Generating a Residual Intersection

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Let I be an ideal of height $g > 0$ in a commutative noetherian ring R, and let $\mathfrak{A}$ be a proper sub-ideal of I generated by $s \geq g$ elements. If the ideal $J = (\mathfrak{A} : I)$ has height at least s, then J is called an $s-$residual intersection of I. (See Artin and Nagata [1], Huneke [19], or Huneke and Ulrich [22].) The case $s = g$ corresponds to the theory of linkage, and is well understood, at least from the homological point of view. The cornerstone of this theory is that, if I is perfect, then a free resolution of R/J can be produced from a free resolution of R/I and the Koszul complex on a set of generators of $\mathfrak{A}$. Very little is known about resolutions of $s-$residual intersections for $s > g$. A few special cases have been resolved ([5], [10], [30]); and the end of the resolution, or, in other words, a set of generators for the canonical module of R/J, has been described ([22]). However, in general, even the generators of J remain unknown. Here, we show that J can be approximated by a "sum of links" of I if $s = g + 1$, and we give conditions under which this approximation in fact gives a full set of generators of J. Since these generators arise from comparison maps from Koszul complexes to known resolutions (at least assuming that the resolution of R/I is in some sense known), it is feasible to count them, compute their degrees in the graded case, or even to give explicit formulas for them. After stating our main results (Theorems 2.15 and 2.16 and Corollary 2.18) precisely in section 2, we lay out the argument in sections 3, 4 and 5. In section 6, we are concerned with the question of when a $(g + 1)-$residual intersection J

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is Cohen-Macaulay, and in section 8, we estimate the number of generators for J. Section 7 contains various examples and applications.

1. Preliminary Concepts.

Throughout this paper R is a commutative noetherian ring (sometimes it will also be local and Cohen-Macaulay), and Tor denotes Tor^R. An ideal we call K will play a prominent role in our calculations. Throughout the entire paper we let

$$\overline{R} = R/K. \tag{1.1}$$

If r is an element of R, and N is an ideal of R, then we write

$$r + N \tag{1.2}$$

for the image of r in R/N. If M is an $R-$module, then M^* denotes $\mathrm{Hom}_R(M, R)$, and

$$\langle\ ,\ \rangle \colon M^* \otimes M \to R \tag{1.3}$$

is the evaluation map.

An ideal I in a ring R is unmixed if all of the associated prime ideals of I have the same height. The grade of a (proper) ideal I in a ring R is the length of the longest regular sequence on R in I. The ideal I of R is called perfect if the grade of I is equal to the projective dimension, $\mathrm{pd}_R(R/I)$, of the $R-$module R/I. The grade g perfect ideal I is called Gorenstein if $\mathrm{Ext}^g_R(R/I, R) \cong R/I$. If I is a grade g Gorenstein ideal and A is any ideal, then there is an isomorphism

$$\tau \colon \mathrm{Tor}_g(R/I, R/A) \xrightarrow{\cong} (A \colon I)/A. \tag{1.4}$$

If R is local, then one can easily establish (1.4) by applying $— \otimes (R/A)$ to a minimal resolution of R/I. We return to this isomorphism in Lemma 3.2.

If A and B are ideals, then we will make much use of the isomorphism

$$\vartheta \colon \mathrm{Tor}_1(R/A, R/B) \to (A \cap B)/AB. \tag{1.5}$$

To be explicit, if (F, d) is a projective resolution of R/A and x is an element of F_1 with $x \otimes 1$ a cycle in $F \otimes (R/B)$, then $\vartheta[x \otimes 1] = [d_1(x)]$.

