THE RESOLUTION OF THE FROBENIUS MODULE

A. Kustin (University of South Carolina)
C. Miller (Syracuse University)
A. Vraciu (University of South Carolina)

(preliminary report)

The ring R has positive prime characteristic p, $e \geq 1$ is an integer, $\varphi_R \colon R \to R$ is the Frobenius homomorphism, and $q = p^e$. We write ${}^{\varphi_R^e}R$ to mean R viewed as as R-module by way of the homomorphism φ_R^e.

The Starting Point.

The following Theorem plays a critical role in a project that Adela and I studied.

Theorem (L. Avramov and C. Miller). *Let M be a finitely generated module over a complete intersection local ring R. If* $\operatorname{Tor}_j^R(M, {}^{\varphi^e_R}R) = 0$ *for any fixed $j \geq 1$ and any fixed $e \geq 1$, then* $\operatorname{pd}_R M < \infty$.

We wonder if the hypothesis that R is a complete intersection can be weakened.

I am so fond of the A-M theorem because if I know something about a particular R-module M I have a chance of calculating $\operatorname{Tor}_1^R(M, {}^{\varphi^e_R}R)$ even if I appear to have insufficient information to calculate the entire R-resolution of M.

The context.

Let R be a local ring of prime characteristic $p > 0$ and M be a finitely generated R-module.

Kunz: The ring R is regular if and only if ${}^{\varphi^e_R}R$ is a flat R-module. In other words, R is regular if and only if $\operatorname{Tor}_i^R(M, {}^{\varphi^e_R}R) = 0$ for all M and all $e, i \geq 1$.

Peskine and Szpiro: If $\operatorname{pd}_R M < \infty$, then $\operatorname{Tor}_i^R(M, {}^{\varphi^e_R}R) = 0$ for all $e, i \geq 1$.

Herzog: If $\operatorname{Tor}_i^R(M, {}^{\varphi^e_R}R) = 0$ for all $i \geq 1$ and infinitely many e, then $\operatorname{pd}_R M < \infty$.

Koh and Lee: If $\operatorname{Tor}_i^R(M, {}^{\varphi^e_R}R) = 0$ for $\operatorname{depth} R + 1$ consecutive i and sufficiently large e, then $\operatorname{pd}_R M < \infty$.

Avramov and Miller: If R is a $\boxed{\text{complete intersection}}$ and $\operatorname{Tor}_i^R(M, {}^{\varphi^e_R}R) = 0$ for one i and one e, then $\operatorname{pd}_R M < \infty$.

The factorization. Suppose $R = Q/I$ where Q is a polynomial ring or a power series ring over the perfect field k. The map $\varphi_Q^e\colon Q \to Q$ exhibits ${}^{\varphi_Q^e}Q$ as a free Q-module. A base change gives makes $Q/I \to Q/I^{[q]}$ a free extension. Furthermore, the original $\varphi_R\colon R \to R$ factors as

$$R = Q/I \xrightarrow{\text{free module ext.}} Q/I^{[q]} \xrightarrow{\text{natural quot. map}} Q/I = R.$$

The map on the left is faithfully flat so

$$\operatorname{Tor}^R(M, {}^{\varphi_R^e}R) = \operatorname{Tor}^{Q/I^{[q]}}(M \otimes_R Q/I^{[q]}, Q/I).$$

We just saw that

$$\operatorname{Tor}^R(M, {}^{\varphi_R^e}R) = \operatorname{Tor}^{Q/I^{[q]}}(M \otimes_R Q/I^{[q]}, Q/I).$$

We are lead to ask questions about

$$\operatorname{Tor}^{Q/I^{[q]}}(\_\_, Q/I),$$

where Q/I is a $Q/I^{[q]}$-module by way of the natural quotient map

$$Q/I^{[q]} \to Q/I.$$

In particular, we ask if Q/I is rigid as a $Q/I^{[q]}$-module.

In fact, we do not yet know the answer to that question, but as we worked on it, we saw that we knew how to resolve Q/I as a $Q/I^{[q]}$-module.

Eventually, we realized that our resolution has nothing to do with the Frobenius map.

The set up. *Let Q be a ring, J be an ideal of Q, and M be a Q/J-module. Suppose that the Q-resolution $\mathfrak{F}$ of Q/J is a DG-algebra and that the Q-resolution $\mathbb{F}$ of M is a DG $\mathfrak{F}$-module. I will describe Q-modules $\mathbb{L}$ and Q-module maps $\mathbb{L}_i \to \mathbb{L}_{i-1}$ so that $\bar{\mathbb{L}}$ is the Q/J-resolution of M, where $^-$ means $\_\_ \otimes_Q Q/J$.*

Remarks.

