$. We have

$$\Gamma_i = \Delta_i + \sum |f_j| = pN - (p-1)(-\sum |x_j|) - D_i + \sum |f_j|$$

$$= pN - (p-1)(-\sum |x_j|) - \Big(pd_i - (p-1)(\sum |f_j| - \sum |x_j|)\Big) + \sum |f_j|$$

$$= pN - pd_i + (p-1)\sum |f_j| + \sum |f_j| = pN - pd_i + p\sum |f_j|$$

$$= p((N - d_i) + \sum |f_j|) = p(\delta_i + \sum |f_j|) = p\gamma_i,$$

as claimed. $\square$

Proof of Step 7. We want to prove that

$$(***) \qquad I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C$$

implies $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$. We show that $\operatorname{Tor}_1^R(R/IR, {}^{\varphi}R) = 0$ by showing that

$$(**) \qquad \begin{aligned} &R^{b_2} \xrightarrow{d_2} R^{b_1} \xrightarrow{d_1} R \to R/I \to 0 \quad \text{is exact} \\ &\implies R^{b_2} \xrightarrow{d_2^{[p]}} R^{b_1} \xrightarrow{d_1^{[p]}} R \to R/I^{[p]} \to 0 \quad \text{is exact.} \end{aligned}$$

We show that (***) implies (**).

I make my calculation at the P-level. Let $a_1, \dots, a_{b_1}$ generate I in P; so,

$$d_1 = [\, a_1 \quad \dots \quad a_{b_1} \,]$$

and

$$d_1^{[p]} = [\, a_1^p \quad \dots \quad a_{b_1}^p \,]\,.$$

We think of d_2 as having two pieces:

$$d_2 = [\, d_2' \quad d_2'' \,]$$

where

$$P^{b_2{}'} \xrightarrow{d_2'} P^{b_1} \xrightarrow{d_1} P$$

is exact (and d_2'' is all of the extra columns that describe elements of I which are also in C.) Recall that Kunz's Theorem (ingredient (B) of the other direction) tells us that

$$P^{b_2{}'} \xrightarrow{(d_2')^{[p]}} P^{b_1} \xrightarrow{d_1^{[p]}} P$$

is exact.

Suppose v is in P^{b_1} with $d_1^{[p]}(v) \in C$. In other words,

$$d_1^{[p]}(v) \in I^{[p]} \cap C = (I \cap C)^{[p]} + I^{[p]}C.$$

So, there exist $s_1, \dots, s_t \in I \cap C$; $\alpha_1, \dots, \alpha_t$ in P; and $c_1, \dots c_{b_1}$ in C so that

$$d_1^{[p]}(v) = \sum_{i=1}^{t} \alpha_i s_i^p + \sum_{i=1}^{b_1} a_i^p c_i.$$

Of course, there exists $v_i \in P^{b_1}$ with $d_1(v_i) = s_i$ (and therefore also $d_1^{[p]} v_1^{[p]} = s_i^p$). So,

$$d_1^{[p]}(v) = d_1^{[p]} \left(\sum_{i=1}^{t} \alpha_i v_i^{[p]} + \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix} \right).$$

So,

$$v - \sum_{i=1}^{t} \alpha_i v_i^{[p]} - \begin{bmatrix} c_1 \\ \vdots \\ c_{b_1} \end{bmatrix}$$

is killed by $d_1^{[p]}$; hence is in the image of $(d_2')^{[p]}$. Finally, $d_1(v_i) = s_i \in I \cap C$, so $v_i = d_2''(w_i)$ for some w_i; hence, $v_i^{[p]} = (d_2'')^{[p]}(w_i^{[p]})$. Thus,

$$v \in \operatorname{im} d_2^{[p]} + CP^{b_1},$$

as desired.