The theory of linkage appears everywhere in this paper. Let I and J be proper ideals in a noetherian ring R, and let $\mathbf{x}$ be an $R-$regular sequence contained in $I \cap J$. If $J = ((\mathbf{x}) \colon I)$ and $I = ((\mathbf{x}) \colon J)$, then I and J are linked by $(\mathbf{x})$. Let I be an $R-$ideal of grade g, and let $\mathbf{x}$ be an $R-$regular sequence of length g with $(\mathbf{x})$ properly contained in I. Set $J = ((\mathbf{x}) \colon I)$. If, either, I is perfect, or else R is Gorenstein, I is unmixed, and R/I is Cohen-Macaulay, then J has grade g, and

I and J are linked by $(\mathbf{x})$. Furthermore, in the first case, J is perfect and, in the second case, J is unmixed and R/J is Cohen-Macaulay. (See, for example, [33, Propositions 1.3 and 2.6] or [7, section 5].) An ideal I is said to be licci (that is, in the linkage class of a complete intersection) if there is a sequence of ideals $I = I_0, \ldots, I_n$ such that I_n is generated by a regular sequence, and I_i and I_{i+1} are linked for all i. Now assume that I and J are linked by $(\mathbf{x})$ where either I is perfect, or else R is Gorenstein, I is unmixed, and R/I is Cohen-Macaulay. Then it is well-known (see, for example, [28, Proposition 3.4]) that the following statements are equivalent:

(a) $I \cap J = (\mathbf{x})$,

(b) $(\mathbf{x})_P = I_P$ for all associated prime ideals P of R/I,

(c) $\operatorname{Ass}(R/I) \cap \operatorname{Ass}(R/J) = \emptyset$.

If these conditions hold, then I and J are said to be geometrically linked.

Following the lead of [1], we say that an ideal I in a ring R satisfies the condition G_k if $\mu(I_P) \leq \dim(R_P)$ for all prime ideals P of R with $I \subseteq P$ and $\dim(R_P) < k$. (In the present context, $\mu(M)$ means the minimal number of generators of the module M.) The ideal I satisfies G_∞ if it satisfies G_k for all k. An ideal I is said to be strongly Cohen-Macaulay if the Koszul homology modules on any set of generators of I are Cohen-Macaulay modules. The main theorem of [18] guarantees that every licci ideal in a Gorenstein ring is strongly Cohen-Macaulay.

Geometric residual intersection (introduced in [22]) is analogous to geometric linkage. The $s-$residual intersection $J = (\mathfrak{A} : I)$ is called geometric if $I_P = \mathfrak{A}_P$ for all prime ideals P of R with $I \subseteq P$ and $\dim R_P \leq s$. In this case it follows immediately from the definition that

$$\operatorname{ht}(I + J) \geq s + 1. \tag{1.6}$$

Let (R, I) be a pair consisting of a local ring R and an $R-$ideal I. A pair $(\widetilde{R}, \widetilde{I})$ is a deformation of (R, I) if

$$(R, I) = (\widetilde{R}/(\mathbf{x}),\ (\widetilde{I}, \mathbf{x})/(\mathbf{x}))$$

for some sequence $\mathbf{x}$ in $\widetilde{R}$ which is regular on both $\widetilde{R}$ and $\widetilde{R}/\widetilde{I}$.

Finally, we recall that an ideal I is said to satisfy a property generically if I_Q satisfies the property for every associated prime Q of I. If M is a matrix with entries in the ring R, then $I_t(M)$ is the ideal of R generated by the $t \times t$ minors of M. If $X = (x_{ij})$ is a matrix of indeterminates, then we let $R[X]$ denote the polynomial ring $R[\{x_{ij}\}]$.

2. The Main Theorems.

Assume that $(R, \mathfrak{M}, k)$ is a local Cohen-Macaulay ring and that I is a perfect ideal of height $g > 0$ with I generically a complete intersection. Suppose $\mathfrak{A}$ is an ideal, contained in I, generated by $g+1$ elements; and $J = (\mathfrak{A} : I)$ is a $(g+1)-$residual intersection. Consequently, $I_p = \mathfrak{A}_p$ for ht $P = g$. Assume that generators $a_1, \dots, a_{g+1}$ for $\mathfrak{A}$ have been chosen so that the first g elements form a regular sequence and $I_p = (a_1, \dots, a_g)_P$ for primes P minimal over I. (If k is infinite, then one may either prove this general position statement directly or consult [4].) Let $A = (a_1, \dots, a_g)$, $y = a_{g+1}$, and $K = (A : I)$. (This ideal K is the ideal of (1.1).) By the choice of A, we see that K is geometrically linked to I.

