SOCLE DEGREES, RESOLUTIONS, AND FROBENIUS POWERS

The set up.
- k is a field of characteristic $p > 0$.
- R is a graded ring over k.
- $\mathfrak{m}$ is the maximal homogeneous ideal of R.
- J is a homogeneous $\mathfrak{m}$-primary ideal of R.

The question.
- *Adela asked "How are the socle degrees of $R/J^{[q]}$ related to the socle degrees of R/J?*

The notation of the question.
- The socle of the ring S is $\{s \in S \mid \mathfrak{m}_S s = 0\}$.
- $q = p^e$ for some exponent e.
- $J^{[q]} = (\{j^q \mid j \in J\})$.

Example. We calculate the socle degrees of $R/J^{[p^e]}$ for for $R = \mathbb{Z}/2[x,y,z]/(f)$, where $f = x^5 + y^5 + z^5$ and $J = (x,y,z)$. We learn

e	socle degrees	socle basis
0	$0\colon 1$	1
1	$3\colon 1$	xyz
2	$9\colon 1$	$x^3y^3z^3$
3	$12\colon 1$ $16\colon 1$	$x^4y^4z^4, x^2y^7z^7$
4	$22\colon 1$ $30\colon 1$	$x^4y^{14}z^4 + x^4y^9z^9 + x^4y^4z^{14}, y^{15}z^{15}$
5	$42\colon 1$ $58\colon 1$	$x^4y^{29}z^9 + x^4y^{19}z^{19} + x^4y^9z^{29}, xy^{26}z^{31}$

After a while: if the socle degrees of $R/J^{[q]}$ are $\{d_i\}$, then the socle degree of $R/J^{[pq]}$ are $\{pd_i - (p-1)2\}$.

Folklore. *If* $\mathrm{pd}_R\, R/J < \infty$*, then the socles of* R/J *and* $R/J^{[q]}$ *have the same dimension and if the socle degrees of* R/J *are* $d_1 \le \cdots \le d_s$ *then the socle degrees of* $R/J^{[q]}$ *are* $D_1 \le \cdots \le D_s$ *with* $D_i = qd_i - (q-1)a$*.*

Reason. In the above notation, the generator degrees of the canonical module ω of R/J are

$$-d_s \le \cdots \le -d_1.$$

The canonical module is

$$\mathrm{Ext}_R^{\mathrm{top}}(R/J, \omega_R),$$

and $\omega_R = R(a(R))$ if R is Gorenstein; thus, the degrees of the generators of ω are given by the back twists in the R-resolution $\mathbb{F}$ of R/J. The resolution of $R/J^{[q]}$ is $\mathbb{F}^{[q]}$.

Theorem [K,V]. *If* R *is a complete intersection, then the converse of folklore is true.*

Moral.

1. *At least sometimes, if you know the socle degrees of* R/J*, then you know the graded betti numbers in the tail of the resolution of* R/J*.*
2. *At least sometimes, if the socle degrees grow "correctly" as you apply the Frobenius homomorphism, then the tail of the resolution of* $R/J^{[p^e]}$ *is independent of* e*.*

Example. Adela and I found other examples in which the numbers made it **look like** the tail of the resolution of $R/J^{[p^e]}$ is independent of e

Let P be the polynomial ring $\frac{\mathbb{Z}}{(5)}[x,y,z]$, f be the element $x^3+y^3+z^3$ of P, R be the hypersurface ring $P/(f)$, and J be the ideal (x^5,y^5,z^5) of R. The graded betti numbers in the R-resolution of $R/J^{[p^e]}$ are:

$$\cdots \to \begin{matrix} R(-9)^1 \\ \oplus \\ R(-10)^3 \end{matrix} \to \begin{matrix} R(-8)^3 \\ \oplus \\ R(-9)^1 \end{matrix} \to R(-5)^3 \to R \to R/J^{[5^0]} \to 0.$$

$$\cdots \to \begin{matrix} R(-39)^1 \\ \oplus \\ R(-40)^3 \end{matrix} \to \begin{matrix} R(-38)^3 \\ \oplus \\ R(-39)^1 \end{matrix} \to R(-25)^3 \to R \to R/J^{[5^1]} \to 0.$$

