Socle degrees of Frobenius powers
Lecture 1 — January 18, 2006
talk by A. Kustin

I will talk about recent joint work with Adela Vraciu. A preprint is available if you want all of the details. I will only talk about parts of the proofs. The rings that we study all have the form $S = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$, where k is a field of positive characteristic p and each f_i is a homogeneous polynomial. Often S will be finite dimensional as a vector space over k.

Definition. If the ring S is a finite dimensional vector space over k, then the *socle* of S is the set of elements in S that are killed by the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$.

Remark. According to Eisenbud, "socle" is an architectural term. It refers to the base of a column.

Examples. A. If $S = k[x, y]/(x^2, y^2)$, then the socle of S has basis xy.

B. If $S = k[x, y]/(x^2, xy, y^2)$, then the socle of S has basis x, y.

C. If $S = k[x, y, z]/(x^2, xz, xy + z^2, yz, y^2)$, then the socle of S has basis xy.

Comment. If one thinks about the socle of S, then one is viewing S in some upside-down (i.e., dual) sense. Each (homogeneous) element in S is reached from the top by multiplication (and addition) starting at 1. On the other hand, if you start with an arbitrary (homogeneous) element of S, then by multiplying by the appropriate sequence of generators of $\mathfrak{m}$, then you will land in the socle of S.

The question. Let $R = P/C$, where C is a homogeneous ideal in the polynomial ring $P = k[x_1, \ldots, x_n]$, and let J be a homogeneous ideal of R, with R/J a finite dimensional vector space over k. Adela asked, how do the degrees of the basis for the socle of $R/J^{[p^e]}$ vary as e increases? Adela is especially interested in asymptotic behavior. (**Note:** If the ideal J is generated by $a_1, \ldots, a_t$, then $J^{[q]}$ is the ideal generated by $a_1^q, \ldots, a_t^q$. We see that the ideal $J^{[q]}$ depends on the ideal J and not the particular generating set, provided q is a power of the characteristic p of k.)

Examples. 1. If $P = k[x, y]$, $C = 0$, $R = P/C = P$, and $J = (x^2, y^2)$, then $J^{[q]} = (x^{2q}, y^{2q})$, and $x^{2q-1}y^{2q-1}$ is a basis for the socle of $R/J^{[q]}$, where $q = p^e$. We see

e	socle degree of $R/J^{[q]}$
0	2
1	$2(2p - 1)$
2	$2(2p^2 - 1)$
3	$2(2p^3 - 1)$.

There is a very pretty pattern. If the socle degree of $R/J^{[q]}$ is d, then the socle degree of $R/J^{[pq]}$ is $pd+2(p-1)$:

$$\begin{aligned} p2+2(p-1)&=4p-2\\ p(4p-2)+2(p-1)&=4p^2-2\\ p(4p^2-2)+2(p-1)&=4p^3-2. \end{aligned}$$

2. If $P=k[x,y]$, $C=(x^2)$, $R=k[x,y]/(x^2)$, and $J=(x^2,y^2)R$, then $J^{[q]}=(x^{2q},y^{2q})R=(x^2,y^{2q})R$, and xy^{2q-1} is a basis for the socle of $R/J^{[q]}$, where $q=p^e$. We see

e	socle degree of $R/J^{[q]}$
0	2
1	$2p$
2	$2p^2$
3	$2p^3$.

There is a very pretty pattern. If the socle degree of $R/J^{[q]}$ is d, then the socle degree of $R/J^{[pq]}$ is pd.

3. If $P=k[x,y]$, $C=(x^2,y^2)$, $R=k[x,y]/(x^2,y^2)$, and $J=(x^2,y^2)R=(0)$, then $J^{[q]}=(x^2,y^2)$, and xy is a basis for the socle of $R/J^{[q]}$, where $q=p^e$. We see

e	socle degree of $R/J^{[q]}$
0	2
1	2
2	2
3	2.

There is a very pretty pattern. If the socle degree of $R/J^{[q]}$ is d, then the socle degree of $R/J^{[pq]}$ is d.

4. We calculate the socle degrees of $R/J^{[p^e]}$ for $R=\mathbb{Z}/2[x,y,z]/(f)$, where $J=(x^2,xz,y^2,yz,xy+z^2)$ and $f=x^3+y^3+z^3$. We learn

e	socle degrees
0	$2\colon 1$
1	$4\colon 7$
2	$9\colon 12$
3	$19\colon 12$
4	$39\colon 12$.