We define three ideals $H, \mathfrak{K}$ and $\mathfrak{L}$ of R which are all contained in J. (The ideal $\mathfrak{K}$ is defined only if I is Gorenstein.) We think of these ideals as "approximations of J." Of the three ideals, H is the most difficult to compute, but the quotient J/H has been studied often in other contexts. The ideals $\mathfrak{K}$ and $\mathfrak{L}$ are both easy to compute. Indeed, the "new generators" of these ideals are the entries of the last map in a comparison from a Koszul complex to a known resolution. As such, both of these ideals appear to be a "sum of links". Fortunately, in Theorems 2.15 and 2.16, we are able to show that the three approximations $H, \mathfrak{K}$ and $\mathfrak{L}$ are all essentially the same; consequently, we can both calculate our approximations and estimate how close they are to all of J.

We first give our recipe for calculating $\mathfrak{K}$. Assume that I is a Gorenstein ideal. Let (E, d) be the Koszul complex on a generating set of the ideal $\mathfrak{A}$, F be a minimal resolution of R/I, and $\alpha \colon E \to F$ be a complex map which covers the natural map

$$H_0(E) = R/\mathfrak{A} \to R/I = H_0(F).$$

If an identification $F_g \cong R$ is fixed, then

$$\mathfrak{K} = \operatorname{im}(\alpha_g) + \mathfrak{A}. \tag{2.1}$$

It will be useful to identify a particular generating set for $\operatorname{im}(\alpha_g)$. We have chosen a generating set $a_1, \dots, a_{g+1}$ for $\mathfrak{A}$. Let $\varepsilon_1, \dots, \varepsilon_{g+1}$ be a basis for E_1 with $d\varepsilon_i = a_i$ for all i. Define

$$c_i = (-1)^{i+1} \alpha_g(\varepsilon_1 \wedge \dots \wedge \widehat{\varepsilon_i} \wedge \dots \wedge \varepsilon_{g+1}) \in R. \tag{2.2}$$

In (2.1) we said that

$$\mathfrak{K} = (c_1, \dots, c_{g+1}) + \mathfrak{A}. \tag{2.3}$$

The commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & E_{g+1} & \xrightarrow{d_{g+1}} & E_g \\ & & \downarrow \alpha_{g+1} & & \downarrow \alpha_g \\ & & 0 & \longrightarrow & F_g \end{array}$$

with $\alpha_g = [c_1, \dots, c_{g+1}]$ and $d_{g+1} = [a_1, \dots, a_{g+1}]^T$ yields the equation

$$\sum_{i=1}^{g+1} a_i c_i = 0. \tag{2.4}$$

(We use "T" to mean "transpose".) For each i with $1 \le i \le g+1$, let K_i be the ideal

$$K_i = (a_1, \dots, \widehat{a_i}, \dots, a_{g+1}, c_i). \tag{2.5}$$

Observe that $K_{g+1} = K$ is a link of I. If $a_1, \dots, \widehat{a_i}, \dots, a_{g+1}$ is a regular sequence, then K_i is also a link of I. At any rate,

$$\mathfrak{K} = \sum K_i,$$

and we refer to $\mathfrak{K}$ as a "sum of links".

Continuing to assume that I is a Gorenstein ideal, we observe (with generalization in mind) that $\mathfrak{K}$ could have been defined as the ideal of R with the property that $\mathfrak{K}/\mathfrak{A}$ is the image of

$$\textstyle\bigwedge^g \operatorname{Tor}_1(R/I, R/\mathfrak{A}) \xrightarrow{\text{mult.}} \operatorname{Tor}_g(R/I, R/\mathfrak{A}) \xrightarrow{\tau} (\mathfrak{A} : I)/\mathfrak{A} = J/\mathfrak{A}, \tag{2.6}$$

where the first map is Tor−algebra multiplication, the map τ is the isomorphism of (1.4), and the final identification is due to the definition of J.

