"DIVISORS OVER DETERMINANTAL RINGS DEFINED BY TWO BY TWO MINORS"

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When I first thought about giving this talk, I was attracted to the topic because I figured I could say something coherent and interesting. As I started to prepare the talk, I realized that there is an extra reason to give it here at Purdue: two of the players in talk were students at Purdue Anna Guerrieri (student of Huneke) and Alex Tchernev (student of Avramov).

Let G be a free module over the ring R. The complex

$$\mathbb{C}_n\colon \quad \cdots \to \operatorname{Sym}_{n-2} G \otimes \bigwedge^2 G \to \operatorname{Sym}_{n-1} G \otimes \bigwedge^1 G \to \operatorname{Sym}_n G \otimes \bigwedge^0 G \to 0$$

is well known and well-understood. This is one graded strand of the Koszul complex. If $x_1, \dots, x_g$ is a basis for G, then $\mathbb{C}_n$ is the strand from the $R[x_1, \dots, x_g]$-resolution of R which consists of the homogeneous elements of total degree n. The homology of $\mathbb{C}_0$ is R in position zero, and each of the other $\mathbb{C}_n$ are split exact.

The situation is much different if two free modules are involved: E^* and G. In this case, I look at

$$(*) \qquad \cdots \to \operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G \otimes \bigwedge^p (E^* \otimes G) \to \operatorname{Sym}_{m+1} E^* \otimes \operatorname{Sym}_{n+1} G \otimes \bigwedge^{p-1} (E^* \otimes G) \to \dots .$$

• Many of the complexes (*) have homology.
• The homology of (*) may occur in the middle and not necessarily at the left or right end.
• (*) may have homology in more than one position.
• The homology of (*) is not always a free R-module.
• If R is a field, then the dimension of the homology of (*) depends on the characteristic of R.

The first three statements are not particularly shocking. The fourth and fifth statements are eye-opening (I think) and are essentially the same assertion.

The fourth and fifth statements are due to Hashimoto (1990) – representation theory argument. An argument from algebraic topology is given by Björner, Lovász, Verćia, Živaljević (1994). I don't think that [BLVZ] were aware of [H].

Here is the plan of today's talk.

(1) Interpret the homology of (*) in terms of the resolution of divisors over determinantal rings. Bruns and Guerrieri call these divisors $\{M_\ell\}$; so we connect the homology of (*) at (m, n, p) to $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, R)$. ($\mathcal{P}$ is a polynomial ring to be described later). This is why [H] applies.
(2) Other contexts in which the homology of (*) arises.
  (a) Resolutions of universal rings (I learned about this from Tchernev).
  (b) Chessboard complexes (here "complex" means "simplicial complex") from [BLVZ].
  (c) Matching graphs.
(3) The resolution of M_ℓ (or the calculation of $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, R)$).
  (a) Lascoux approach (R is a field of characteristic zero).
  (b) The beautiful description given by Reiner-Roberts (R is a field of characteristic zero).
  (c) A consequence of the R-R description at the CM-boundary (This works over all rings. I had established it before I knew about R-R, but R-R gives a very cute proof when it applies!)
  (d) Repair (*) to make it become exact. (This works over all rings.)

Open Question. $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, R)$ *is known when* R *is a field of characteristic zero. However,* $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, R)$ *is not known in general* (*i.e., when* $R = \mathbb{Z}$ *or* R *is a field of prime characteristic.*)

Amplification.
a. If $\min\{e, g\} = 2$, then $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, \mathbb{Z})$ is known (by Eagon Northcott).
b. If $\min\{e, g\} = 3$, then I have calculated that $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, \mathbb{Z})$ is a free abelian group (and hence is described by R-R).
c. I GUESS that if $\min\{e, g\} = 4$, then $\mathrm{Tor}^{\mathcal{P}}_{p,q}(M_\ell, \mathbb{Z})$ is a free abelian group (and hence is described by R-R), but I do not know if anyone has established this.
d. So the open question "really" is about $\min\{e, g\} \geq 5$.

1. Interpret the homology of (*) in terms of $\operatorname{Tor}^{\mathcal{P}}_{p,q}(M_\ell, R)$.

• Let S be the R-algebra $\operatorname{Sym}_\bullet E^* \otimes \operatorname{Sym}_\bullet G$. If we fix bases $v_1, \dots, v_e$ for E^*, and $x_1, \dots, x_g$ for G, then one may think of S as the polynomial ring

$$S = R[v_1, \dots, v_e, x_1, \dots, x_g].$$

• Let T be the subring

$$T = \sum_m \operatorname{Sym}_m E^* \otimes \operatorname{Sym}_m G$$

of S. One may think of T as the subring $R[v_i x_j]$ of S.