1. We apply this technique with $J = I^{[q]}$ and $M = Q/I$.

2. The hypotheses that $\mathfrak{F}$ is a DG algebra and $\mathbb{F}$ is a DG-module over $\mathfrak{F}$ can always be attained, at the expense of any pretense of minimality.

3. In practice, it appears that we don't really need associativity or "associativity". The modules stay the same, but the differentials get more complicated (with various homotopy maps coming into play). We are not finished with this thought.

$\mathbb{L}$ **as a module.** Every module of the form

$$F_{b_0} \otimes \mathfrak{F}_{b_1} \otimes \ldots \otimes \mathfrak{F}_{b_t},$$

with $0 \leq t$, $0 \leq b_0$ and $1 \leq b_i$, for $1 \leq i \leq t$ is a summand of $\mathbb{L}$. This particular summand sits in position $\sum_{i=0}^{t} b_i + t$.

An alternate description of $\mathbb{L}$ as a module. For each $i \geq 1$, pick a basis for the module $\mathfrak{F}_i$, say: $x_{i,1}, x_{i,2} \ldots$. Then $\mathbb{L}$ is $\mathbb{F}$ with **non-commuting** variables

$$\begin{array}{lll} X_{1,1}, & X_{1,2}, & \ldots \\ X_{2,1}, & \ldots & \\ X_{3,1}, & \ldots & \\ \vdots & & \end{array}$$

adjoined, where $X_{i,*}$ contributes $i+1$ to the position.

The differential. The map $d\colon \mathbb{L} \to \mathbb{L}$ carries

$$Y_0 \otimes Y_1 \otimes Y_2 \otimes \ldots \otimes Y_t$$

to

$$\begin{cases} \;\; d(Y_0) \otimes Y_1 \otimes Y_2 \otimes \ldots \otimes Y_t \\ \pm Y_0 Y_1 \otimes Y_2 \otimes \ldots \otimes Y_t \\ \pm \chi(2 \leq |Y_1|) Y_0 \otimes d(Y_1) \otimes Y_2 \otimes \ldots \otimes Y_t \\ \pm Y_0 \otimes Y_1 Y_2 \otimes Y_3 \ldots \otimes Y_t \\ \pm \chi(2 \leq |Y_2|) Y_0 \otimes Y_1 \otimes d(Y_2) \otimes \ldots \otimes Y_t \\ \quad\;\; \vdots \end{cases}$$

Remarks.

1. $d(Y_0)$ is the differential in $\mathbb{F}$

2. $Y_0 Y_1$ is the (right) module action of $\mathfrak{F}$ on $\mathbb{F}$.

3. For $1 \leq i$, $Y_i \in \mathfrak{F}_{|Y_i|}$.

4. I use "χ" like a Kronecker delta. The value of $\chi(S)$ is 1 if S is true, but 0 is S is false.

5. The point of the χ factors is that $\mathfrak{F}_0$ is NOT used in $\mathbb{L}$.

6. $d(Y_1)$ is the differential in $\mathfrak{F}$.

7. $Y_1 Y_2$ is multiplication if $\mathfrak{F}$.

8. The above sum consists of $2t + 1$ terms.

Theorem. *Let Q be a ring, J be an ideal of Q, and M be a Q/J-module. If the Q-resolution $\mathfrak{F}$ of Q/J is a DG-algebra and the Q-resolution $\mathbb{F}$ of M is a DG $\mathfrak{F}$-module, then $\bar{\mathbb{L}}$ is the Q/J-resolution of M, where $^-$ means $\_\_ \otimes_Q Q/J$.*

Example. Let I be a perfect grade 2 ideal in the ring Q. Let $J = I^{[q]}$ and $M = Q/I$. In this case, $\mathbb{F}$ looks like

$$0 \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0,$$

and $\mathfrak{F}$ looks like

$$0 \to F_2^{[q]} \xrightarrow{d_2^{[q]}} F_1^{[q]} \xrightarrow{d_1^{[q]}} F_0.$$

The resolutions $\mathbb{F}$ and $\mathfrak{F}$ are DG algebras and any comparison map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_2^{[q]} & \xrightarrow{d_2^{[q]}} & F_1^{[q]} & \xrightarrow{d_1^{[q]}} & F_0 & \longrightarrow & Q/I^{[q]} \\
 & & \downarrow & & \downarrow & & =\downarrow & & \downarrow \text{nat. quot. map} \\
0 & \longrightarrow & F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \longrightarrow & Q/I
\end{array}$$

gives $\mathbb{F}$ the structure of a DG-module over $\mathfrak{F}$.