Dropping the Gorenstein hypothesis on I, we define the ideal $\mathfrak{L}$ in a similar manner. Recall, from [33, Remarque 1.4], that the ideal $I + K$ is a Gorenstein ideal of height $g+1$. It was observed in [19], that

$$J = (K, y) : (I + K). \tag{2.7}$$

(Indeed, if $rI \subset (K, y)$, then $r \in ((A, y) : I) = J$ because $y \in I$ and $(I \cap K) = A$.) Let $\mathfrak{L}$ be the ideal in R such that $\mathfrak{L}/(K, y)$ is the image of

$$\textstyle\bigwedge^{g+1} \operatorname{Tor}_1\left(\frac{R}{I+K}, \frac{R}{(K,y)}\right) \xrightarrow{\text{mult.}} \operatorname{Tor}_{g+1}\left(\frac{R}{I+K}, \frac{R}{(K,y)}\right) \xrightarrow{\tau} \frac{(K,y):(I+K)}{(K,y)} = \frac{J}{(K,y)}. \tag{2.8}$$

The map labeled mult. is Tor−algebra multiplication; the map τ is the isomorphism of (1.4); and the last identification is due to (2.7).

We next give a recipe for calculating $\mathfrak{L}$. Let (K,d) be a Koszul complex on a generating set of the ideal (K,y), C be the minimal resolution of $R/(I+K)$, and $\alpha\colon K\to C$ be a map of complexes which covers the natural map

$$H_0(K)=R/(K,y)\to R/(I+K)=H_0(C).$$

If an identification $C_{g+1}\cong R$ is fixed, then a quick look at (2.8) shows that

$$\mathfrak{L}=\operatorname{im}(\alpha_{g+1})+(K,y).$$

A closer inspection of (2.8) yields a more detailed description of $\mathfrak{L}/(K,y)$. If x is an element of K_1 with $dx=y$, then

$$\mathfrak{L}=\operatorname{im}(\alpha_{g+1}(\text{---}\wedge x))+(K,y). \tag{2.9}$$

Indeed, if $x_1,\dots,x_{g+1}$ are elements of K_1 with $dx_i\in K$ for all i, then

$$\alpha_{g+1}(x_1\wedge\ldots\wedge x_g\wedge x_{g+1})$$

is an element of (K,y). This last observation holds because Tor$-$algebra multiplication factors through

$$\begin{array}{ccc}\bigwedge^{g+1}\operatorname{Tor}_1(R/(I+K),\overline{R}) & \longrightarrow & \operatorname{Tor}_{g+1}(R/(I+K),\overline{R})=0\\ \downarrow & & \downarrow\\ \bigwedge^{g+1}\operatorname{Tor}_1(R/(I+K),R/(K,y)) & \longrightarrow & \operatorname{Tor}_{g+1}(R/(I+K),R/(K,y)).\end{array} \tag{2.10}$$

(The module in the upper right hand corner is zero because K is a perfect ideal of grade g.)

Let $K=(z_1,\dots,z_m)$. If (i) is a $g-$multi-index and $z_{(i)}$ denotes

$$z_{i_1},\dots,z_{i_g},$$

then let $L_{(i)}$ be the ideal $(z_{(i)},y,w_{(i)})$ for

$$w_{(i)}=\alpha_{g+1}(x_{i_1}\wedge\ldots\wedge x_{i_g}\wedge x),\ \text{ where }\ x_{i_j}\in K_1\ \text{ and }\ dx_{i_j}=z_{i_j}.$$

It is clear that the equality

$$\mathfrak{L}=\sum_{(i)}L_{(i)} \tag{2.11}$$

always holds. If the elements $z_{(i)},y$ form a regular sequence, then $L_{(i)}$ is an almost complete intersection which is linked to the Gorenstein ideal $I+K$. The fact that

(2.11) holds and that the ideals $L_{(i)}$ are sometimes links has lead us to refer to $\mathfrak{L}$ as a "sum of links".