• Let $\mathcal{P}$ be the R-algebra $\operatorname{Sym}_\bullet(E^* \otimes G)$. One may think of $\mathcal{P}$ as a polynomial ring over R in the eg indeterminates $\{v_i \otimes x_j\}$. It is convenient to let z_{ij} represent the element $v_i \otimes x_j$ of $\mathcal{P}$.

• The identity map on $E^* \otimes G$ induces a surjective map $\varphi\colon \mathcal{P} \to T$. Let Z be the $e \times g$ matrix whose entry in row i column j is the indeterminate z_{ij}. The kernel of φ is the ideal $I_2(Z)$ generated by the 2×2 minors of Z; and therefore, T is isomorphic to the determinantal ring $\mathcal{P}/I_2(Z)$.

• Hashimoto proved that if e and g are both at least five, then $\operatorname{Tor}^{\mathcal{P}}_{3,5}(T, \mathbb{Z})$ is not a free $\mathbb{Z}$-module.

• On the other hand, the Koszul complex $\mathcal{P} \otimes_R \bigwedge^\bullet(E^* \otimes G)$ is a homogeneous resolution of the $\mathcal{P}$-module R. It follows that

$$\operatorname{Tor}^{\mathcal{P}}_{p,n+p}(T, R) = \mathrm{H} \text{ of } (*) \text{ at } (n, n, p).$$

• There is a determinantal interpretation of the complexes (*), even when $m \neq n$. For each integer ℓ, let M_ℓ be the T-submodule

$$M_\ell = \sum_{m-n=\ell} \operatorname{Sym}_m E^* \otimes \operatorname{Sym}_n G$$

of S. **View M_ℓ as a graded T-module by giving $\operatorname{Sym}_{n+\ell} E^* \otimes \operatorname{Sym}_n G$ grade n.** The same reasoning we used before shows that

$$\operatorname{Tor}^{\mathcal{P}}_{p,n+p}(M_{m-n}, R) = \mathrm{H} \text{ of (*) at } (m,n,p).$$

• Take $R = \mathbb{Z}$. The divisor class group of T is known to be $\mathbb{Z}$ and [BG] shows why $\ell \mapsto [M_\ell]$ is an isomorphism from $\mathbb{Z} \to \mathrm{C}\ell\,(T)$. This numbering satisfies $M_0 = T$, M_{g-e} is equal to the canonical class of T, and M_ℓ is a Cohen-Macaulay T-module if and only if $1 - e \le \ell \le g-1$.

2. Other contexts in which the homology of (*) arises.

(a) Resolutions of universal rings. This is how I became interested in the subject.

For any triple of parameters e, f, and g, subject to the obvious constraints, Hochster established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution

$$\mathbb{U}\colon \quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g,$$

such that for any commutative noetherian ring S and any resolution

$$\mathbb{V}\colon \quad 0 \to S^e \to S^f \to S^g,$$

there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$.

In the border case $f = e + g$ (one of the obvious constraints is $g - e + f \ge 0$), $\mathcal{R}$ is $\mathfrak{P}/\mathcal{J}$, where

$$\mathfrak{P} = \mathbb{Z}[\text{ the entries of each matrix, and one Buchsbaum-Eisenbud multiplier}]$$

and $\mathcal{J}$ sets the entries of the composition equal to zero and makes the multiplier be a multiplier.

Theorem. *If $\boldsymbol{K}$ is a field, then the minimal resolution of $\mathcal{R} \otimes_{\mathbb{Z}} \boldsymbol{K}$ by free $\mathfrak{P} \otimes_{\mathbb{Z}} \boldsymbol{K}$-modules "is"*

$$\begin{cases} \displaystyle\bigoplus_{\substack{p,q \\ -e \le \ell \le g-1}} \mathfrak{P} \otimes_{\boldsymbol{K}} \operatorname{Tor}^{\mathcal{P}}_{p,q}(M_\ell, \boldsymbol{K}) \\ \qquad\qquad \oplus \\ \displaystyle\bigoplus_{p,q} \mathfrak{P} \otimes_{\boldsymbol{K}} \frac{\bigwedge^{\#}(E^* \otimes G)}{\operatorname{Tor}^{\mathcal{P}}_{p,q}(M_g, \boldsymbol{K})} \end{cases}$$

(b) Chessboard complexes. Consider all legal rook configurations an an $e \times g$ chess board – no more than one rook per row, no more than one rook per column. Create a simplicial complex $\Delta_{e,g}$. The vertices are the squares of the Chessboard. The simplicies are the legal rook configurations. [BLVZ] proved that H_2 of $\Delta_{5,5}$ has 3-torsion.

