An example of using Representation Theory to find a resolution

I plan to give three 50 minute talks. Here is an outline of my plan.

Section 1. A brief introduction to irreducible representations of $\mathrm{GL}(V)$.

Section 2. A family of complexes associated to a generic map $\varphi\colon F \to G$.

Section 3. The Littlewood-Richardson rule shows that the objects of section 2 are complexes.

Section 4. The Acyclicity Lemma (together with LR) yields exactness!

Section 1. A brief introduction to irreducible representations of $\mathrm{GL}(V)$.

Let K be a field of characteristic zero (like $\mathbb{Q}$) and V be a finite dimensional vector space over K. There are many K-module maps $\theta\colon V \to V$. However, if I restrict my attention to coordinate-free maps θ (and this is the natural thing to do in my business – if I am trying to tell somebody a large sequence of maps (i.e., a resolution), I have to make these maps be as transparent as possible, I don't want to have to start by telling my favorite basis!), then there are many fewer choices for θ. Indeed, in this case, θ is multiplication by a scalar from K. (This is an easy exercise in Linear Algebra. The words "$\theta\colon V \to V$ is a coordinate-free map" mean that for every $g \in \mathrm{GL}(V)$, $g \circ \theta \circ g^{-1} = \theta$. So, the matrix for θ, with respect to your favorite basis, commutes with all invertible matrices.)

The basic building blocks in this sequence of lectures are irreducible representations of $\mathrm{GL}(V)$. The first paragraph establishes V as one of these. As soon as I tell you what I am talking about, and why I care, I will tell you many more examples.

The vector space L is a representation of $\mathrm{GL}(V)$ (or is a $\mathrm{GL}(V)$-module, or is a $K[\mathrm{GL}(V)]$-module, or admits a $\mathrm{GL}(V)$-action) if every change of basis in V gives rise to a corresponding change of basis in L (in a coherent manner).

Furthermore, the representation L is irreducible if the L does not contain any proper sub-representations. Notice that if L is an irreducible representation of $\mathrm{GL}(V)$, and $\theta\colon L \to L$ is a K-linear map which is independent of the choice of basis for V, (i.e., θ is a $\mathrm{GL}(V)$-equivariant map), then θ is multiplication by a scalar. Similarly, if L and L' are non-isomorphic irreducible representations of $\mathrm{GL}(V)$, and $\theta\colon L \to L'$ is a $\mathrm{GL}(V)$-equivariant K-module homomorphism, then θ is zero.

Remark. I believe that the last two sentences (about homomorphisms of irreducible $\mathrm{GL}(V)$-modules) is known as Schur's Lemma. For the first fifty years of my life – before I made any sense out of Representation Theory – I would tell students that Schur's lemma said that any ring homomorphism from a field to itself is either identically zero or and isomorphism. Then I would stand back, scratch my head, and wonder why a name was associated to such a trivial observation. Technically, I suppose, I was telling the truth in the old days. The intersection of what Schur proved and what I understood; was indeed, a trivial comment. However, I am more impressed with the result now that I understand more of it. The two sentences in the preceding paragraph are the key to the whole sequence of lectures.)

Example. The module $\mathrm{Sym}_d(V)$ is an irreducible $\mathrm{GL}(V)$-representation for all non-negative integers d. The easiest way to deal with $\mathrm{Sym}_d(V)$ is to pick a basis $v_1, \dots, v_n$ for V. The vector space $\mathrm{Sym}_d(V)$ is the vector space of all homogeneous polynomials of degree d in the n symbols $v_1, \dots, v_n$. Do notice that $\mathrm{Sym}_d(V)$ admits a $\mathrm{GL}(V)$-action. (If you decide to use a new basis for V, you will be looking at the same set of polynomials.)

Example. The module $\bigwedge^d(V)$ is an irreducible $\mathrm{GL}(V)$-representation for all non-negative integers d. My favorite basis for $\bigwedge^d(V)$ is

$$\{v_{i_1} \wedge \dots \wedge v_{i_d} \mid 1 \le i_1 < \dots < i_d \le n\}.$$

Once again, notice that $\bigwedge^d(V)$ admits a $\mathrm{GL}(V)$-action.

Example. The Schur module $L_\lambda V$ is an irreducible $\mathrm{GL}(V)$-representation for all partitions $\lambda = (\lambda_1, \dots, \lambda_\ell)$. (The words "$\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a partition" mean $\lambda_1 \ge \dots \ge \lambda_\ell$ are non-negative integers.) The coordinate-free definition is that $L_\lambda V$ is the image of the natural map

$$\bigwedge^{\lambda_1} V \otimes \dots \otimes \bigwedge^{\lambda_\ell} V \to \mathrm{Sym}_{\lambda'_1} V \otimes \dots \otimes \mathrm{Sym}_{\lambda'_{\mathrm{last}}} V \tag{*}$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_{\mathrm{last}})$ is the partition dual to λ. In particular, $L_i V = \bigwedge^i V$ and $L_{1^i} V = \mathrm{Sym}_i(V)$.