Let H be the ideal in R such that H/K is the image of the following composition:

$$\bigwedge^g \operatorname{Tor}_1(\overline{R},\overline{R}) \xrightarrow{\mu} \operatorname{Tor}_g(\overline{R},\overline{R}) \xrightarrow{\lambda} J/K. \tag{2.12}$$

The map μ is Tor$-$algebra multiplication and the isomorphism λ is described in Lemma 3.5. (Roughly speaking, there is a familiar identification of $\operatorname{Tor}_g(\overline{R},\overline{R})$ with the dual of the canonical module, $\omega_{\overline{R}}^* = \operatorname{Hom}(I/A,\overline{R})$; evaluation at y provides an isomorphism from this last module to J/K.) Observe, for future reference, that

$$J/H \cong \operatorname{coker}(\mu). \tag{2.13}$$

Our main theorems are valid in a setting that is somewhat more general than has been discussed so far. Indeed, they are valid when:

(2.14) R is a commutative noetherian ring; I and K are geometrically linked perfect ideals of grade $g>0$; y is an element of I that is regular on $\overline{R}=R/K$; and $A=I\cap K$, $\mathfrak{A}=(A,y)$, and $J=(\mathfrak{A}:I)$.

Recall that A, y, and K can be produced, given a $(g+1)-$residual intersection $J=(\mathfrak{A}:I)$, provided I is a perfect ideal of height $g>0$ in a Cohen-Macaulay local ring $(R,\mathfrak{M},k)$, I is generically a complete intersection and k is an infinite field. Observe that this manufactured data does satisfy the hypotheses of (2.14). (Indeed, it is only necessary to show that y is regular on $\overline{R}$. However, if y were a zero divisor on $\overline{R}$, then $(A,y)=\mathfrak{A}$ would be contained in a minimal prime of K. On the other hand, every height g prime of R which contains $\mathfrak{A}$ also contains I since $(\mathfrak{A}:I)=J$ is a $(g+1)-$residual intersection. This is a contradiction because geometrically linked ideals do not have any common components.)

In section 3 we prove that the ideals $H, \mathfrak{L}$, and $\mathfrak{K}$ of (2.12), (2.8) and (2.6) are defined whenever the assumptions of (2.14) hold. Furthermore, the recipes of (2.9) and (2.1) hold under the assumptions of (2.14) provided that the free resolutions C and F have the correct length and $\operatorname{rank} C_{g+1} = \operatorname{rank} F_g = 1$.

Theorem 2.15. Assume the notation and hypotheses of (2.14). Define $H, \mathfrak{L}$, and μ as in (2.12) and (2.8). Then

(a) The ideal $\mathfrak{L}$ is equal to (H,y).

(b) There is a short exact sequence

$$0 \to (H,y)/H \to \operatorname{coker}(\mu) \to J/\mathfrak{L} \to 0.$$

(c) If R is a power series ring over an infinite perfect field and $\overline{R}$ is reduced, then $\mathrm{coker}(\mu)$ is isomorphic to the ratio $\mathfrak{D}_D/\mathfrak{D}_K$ of the Dedekind and Kähler differents of $\overline{R}$.

Theorem 2.16. In addition to the hypotheses of Theorem 2.15 (a), assume that I is a Gorenstein ideal and define $\mathfrak{K}$ as in (2.6). Then

(a) The ideals $H, \mathfrak{K}$, and $\mathfrak{L}$ are all equal.

(b) The module $J/\mathfrak{K}$ is isomorphic to $\mathrm{coker}(\mu)$.

(c) The module $\mathrm{coker}(\mu)$ is isomorphic to the cotangent cohomology $T^2(\overline{R}/R, \overline{R})$.

Under certain circumstances it is known that $\mathrm{coker}(\mu) = 0$.

Corollary 2.17. Assume the hypotheses of Theorem 2.15 (a). If R is a Cohen-Macaulay ring which contains the rational numbers and $g = 2$, then $J = \mathfrak{L}$.

Proof. If suffices to assume that R is local. In this case Herzog [15, Corollary 4.12] has shown that $\mathrm{coker}(\mu) = 0$ if K is perfect of grade two and is generically a complete intersection.

Corollary 2.18. Let I be a grade $g > 0$ Gorenstein ideal in a Gorenstein ring R such that I is generically a complete intersection. Let $J = (\mathfrak{A} : I)$ be a $(g+1)-$residual intersection in R. Form $\mathfrak{K}$ as in (2.6). Then $J = \mathfrak{K}$ if and only if $T^2((R/I)/R, R/I) = 0$.