If γ and δ are vectors of non-negative integers, then one can focus on the homogeneous submodule of $\operatorname{Tor}_p^{\mathcal{P}}(M_\ell, \boldsymbol{K})$ which involves $v_i^{\gamma_i}$ and $x_j^{\delta_j}$ for all i and j. Call this submodule $\operatorname{Tor}_p^{\mathcal{P}}(M_\ell, \boldsymbol{K})_{\gamma,\delta}$. One can also focus on the "chessboard with multiplicities" (named by Bruns and Herzog) $\Delta_{\gamma,\delta}$ where no more than γ_i squares from row i and no more than δ_j squares from column j are used (so, $\Delta_{e,g} = \Delta_{\gamma,\delta}$ where each γ_i and each δ_j is 1). A fairly straightforward calculation about modules defined over semigroup rings (this result was published by Bruns-Herzog, Stanley, Reiner-Roberts, and Sturmfels) shows that

$$\operatorname{Tor}_p^{\mathcal{P}}(M_\ell, \boldsymbol{K})_{\gamma,\delta} = \widetilde{\mathrm{H}}_{p-1}(\Delta_{\gamma,\delta}, \boldsymbol{K}).$$

(c) The matching complex of a complete bipartite graph.

Let G be a graph. The matching complex of G is the simplicial complex whose vertex set is the set of edges of G and whose faces are sets of edges with no two edges meeting at a vertex. For example, if G is the complete bipartite graph on $\{1,2,3\}$ and $\{a,b,c\}$, then the simplicies of the corresponding matching complex exactly correspond to legal rook configurations on the chessboard labeled $\{1,2,3\}$ down the side and $\{a,b,c\}$ across the top. We conclude that the matching complex of a complete bipartite graph is a chessboard complex.

3. The resolution of M_ℓ (i.e., the calculation of $\operatorname{Tor}_{p,q}^{\mathcal{P}}(M_\ell, \boldsymbol{K})$.

(a) The Lascoux approach. Lascoux knows how to resolve determinantal rings over fields of characteristic zero. Resolve the singularity. Now the resolution is given by a Koszul complex. Use the Bott isomorphism Theorem to push the Koszul complex back to the original polynomial ring. Weyman's book shows how to use the same basic approach to resolve M_ℓ for all ℓ.

Weyman and I used the Lascoux approach to resolve the Universal rings $\mathcal{R}$ when $\boldsymbol{K}$ is a field of characteristic zero. So, in fact we resolved the M_ℓ; although we did not stop and circle: here is our formula for the resolution of M_ℓ. We did tidy our answer a great deal; nonetheless, we did not get an answer that is nearly as pretty as the Reiner-Roberts answer. (As soon as I saw the R-R answer, I saw how to tidy the KW answer into their form.)

(b) THE BEAUTIFUL DESCRIPTION OF $\operatorname{Tor}^{\mathcal{P}}_{p,q}(M_\ell, \boldsymbol{K})$ GIVEN BY REINER-ROBERTS.

Theorem. *Let $\boldsymbol{K}$ be a field of characteristic zero. For each integer ℓ,*

$$\operatorname{Tor}^{\mathcal{P}}_{\bullet,\bullet}(M_\ell, \boldsymbol{K}) = \bigoplus_{(\lambda,\mu)} S_\lambda E^* \otimes_{\boldsymbol{K}} S_\mu G,$$

where (λ, μ) have the form

$$\left(\begin{array}{ll} \boxed{}_{(s+1-\ell)\times s} & \beta \\ \alpha & \end{array} \quad , \quad \begin{array}{ll} \boxed{}_{(s+1)\times(s-\ell)} & \alpha' \\ \beta' & \end{array} \right),$$

for some integer s and partitions α and β.

We use $S_\lambda G = L_{\lambda'} G = K_\lambda G$.

(c) A CONSEQUENCE OF THE R-R DESCRIPTION AT THE CM-BOUNDARY.