A Macaulay session which gave this information may be viewed from the seminar webpage. **After a while:** if the socle degrees of $R/J^{[q]}$ are $\{d_i\}$, then the socle degree of $R/J^{[pq]}$ are $\{pd_i + 1\}$.

5. We calculate the socle degrees of $R/J^{[p^e]}$ for $R = \mathbb{Z}/2[x, y, z]/(f)$, where $f = x^3 + y^3 + z^3$ and $J = (x, y, z)$. We learn

e	socle degrees
0	$0{:}1$
1	$3{:}1$
2	$6{:}2$
3	$12{:}2$
4	$24{:}2$.

A Macaulay session which gave this information may be viewed from the seminar webpage. **After a while:** if the socle degrees of $R/J^{[q]}$ are $\{d_i\}$, then the socle degree of $R/J^{[pq]}$ are $\{pd_i\}$.

6. We calculate the socle degrees of $R/J^{[p^e]}$ for for $R = \mathbb{Z}/2[x, y, z]/(f)$, where $f = x^5 + y^5 + z^5$ and $J = (x, y, z)$. We learn

e	socle degrees
0	$0{:}1$
1	$3{:}1$
2	$9{:}1$
3	$12{:}1$ $16{:}1$
4	$22{:}1$ $30{:}1$
5	$42{:}1$ $58{:}1$.

A Macaulay session which gave this information may be viewed from the seminar webpage. **After a while:** if the socle degrees of $R/J^{[q]}$ are $\{d_i\}$, then the socle degree of $R/J^{[pq]}$ are $\{pd_i - (p-1)2\}$.

7. One of the first things that Adela told me about this game is that if $S = R/J$ has finite projective dimension as an R-module, then then the socle degrees of $R/J^{[p]}$ are $\{pd_i - (p-1)a(R)\}$ where the socle degrees of $S = R/J$ are $\{d_i\}$ and $a(R)$ is the a-invariant of R; furthermore, if $R = k[x_1, \dots, x_n]/(f_1, \dots, f_m)$ and the f's are a regular sequence, then

$$a(R) = \sum |f_i| - \sum |x_i|.$$

In particular, in

Example	$a(R)$	Does $D_i = pd_i - (p-1)a(R)$?	Proj. dim. of $R/J^{[q]}$ over R
1	-2	yes	Of course, finite.
2	0	yes	In fact, finite.
3	2	Actually, yes	finite, since $R = S$!
4	0	eventually, off by 1	infinite, maybe the resolution stabalizes as e varies! This is too crazy a thought to be called a conjecture.
5	0	eventually, yes	Macaulay says finite at the moment that the formula starts to hold.
6	2	eventually, yes	Macaulay says finite at the moment that the formula starts to hold.

Examples like the above caused Adela and me to wonder if the converse to Example 7 might possibly be try. That is:

Question. *Let R be P/C, where P is a polynomial ring $k[x_1, \dots, x_n]$ and C is a homogeneous ideal in P. Let J be a homogeneous ideal in R with R/J a finite dimensional vector space over k. Suppose that socle degrees of R/J are $d_1 \le \cdots \le d_\ell$ and that the socle degrees of $R/J^{[p]}$ are $D_1 \le \cdots \le D_\ell$. Suppose further that*

$$D_i = pd_i - (p-1)a(R)$$

for all i. Does it follow that R/J has finite projective dimension over R?

The answer is yes, provided C is generated by a regular sequence. (Notice: One still might ask what happens if C is a Gorenstein ideal but not a complete intersection!?!)

Theorem. *Let R be P/C, where P is a polynomial ring $k[x_1, \dots, x_n]$ and C is a homogeneous complete intersection ideal in P. Let J be a homogeneous ideal in R with R/J a finite dimensional vector space over k. Suppose that socle degrees of R/J are $d_1 \le \cdots \le d_\ell$ and that the socle degrees of $R/J^{[p]}$ are $D_1 \le \cdots \le D_L$. Then the following statements are equivalent:*

1. $L = \ell$ and $D_i = pd_i - (p-1)a(R)$ for all i, and

2. The ring R/J has finite projective dimension as an R-module.

In these lectures, I will

1. prove a special case of (2) $\implies$ (1) (this is Example 7), and

2. prove parts of (1) $\implies$ (2).

You might find the order in which we established the result to be interesting. In May 2004, we proved the result when C had one generator and $J = IR$ where I is a Gorenstein ideal of P. (So, the socle of P/I had dimension one. In this situation there is the maximum amount of duality.) We stayed focused on the hypersurface case (C has one generator) but we were interested in what happens when the magic formula misses by one. We made no progress there and drifted to other projects.