Note. If I is a licci ideal which is generically a complete intersection, then Theorem 2.19 (a) guarantees that $T^2((R/I)/R, R/I) = 0$.

Proof. Since $\mathfrak{A} \subseteq \mathfrak{K} \subseteq J$, there is no loss of generality if we assume that R is local. If necessary we may make a faithfully flat extension of R in order assume that the residue field of R is infinite. The data of (2.14) can now be manufactured. We apply Theorem 2.16 in order to see that $J/\mathfrak{K} \cong T^2(\overline{R}/R, \overline{R})$. The proof is completed by applying Theorem 2.19 (b).

Theorem 2.19. Let R be a Gorenstein ring.

(a) ([8]) If I is a licci ideal of R which is generically a complete intersection, then $T^2((R/I)/R, R/I) = 0$.

(b) ([9]) Suppose I and K are perfect ideals of R which are in the same linkage class. If each of these ideals is generically a complete intersection, then

$$T^2((R/K)/R, R/K)_{\mathfrak{M}} \cong T^2((R/I)/R, R/I)_{\mathfrak{M}}$$

for all maximal ideals $\mathfrak{M}$ of R. In particular,

$$T^2((R/K)/R, R/K) = 0 \Leftrightarrow T^2((R/I)/R, R/I) = 0.$$

There are two parts to the proof of Theorems 2.15 and 2.16. In sections 3 and 4 we establish the relationship between the ideals $H, \mathfrak{K}$, and $\mathfrak{L}$. In section 5 we identify $\operatorname{coker}(\mu)$ with other classical objects. Observe that the assertions of 2.15 (b) and 2.16 (b) are immediate consequences of (2.13), 2.15 (a), and 2.16 (a).

3. The Approximation Ideals are Well Defined.

In this section R is an arbitrary commutative noetherian ring. The isomorphisms

$$\begin{aligned} &\lambda\colon \operatorname{Tor}_g(\overline{R}, \overline{R}) \cong J/K, \\ (3.1) \qquad &\tau\colon \operatorname{Tor}_{g+1}\left(\tfrac{R}{I+K}, \tfrac{R}{(K,y)}\right) \cong \tfrac{(K,y):(I+K)}{(K,y)}, \quad \text{and} \\ &\tau\colon \operatorname{Tor}_g(R/I, R/\mathfrak{A}) \cong (\mathfrak{A}\colon I)/\mathfrak{A} \end{aligned}$$

of (2.12), (2.8), and (2.6) are **not** canonical. For the time being, let N be K, $I+K$, or I, respectively; let M be K, (K, y) or $\mathfrak{A}$, respectively, and let $n = \operatorname{pd}_R(R/N)$. If A is a projective resolution of R/N of length n, then the **isomorphisms** of (3.1) depend on an identification of $\operatorname{Ext}^n_R((R/N), R)$ with a submodule of R/N, and on a choice of augmentation m from A^* to this submodule. However, if T is a submodule of $\operatorname{Tor}_n(R/N, R/M)$, then the **image** of T in R/M does not depend on the particular isomorphism used in (3.1). In this section we establish the above assertions in order to know that the ideals $H, \mathfrak{K}$, and $\mathfrak{L}$ of section 2 are well-defined whenever the hypotheses of (2.14) are satisfied. We begin by establishing isomorphism (1.4) whenever I is a grade g Gorenstein ideal.

Lemma 3.2. Let I be a Gorenstein ideal of grade g in the ring R, and let L be an arbitrary ideal of R. If (A, a) is a length g resolution of R/I by projective $R-$modules and $m\colon A_g^* \to R/I$ is an augmentation of the resolution A^* onto R/I, then m induces an isomorphism

$$\tau_m\colon \operatorname{Tor}_g(R/I, R/L) \to (L\colon I)/L.$$

Proof. Select $u \in A_g^*$ with $m(u) = \bar{1}$ in R/I. We show that the map

$$(u \otimes 1)\colon A_g \otimes (R/L) \to (R/L)$$

restricts to give an isomorphism

$$\tau_m\colon \ker(a_g \otimes 1_{R/L}) \to (L\colon I)/L.$$

(Notice that τ_m does not depend on the choice of u. If v is another element of A_g^* with $m(v) = \bar{1}$, then $u - v \in \ker(m) = \operatorname{im}(a_g^*)$. Thus, $u \otimes 1$ and $v \otimes 1$ are the same function when restricted to the domain of τ_m.)