$$\cdots \to \begin{matrix} R(-189)^1 \\ \oplus \\ R(-190)^3 \end{matrix} \to \begin{matrix} R(-188)^3 \\ \oplus \\ R(-189)^1 \end{matrix} \to R(-125)^3 \to R \to R/J^{[5^2]} \to 0.$$

$$\cdots \to \begin{matrix} R(-939)^1 \\ \oplus \\ R(-940)^3 \end{matrix} \to \begin{matrix} R(-938)^3 \\ \oplus \\ R(-939)^1 \end{matrix} \to R(-625)^3 \to R \to R/J^{[5^3]} \to 0.$$

$$\cdots \to \begin{matrix} R(-4689)^1 \\ \oplus \\ R(-4690)^3 \end{matrix} \to \begin{matrix} R(-4688)^3 \\ \oplus \\ R(-4689)^1 \end{matrix} \to R(-3125)^3 \to R \to R/J^{[5^4]} \to 0.$$

It **looks like** there is a resolution

$$\mathbb{F}: \quad \cdots \to \begin{matrix} R(-1)^1 \\ \oplus \\ R(-2)^3 \end{matrix} \to \begin{matrix} R^3 \\ \oplus \\ R(-1)^1 \end{matrix},$$

which is independent of e so that for each e there exists t_e so that the resolution of $R/J^{[p^e]}$ is

$$\mathbb{F}(-t_e) \to R(-5^{e+1}) \to R \to R/J^{[5^e]} \to 0.$$

In these examples I did row and column operations to the matrix in position 3. Each matrix can be transformed into

$$\begin{bmatrix} 0 & -x^2 & -y^2 & -2z \\ x^2 & 0 & -z^2 & 2y \\ y^2 & z^2 & 0 & -2x \\ 2z & -2y & 2x & 0 \end{bmatrix}.$$

The purpose of my talk.

1. *I will show a situation where the graded betti numbers in the tail of the resolution of R/J are completely determined the socle degrees of R/J.*
2. *I will apply apply (1) twice and obtain a situation where the tail of the resolution of $R/J^{[p^e]}$ is a shift of the tail of the resolution of R/J* **as a graded module – I make no claim about the differential at this point**.

Theorem [K,U]. *Let $P = k[x,y,z]$, $f \in P$ homogeneous , $R = P/(f)$, and $a = a(R) = |f| - 3$. Let I be a homogeneous grade three Gorenstein ideal in P, b_0 be the back twist in the P-resolution of $\frac{P}{I}$, and $J = IR$. Let*

$$\mathbb{F}_{0,\bullet}: \quad \dots \xrightarrow{d_{0,4}} \mathbb{F}_{0,3} \xrightarrow{d_{0,3}} \mathbb{F}_{0,2} \xrightarrow{d_{0,2}} \mathbb{F}_{0,1} \xrightarrow{d_{0,1}} R \to R/J \to 0$$

be the graded minimal R-resolution of R/J, and $\{\sigma_{0,i} \mid 1 \le i \le s_0\}$ be the socle degrees of $\frac{R}{J}$. Assume

(1) $\mu(I) = \mu(J)$,
(2) $\operatorname{rank} \mathbb{F}_{0,2} = \dim_k \operatorname{soc} \frac{R}{J}$, *and*
(3) $\sigma_{0,i} + \sigma_{0,j} \neq b_0 + 2a$ *for any pair* (i,j). *Then*

$$\begin{aligned} \mathbb{F}_{0,2} &= \bigoplus_{i=1}^{s_0} R(-(b_0 + a - \sigma_{0,i})), \\ \mathbb{F}_{0,3} &= \bigoplus_{i=1}^{s_0} R(-(\sigma_{0,i} + 3)), \text{ and} \\ \mathbb{F}_{0,i+2} &= \mathbb{F}_{0,i}(-|f|). \end{aligned}$$

Corollary [K,U]. *Assume all of the above and that $\mu(J^{[q]}) = \mu(I)$. If $\operatorname{soc} R/J^{[q]} = \operatorname{soc} R/J\left[-\frac{b_0(q-1)}{2}\right]$, then*

$$\mathbb{F}_{e,i} = \mathbb{F}_{0,i}\left[-\frac{b_0(q-1)}{2}\right], \quad \forall i \geq 2.$$

Proof of Corollary. The Corollary follows quickly from the Theorem. Make sure that all of the hypothesis apply to J and $J^{[q]}$.