Return to (*) with $m-n = -e$ or $m-n = g$. It turns out that the only homology in

$$0 \to \operatorname{Sym}_0 E^* \otimes \operatorname{Sym}_e G \otimes \textstyle\bigwedge^p (E^* \otimes G) \to \cdots \to \operatorname{Sym}_p E^* \otimes \operatorname{Sym}_{p+e} G \otimes \textstyle\bigwedge^0 (E^* \otimes G) \to 0$$

and

$$0 \to \operatorname{Sym}_g E^* \otimes \operatorname{Sym}_0 G \otimes \textstyle\bigwedge^p (E^* \otimes G) \to \cdots \to \operatorname{Sym}_{g+p} E^* \otimes \operatorname{Sym}_p G \otimes \textstyle\bigwedge^0 (E^* \otimes G) \to 0$$

occurs at the left side and that $\bigwedge^{e+p}(E^* \otimes G)$ (and $\bigwedge^{g+p}(E^* \otimes G)$) maps onto to this homology and if one pairs one of these complexes with the dual of the appropriate other complex one creates a split exact complex

$$\cdots \to D_g E \otimes D_0 G^* \otimes \textstyle\bigwedge^{p'} (E \otimes G^*) \to \textstyle\bigwedge^{e+p} (E^* \otimes G) \to \operatorname{Sym}_0 E^* \otimes \operatorname{Sym}_e G \otimes \textstyle\bigwedge^p (E^* \otimes G) \to \dots,$$

where $p + p' = eg - e - g$. (I proved this. It works over every ring.)

It turns out that one can deduce the numerical consequences of the above fact, when R is a field of characteristic zero, from R-R:

$$\dim \operatorname{Tor}_{p,p+e}(M_{-e}, \boldsymbol{K}) + \dim \operatorname{Tor}_{p',p'}(M_g, \boldsymbol{K}) = \dim \textstyle\bigwedge^{e+p} (E^* \otimes G).$$

Proof. Use the R-R description to see that

$$\dim \operatorname{Tor}_{p,p+e}(M_{-e},\boldsymbol{K}) = \sum_{\substack{\beta\subseteq e\times g\\ |\beta|=p+e\\ \beta'_1=e}} \dim\left(S_\beta E^* \otimes S_{\beta'} G\right) \quad s=0$$

and

$$\dim \operatorname{Tor}_{p',p'}(M_g,\boldsymbol{K}) = \sum_{\substack{\alpha\subseteq e\times g\\ |\alpha|=p'+g\\ \alpha_1=g}} \dim\left(S_\alpha E^* \otimes S_{\alpha'} G\right) \quad s=g.$$

The dual of $S_\alpha E^*$ is

$$S_{-\alpha_e,\dots,-\alpha_1}E = S_{g-\alpha_e,\dots,g-\alpha_1}E \otimes \underbrace{(\textstyle\bigwedge^e E^*)^{\otimes g}}_{\dim=1}.$$

Let $\beta = g-\alpha_e,\dots,g-\alpha_1$. Observe that $\beta\subseteq e\times g$, $|b| = eg-|\alpha|$, and $\alpha_1=g \iff \beta'_1<e$. We conclude that

$$\dim \operatorname{Tor}_{p,p+e}(M_{-e},\boldsymbol{K}) + \dim \operatorname{Tor}_{p',p'}(M_g,\boldsymbol{K}) = \sum_{\substack{\beta\subseteq e\times g\\ |\beta|=p+e}} \dim\left(S_\beta E^*\otimes S_{\beta'}G\right) = \dim \textstyle\bigwedge^{e+p}(E^*\otimes G).$$

(The last equality is the Cauchy Formula.) $\square$

(d) Repair (*) to make it become split exact.

Let the homology of (*) at (m,n,p) be called $\mathrm{H}_{m,n,p}$ and let the cohomology of the dual of (*):

$$\cdots \to D_m E\otimes D_n G^*\otimes \textstyle\bigwedge^p(E\otimes G^*)\to \dots$$

at (m,n,p) be called $\mathrm{H}^{m,n,p}$. It turns out that in the Cohen-Macaulay range $1-e\le m-n\le g-1$,

$$\mathrm{H}_{m,n,p}\cong \mathrm{H}^{m',n',p'}$$

provided $m+m'=g-1$, $n+n'=e-1$ and $p+p'=(e-1)(g-1)$. Furthermore, there exists a map of complexes

$$\begin{array}{ccccc} \dots & \longrightarrow & D_{m'}E\otimes D_{n'}G^*\otimes \bigwedge^{p'}(E\otimes G^*) & \longrightarrow & \dots \\ & & \big\downarrow & & \\ \dots & \longrightarrow & \operatorname{Sym}_m E^*\otimes \operatorname{Sym}_n G\otimes \bigwedge^p(E^*\otimes G) & \longrightarrow & \dots \end{array}$$

For example, the complex

$$\cdots \to S_{g-1}E^* \otimes S_{e-1}G \otimes \bigwedge^{(e-1)(g-1)}(E^* \otimes G) \to \cdots \to S_{(g-1)e}E^* \otimes S_{(e-1)g}G \otimes \bigwedge^0(E^* \otimes G) \to 0$$

has free homology of rank one concentrated in position $(g-1, e-1, (e-1)(g-1))$.