At any rate, we show that τ_m is an isomorphism by showing that $(\tau_m)_{\mathfrak{M}}$ is an isomorphism for all maximal ideals $\mathfrak{M}$ of R. Assume that $(R, \mathfrak{M})$ is local. There exist bases $x_1, \ldots, x_r$ for A_g and $y_1, \ldots, y_s$ for A_{g-1} such that

(a) $u(x_j) = \delta_{1\,j}$, for all j, and

(b) the matrix of a_g with respect to these bases is

$$\left[\begin{array}{c|c} r_1 & \\ \vdots & 0 \\ r_n & \\ \hline 0 & \mathrm{id} \end{array}\right]$$

for some generating set $r_1, \ldots, r_n$ for I.

It is now clear that τ_m is an isomorphism.

Lemma 3.3. Retain the notation of Lemma 3.2. If T is a submodule of $\operatorname{Tor}_g(R/I, R/L)$, then the image $\tau_m(T)$ does not depend on the choice of m.

Proof. We may assume that R is local since it suffices to prove the claim locally at every maximal ideal containing I. Let (B, b) be a resolution of R/I and let $n : B_g^* \to R/I$ be an augmentation of B^* onto R/I. We only need to show that the map $\tau_n\tau_m^{-1}\colon (L\colon I)/L \to (L\colon I)/L$ is multiplication by a unit t of R. The augmentation $n\colon B_g^* \to (R/I)$ induces an isomorphism $\bar{n}\colon \operatorname{coker}(b_g^*) \to (R/I)$ with the property that the composition

$$B_g^* \xrightarrow{\text{nat.}} \operatorname{coker}(b_g^*) \xrightarrow{\bar{n}} R/I$$

is n. Similarly, there is an isomorphism $\overline{m}\colon \operatorname{coker}(a_g^*) \to R/I$ with the composition

$$A_g^* \xrightarrow{\text{nat.}} \operatorname{coker}(a_g^*) \xrightarrow{\overline{m}} R/I$$

equal to m.

Let $\alpha\colon A \to B$ be a complex map which covers the identity map on R/I:

$$\begin{array}{ccccccccccc}
0 \to & A_g & \xrightarrow{a_g} & A_{g-1} & \to \dots \to & A_1 & \xrightarrow{a_1} & A_0 & \xrightarrow{a_0} & R/I \\
& \downarrow \alpha_g & & \downarrow \alpha_{g-1} & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \text{id.} \\
0 \to & B_g & \xrightarrow{b_g} & B_{g-1} & \to \dots \to & B_1 & \xrightarrow{b_1} & B_0 & \xrightarrow{b_0} & R/I.
\end{array}$$

The dual of α covers the identity map on $\operatorname{Ext}^g_R(R/I, R)$; however, we picked the augmentations m and n **before** we chose α. Consequently, the composition

$$R/I \xrightarrow{\bar{n}^{-1}} \operatorname{coker}(b_g^*) \xrightarrow{\alpha_g^*} \operatorname{coker}(a_g^*) \xrightarrow{\overline{m}} R/I$$

is an automorphism of the $(R/I)-$module R/I, but not necessarily the identity. Thus, there is a unit t of R so that the diagram

$$\begin{array}{cccccc}
\dots & \to & B_{g-1}^* & \xrightarrow{b_g^*} & B_g^* & \xrightarrow{n} & R/I \to 0 \\
& & \downarrow \alpha_{g-1}^* & & \downarrow \alpha_g^* & & \downarrow t \\
\dots & \to & A_{g-1}^* & \xrightarrow{a_g^*} & A_g^* & \xrightarrow{m} & R/I \to 0
\end{array}$$

commutes.