Example. In the earlier example, $b_0 = 15$, $a = 0$, and the shift from J to $J^{[p^e]}$ is $\frac{15(5^e-1)}{2}$ and this is 30, 180, 930, and 4680 for e equal to 1, 2, 3, and 4.

Outline of the proof of the Theorem. Let $Z = \operatorname{im} d_{0,2}$. There are three parts to the proof.

Part 1. There exists $Z' \subset Z$ such that

$$\boxed{\omega(-b_0 - a) \cong \frac{I : f}{I}(-|f|) \cong \frac{Z}{Z'}.}$$

• Knowledge of the generator degrees of ω is equivalent to knowledge of the socle degrees.
• The generators of Z have the same degrees as the generators of $\mathbb{F}_{0,2}$.
• The hypothesis $\operatorname{rank} \mathbb{F}_{0,2} = \dim_k \operatorname{soc} \frac{R}{J}$ tells us that Z and $\frac{Z}{Z'}$ have the same generators.
• This finishes the $\mathbb{F}_{0,2}$ part of the argument.

Part 2. Eisenbud proved that if $R = P/(f)$ is a hypersurface ring and M is a maximal Cohen-Macaulay module over with no free summands, then M has a periodic resolution of period two given by a matrix factorization of "f". Our Z is $Z_{\text{periodic}} \oplus Z_{\text{free}}$. The maps $\mathbb{F}_{0,3} \to \mathbb{F}_{0,2} \to Z$ decompose as

$$\mathbb{F}_{0,3} \xrightarrow{\begin{bmatrix} d_{0,3,\text{periodic}} \\ 0 \end{bmatrix}} \begin{matrix} \mathbb{F}_{0,2,\text{periodic}} \\ \oplus \\ \mathbb{F}_{0,2,\text{free}} \end{matrix} \xrightarrow{\begin{bmatrix} d_{0,2,\text{periodic}} & 0 \\ 0 & \text{iso} \end{bmatrix}} \begin{matrix} Z_{\text{periodic}} \\ \oplus \\ Z_{\text{free}}. \end{matrix}$$

• Now one makes 2 fairly easy homological calculations:

$$\boxed{\begin{matrix} Z^*(a) \twoheadrightarrow \omega \quad \text{and} \\ \mathbb{F}_{0,3}^*(|f|) \quad \text{and} \quad (Z_{\text{periodic}})^* \quad \text{have the same generator degrees} \end{matrix}}$$

• At this point we know

$$\begin{aligned} \mu(Z^*) = \mu((Z_{\text{periodic}})^*) + \mu((Z_{\text{free}})^*) &= \operatorname{rank} \mathbb{F}_{0,2,\text{periodic}} + \operatorname{rank} \mathbb{F}_{0,2,\text{free}} \\ &= \operatorname{rank} \mathbb{F}_{0,2} = \mu(\omega). \end{aligned}$$

• So, $Z^*(a)$ and ω have the same generator degrees.
• As soon as we show that $Z_{\text{periodic}} = Z$, then we know the the relationship between the generator degrees of $\mathbb{F}_{0,3}^*$ and the generator degrees of ω. *This completes the proof of the Theorem.*

Part 3. If $\mathfrak{z}$ generates a free summand of Z, then
• the degree of the corresponding element in $Z^*(a)$ is a generator degree of ω, and
• the degree of $\mathfrak{z}$ is a generator degree of $\omega(-b_0 - a)$; however,
• the hypothesis $\sigma_{0,i} + \sigma_{0,j} \neq b_0 + 2a$ for any pair (i, j) prohibits the existence of such a $\mathfrak{z}$.

• **We prove the homological assertions of Part 2.**

• We produce $Z^*(a) \twoheadrightarrow \omega$.