Continue with the previous example. If I has n generators, then $\mathbb{L}$ is $\mathbb{F}$ with n non-commuting variables of degree 2 and $n-1$ non-commuting variables of degree 3 adjoined; and $\bar{\mathbb{L}}$ is the resolution of Q/I by free $Q/I^{[q]}$ modules. Furthermore, if I is not a complete intersection and the data is local or homogeneous, then $\bar{\mathbb{L}}$ is the minimal resolution of Q/I. On the other hand, if I is a complete intersection, then $\mathfrak{F}$ is an exterior algebra and the product of the basis vectors e_1 and e_2 from $\mathfrak{F}_1$ is equal to the basis vector $e_1 \wedge e_2$ of $\mathfrak{F}_2$. Once the the variable of degree 3 is split off from $\bar{\mathbb{L}}$, the resulting resolution is $\bar{\mathbb{F}}$ with two COMMUTING variables of degree 2 adjoined. This resolution is the same as the Avramov-Buchweitz resolution.

Thoughts towards a proof that $\bar{\mathbb{L}}$ is a resolution.

Observe the recursive nature of $\mathbb{L}$. In particular, $\mathbb{L}_i$ is

$$\begin{array}{c} \mathbb{F}_i \\ \oplus \\ \mathbb{L}_0 \otimes \mathfrak{F}_{i-1} \\ \oplus \\ \mathbb{L}_1 \otimes \mathfrak{F}_{i-2} \\ \oplus \\ \vdots \\ \oplus \\ \mathbb{L}_{i-3} \otimes \mathfrak{F}_2 \\ \oplus \\ \mathbb{L}_{i-2} \otimes \mathfrak{F}_1 \end{array}$$

The key calculation. *The composition $d \circ d \colon \mathbb{L}_i \to \mathbb{L}_{i-2}$ is*

$$\begin{cases} 1 \otimes d & \textit{on the component } \mathbb{L}_{i-2} \otimes \mathfrak{F}_1 \\ 0 & \textit{on every other component} \end{cases}$$

The calculation obviously shows that $\bar{\mathbb{L}}$ is a complex. But in fact, the calculation also shows acyclicity. I illustrate by showing that $\mathrm{H}_2(\bar{\mathbb{L}}) = 0$.

We prove something small. Recall that $\mathbb{F}$ is a Q resolution of M, $\mathfrak{F}$ is a Q-resolution of Q/J, $^{-}$ means $\_\_\otimes_Q Q/J$, and $\mathbb{L}$ is

$$\dots \xrightarrow{d} \begin{matrix} F_3 \\ \oplus \\ \mathbb{L}_0 \otimes \mathfrak{F}_2 \\ \oplus \\ \mathbb{L}_1 \otimes \mathfrak{F}_1 \end{matrix} \xrightarrow{d} \begin{matrix} F_2 \\ \oplus \\ \mathbb{L}_0 \otimes \mathfrak{F}_1 \end{matrix} \xrightarrow{d} \mathbb{L}_1 \xrightarrow{d} \mathbb{L}_0.$$

We show that $\mathrm{H}_2(\bar{\mathbb{L}}) = 0$.

Proof. Suppose $Z \in \mathbb{L}_2$ with $dZ \in J\mathbb{L}_1$. The "key calculation" gives $Y_1 \in \mathbb{L}_1 \otimes \mathfrak{F}_1 \subseteq \mathbb{L}_3$ with $dZ = ddY_1$. Thus, $Z - dY_1$ is a **cycle of** $\mathbb{L}$, not only $\bar{\mathbb{L}}$. The $\mathbb{L}_0 \otimes \mathfrak{F}_1$ component of $Z - dY_1$ is sent to zero in $\mathbb{L}_0$ by $d \circ d = 1 \otimes d$. (We used "kc" again.) But $\mathbb{L}_0 \otimes \mathfrak{F}$ is acyclic, so there exists Y_2 in $\mathbb{L}_0 \otimes \mathfrak{F}_2 \subseteq \mathbb{L}_3$ with

$$Z - dY_1 - dY_2$$

is still a **cycle in** $\mathbb{L}$ and has 0 as its $\mathbb{L}_0 \otimes \mathfrak{F}_1$-component.

Thus, $Z - dY_1 - dY_2$ is in F_2 and is killed by d. The argument is complete because $\mathbb{F}$ is acyclic. $\square$