Eventually, in Fall 05, we proved the above result when C is an arbitrary complete intersection, but J was still IR for some Gorenstein ideal I. Finally, after the Fall 2005 semester ended we learned how to remove the I is Gorenstein hypothesis. The funny thing is the proof is easier without the extraneous Gorenstein hypothesis. With the extra hypothesis we had to keep dualizing to go from one step to the next step. Once we removed the extra hypothesis, what we proved at one step was exactly what we needed at the next step.

Proposition. *Let J be a homogeneous ideal of the polynomial ring $P = k[x_1, \ldots, x_n]$, with P/J a finite dimensional vector space over k, and the characteristic of k equal to $p > 0$. Then, the socles of R/J and $R/J^{[p]}$ have the same dimension and if the socle degrees of R/J are $d_1 \leq \cdots \leq d_\ell$, then the socle degrees of $R/J^{[p]}$ are $D_1 \leq \cdots \leq D_\ell$ with $D_i = pd_i - (p-1)a(R)$.*

Proof. The ring P is regular. Every P-module has a finite resolution by free P-modules. Let

$$\mathbb{F}: \quad 0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0$$

be the minimal homogeneous resolution of P/J by free P-modules, with $F_n = \bigoplus_i P(-b_i)$. There are two ingredients to the proof.

(1) The number of back twists in $\mathbb{F}$ is exactly equal to the dimension of the socle of P/J; furthermore, b_i and $d_i = b_i + a(P)$.

(2) One obtains the minimal free resolution of $P/J^{[p]}$ by applying the Frobenius functor to $\mathbb{F}$.

As soon as you buy (1) and (2), then the proof is complete. Ingredient (2) tells us that the back twists in the P-resolution of P/J are pb_i, with $1 \leq i \leq \ell$. Thus, by (1):

$$D_i = pb_i + a(P) = p(b_i + a(P)) - (p-1)a(P) = pd_i - (p-1)a(P).$$

A quick illustration. Let $P=k[x,y]$ and $J=(x^2,xy,y^2)$. The P-resolution of P/J is

$$0\to P(-3)^2 \xrightarrow{\begin{bmatrix} x & 0\\ y & x\\ 0 & y\end{bmatrix}} P(-2)^3 \xrightarrow{\begin{bmatrix} y^2 & -xy & x^2\end{bmatrix}} P.$$

The resolution of $P/J^{[p]}$ is

$$0\to P(-3p)^2 \xrightarrow{\begin{bmatrix} x^p & 0\\ y^p & x^p\\ 0 & y^p\end{bmatrix}} P(-2p)^3 \xrightarrow{\begin{bmatrix} y^{2p} & -x^py^p & x^{2p}\end{bmatrix}} P.$$

We have $a(P)=-2$. We saw that x and y form a basis for the socle of P/J. So the socle degrees of P/J are $d_1=1\le d_2=1$. The back twists in the resolution of P/J are $b_1=3\le b_2=3$. We see that $3-2=1$, so $b_i+a(P)=d_i$. We also see that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are in the socle of $P/J^{[p]}$. One can show that $x^{p-1}y^{2p-1}$ and $x^{2p-1}y^{p-1}$ are a basis for the socle of $P/J^{[p]}$. So the socle degrees of $P/J^{[p]}$ are $D_1=3p-2\le D_2=3p-2$; the back twists in the resolution of $P/J^{[p]}$ are $B_1=3p\le B_2=3p$; and $a(P)$ is still -2. We have $D_i=B_i+a(P)$ and also $D_i=pd_i-(p-1)a(P)$, for both i.

Ingredient (2), in the present form, is due to Kunz (1969) – this is the paper that got commutative algebraists (especially Peskine, Spziro, Hochster) using Frobenius methods. One could also think of this assertion as an application of "What makes a complex exact?" (John Olmo lectured on this last Fall). The complex $\mathbb{F}$ is a resolution, so the ranks of its matrices behave correctly and the grade of the ideals of matrix minors grow correctly. If one raises each entry of each matrix to the p^{th} power, then the ranks of the new matrices are the same as the ranks of the old matrices (since $\det M^{[p]}=(\det M)^p$ because the characteristic of the ring is p), and the grade of the ideals of minors also remains unchanged!