The surjection $R \twoheadrightarrow R/J$ tells me that

$$\omega_{R/J} = \operatorname{Ext}_R^{\dim R - \dim R/J}(R/J, \omega_R) = \operatorname{Ext}_R^2(R/J, R(a)) = \operatorname{Ext}_R^1(J, R(a)).$$

Apply $\operatorname{Hom}_R(\_\_, R(a))$ to

$$0 \to Z \to \mathbb{F}_{0,1} \to J \to 0$$

to get

$$0 \to J^*(a) \to \mathbb{F}_{0,1}^*(a) \to Z^*(a) \to \operatorname{Ext}_R^1(J, R(a)) \to 0.$$

• We prove that $\mathbb{F}_{0,3}^*(|f|)$ and $(Z_{\text{periodic}})^*$ have the same generator degrees.

• There are two steps. The first is routine. Apply $\operatorname{Hom}(\_\_, R)$ to the exact sequence

$$F_{0,3} \xrightarrow{d_{0,3,\text{periodic}}} \mathbb{F}_{0,2,\text{periodic}} \to Z_{\text{periodic}} \to 0$$

to see that $(Z_{\text{periodic}})^* = \ker(d^*_{0,3,\text{periodic}})$.

• The other step is sneaky. Extend the periodic resolution one step to the right:

$$\to \mathbb{F}_{0,4} \xrightarrow{d_{0,4}} \mathbb{F}_{0,3} \xrightarrow{d_{0,3,\text{periodic}}} \mathbb{F}_{0,2,\text{periodic}} \xrightarrow{d_{0,4}(|f|)} \mathbb{F}_{0,3}(|f|) \xrightarrow{d_{0,3,\text{periodic}}(|f|)} \mathbb{F}_{0,2,\text{periodic}}(|F|) \longrightarrow Z_{\text{periodic}}(|f|) \to 0.$$

The module Z_{periodic} is a maximal Cohen-Macaulay module; so, $\operatorname{Ext}^i_R(Z_{\text{periodic}}, R) = 0$ for all positive i; hence,

$$0 \to (Z_{\text{periodic}}(|f|))^* \to (\mathbb{F}_{0,2,\text{periodic}}(|F|))^* \xrightarrow{(d_{0,3,\text{periodic}}(|f|))^*} (\mathbb{F}_{0,3}(|f|))^* \xrightarrow{(d_{0,4}(|f|))^*} (\mathbb{F}_{0,2,\text{periodic}})^* \xrightarrow{(d_{0,3,\text{periodic}})^*} (\mathbb{F}_{0,3})^*$$

is exact and

$$(Z_{\text{periodic}})^* = \ker(d^*_{0,3,\text{periodic}}) = \frac{(\mathbb{F}_{0,3}(|f|))^*}{\operatorname{im}((d_{0,3,\text{periodic}}(|f|))^*)}.$$

- **We prove the assertions of Part 1.**
  - We connect ω and $\frac{I:f}{I}$.

  The surjection $P/I \to R/J$ gives

$$\omega_{R/J} = \operatorname{Ext}^{\dim P/I - \dim R/J}(R/J, \omega_{P/I}) = \operatorname{Hom}(P/(I,f), P/I(a(P/I)) = \frac{I:f}{I}(b_0-3).$$

  - We connect $\frac{I:f}{I}$ and Z.

  Let $d_{1,0} = [\,\bar{g}_1, \dots, \bar{g}_n\,]$, where $(g_1, \dots, g_n)$ is a minimal generating set for I in P. Of course, Z is the kernel of $d_{1,0}$. If $u \in I : f$, then $uf = \sum_{i=1}^n A_i g_i$ for some A_i in P. The association

$$u \mapsto \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_n \end{bmatrix}$$

induces an isomorphism

$$\frac{I:f}{I}(-|f|) \to \frac{Z}{Z'},$$

where Z' is the submodule of Z which comes from relations on $[\,g_1, \dots, g_n\,]$ in P. $\square$

One Final Remark. The isomorphism $\frac{Z}{Z'} \cong \omega(-b_0 - a)$ shows that

$$\dim \operatorname{soc} R/J \le \operatorname{rank} \mathbb{F}_{0,2}$$

automatically happens and equality occurs if and only if $Z' \subseteq \mathfrak{m}Z$. If $Z' \subseteq \mathfrak{m}Z$ occurs at J, then the corresponding statement for $J^{[q]}$ is even more true. This explains why we did not include

$$\dim \operatorname{soc} R/J^{[q]} = \operatorname{rank} \mathbb{F}_{e,2}$$

as a hypothesis in the Corollary.