I will give two explanations for ingredient (1). The quick argument is that one may commute $\operatorname{Tor}_n^P(P/J,k)$ using either coordinate. If one resolves P/J, then applies $\_\otimes_P k$, and then computes homology, then one sees that

$$\operatorname{Tor}_n^P(P/J,k)=\bigoplus_i k(-b_i).$$

In other words, the generators of Tor have degrees $b_1\le\dots$. On the other hand, if one resolves k, then applies $P/J\otimes_P \_$, and then computes homology, then one sees

that
$$\operatorname{Tor}_n^P(P/J,k)=\bigoplus_i \frac{J:\mathfrak{m}}{J}(a(P)).$$
In other words, the the generators of Tor have degrees $d_1-a(P),\dots$ (where the socle degrees of P/J are $d_1,\dots$). So $d_i=b_i+a(P)$ as claimed.

My second argument is exactly the same as my first, except, instead of stating the abstract result that Tor may computed in either coordinate, I reprove this result, giving a construction which associates an element of the socle of P/J to each basis vector at the back of the resolution of of P/J. The constructive argument takes longer, but shows what is really happening. Let $\mathbb{F}$ be a resolution of P/J, as above. Let $\mathbb{G}$ be the Koszul complex which resolves k. One can directly show that there is an isomorphism

$$(*)\qquad H_n(\mathbb{F}\otimes k)\cong H_n(\operatorname{Tot}(\mathbb{F}\otimes\mathbb{G}))\cong H_n(\tfrac{P}{J}\otimes\mathbb{G}).$$

Note to my audience. Lucho Avramov (UNL) taught me the trick of replacing a module with its resolution. Furthermore, this is the first thought about "triangulated categories". That is, rather than look at a category of modules, one looks at a category of complexes mod quasi-isomorphisms.

Anyhow, I think that the best way to convey the idea of (*) is to work out the example where $J=(x^2,xy,y^2)$. In this case, $\mathbb{F}$ is

$$0\to\underbrace{P(-3)^2}_{F_2}\xrightarrow{f_2=\begin{bmatrix}x&0\\y&x\\0&y\end{bmatrix}}\underbrace{P(-2)^3}_{F_1}\xrightarrow{f_1=\begin{bmatrix}y^2&-xy&x^2\end{bmatrix}}\underbrace{P}_{F_0},$$

$\mathbb{G}$ is

$$0\to\underbrace{P(-2)}_{G_2}\xrightarrow{g_2=\begin{bmatrix}y\\-x\end{bmatrix}}\underbrace{P(-1)^2}_{G_1}\xrightarrow{g_1=\begin{bmatrix}x&y\end{bmatrix}}\underbrace{P}_{G_0},$$

and $\mathbb{F}\otimes\mathbb{G}$ is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & F_2\otimes G_2 \arrow[r, "f_2\otimes 1"] \arrow[d, "1\otimes g_2"'] & F_1\otimes G_2 \arrow[r, "f_1\otimes 1"] \arrow[d, "1\otimes g_2"'] & F_0\otimes G_2 \arrow[d, "1\otimes g_2"'] \\
0 \arrow[r] & F_2\otimes G_1 \arrow[r, "f_2\otimes 1"] \arrow[d, "1\otimes g_1"'] & F_1\otimes G_1 \arrow[r, "f_1\otimes 1"] \arrow[d, "1\otimes g_1"'] & F_0\otimes G_1 \arrow[d, "1\otimes g_1"'] \\
0 \arrow[r] & F_2\otimes G_0 \arrow[r, "f_2\otimes 1"] & F_1\otimes G_0 \arrow[r, "f_1\otimes 1"] & F_0\otimes G_0.
\end{tikzcd}$$

Start with $\begin{bmatrix}1\\0\end{bmatrix}\otimes 1$ in $F_2\otimes G_0$ in the lower left hand corner. We see that this element represents an element of the homology of $H_2(\mathbb{F}\otimes k)$. One can extend this element to get an element of the homology of $H_2(\operatorname{Tot}(\mathbb{F}\otimes\mathbb{G}))$:

$$\begin{tikzcd}
 & & 1\otimes y \arrow[d] \\
 & \begin{bmatrix}1\\0\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\\0\end{bmatrix}\otimes\begin{bmatrix}0\\1\end{bmatrix} \arrow[r] \arrow[d] & 1\otimes\begin{bmatrix}y^2\\-xy\end{bmatrix} \\
\begin{bmatrix}1\\0\end{bmatrix} \arrow[r] & \begin{bmatrix}x\\y\\0\end{bmatrix} &
\end{tikzcd}$$

The indicated element of $H_2(\operatorname{Tot}(\mathbb{F}\otimes\mathbb{G}))$ gives rise to the element y of the socle of P/J. To answer the question that our freshman ask: "Yes, it always works like that." We can use the idea of the snaky game to prove both isomorphisms in (*).