Take bases for A_g, A_{g-1}, B_g, and B_{g-1} as described in (a) and (b) of the proof of Lemma 3.2. Let $(t_{i\,j})$ be the matrix of α_g with respect to these bases. One can easily verify that

(a) $\tau_n\tau_m^{-1}(r + L) = t_{1\,1}(r + L)$ for all $r \in (L\colon I)$, and

(b) $t_{1\,1} - t \in I$.

Thus, the automorphisms of $(L\colon I)/L$ given by: $\tau_n\tau_m^{-1}$, multiplication by $t_{1\,1}$, and multiplication by t, are all equal.

Corollary 3.4. Assume the hypotheses of (2.14). Let τ_m be any of the maps of Lemma 3.2.

(a) Every choice for the map $\tau = \tau_m$ in (2.8) gives rise to the same ideal $\mathfrak{L}$ of R.

(b) If I is Gorenstein, then every choice for the map $\tau = \tau_m$ in (2.6) gives rise to the same ideal $\mathfrak{K}$ of R.

For the rest of this section we study the isomorphism $\lambda\colon \operatorname{Tor}_g(\overline{R}, \overline{R}) \to J/K$ of (2.12). The notation and hypotheses of (2.14) are in effect throughout. Under these assumptions, we have

$$\mathrm{Ext}^g_R(\overline{R},R)\simeq \mathrm{Hom}_R(\overline{R},R/A)=\frac{A\colon K}{A}=\frac{I}{I\cap K}=\frac{I+K}{K}.$$

For each length g resolution A of $\overline{R}$ and each augmentation $m\colon A_g^*\to(I+K)K$, we obtain an isomorphism $\lambda_m\colon \ker(a_g\otimes 1_{\overline{R}})\to J/K$. We also show that if T is a fixed submodule of $\mathrm{Tor}_g(\overline{R},\overline{R})$, then the image $\lambda_m(T)$ in J/K does not depend on m.

If R is Gorenstein local, then it is well-known that $\mathrm{Tor}_g(\overline{R},\overline{R})$ is isomorphic to the dual of the canonical module $\omega^*_{\overline{R}}$. The following lemma amounts to the same thing, absent extraneous hypotheses. The argument we give is implicit in Herzog [15, Proposition 3.5].

Lemma 3.5. Adopt the notation of (2.14). If (B,b) is a length g projective resolution of $\overline{R}$ and $m\colon B_g^*\to(I+K)/K$ is an augmentation of B^* onto $(I+K)/K$, then m induces an isomorphism

$$\lambda_m\colon \mathrm{Tor}_g(\overline{R},\overline{R})\to J/K.$$

Proof. Select an element η on B_g^* with $m(\eta)=y+K$ in $(I+K)/K$. We will prove that $(\eta\otimes 1_{\overline{R}})\colon B_g\otimes\overline{R}\to\overline{R}$ restricts to give an isomorphism

$$\lambda_m\colon \ker(b_g\otimes 1_{\overline{R}})\to J/K.$$

First we establish isomorphisms ρ and θ:

$$\ker(b_g\otimes 1_{\overline{R}})\xrightarrow{\ \rho\ }\mathrm{Hom}\left(\frac{I+K}{K},\overline{R}\right)\xrightarrow{\ \theta\ } J/K, \tag{3.6}$$

then we prove that the composition $\theta\rho$ is in fact equal to the restriction of $\eta\otimes 1_{\overline{R}}$ to $\ker(b_g\otimes 1_{\overline{R}})$.

We have been given a presentation

$$B^*_{g-1}\xrightarrow{\ b_g^*\ }B_g^*\xrightarrow{\ m\ }(I+K)/K\to 0 \tag{3.7}$$

of $(I+K)/K$ by free $R-$modules. Let "$-$" denote reduction modulo K. If we apply $-\otimes\overline{R}$ to (3.7) and identify $\mathrm{Hom}_R(B_i,R)\otimes\overline{R}$ with $\mathrm{Hom}_{\overline{R}}(\overline{B}_i,\overline{R})$ for each projective $R-$module B_i, then we obtain a presentation of $(I+K)/K$ by projective $\overline{R}-$modules:

$$\overline{B}^*_{g-1}\xrightarrow{\ \bar{b}_g^*\ }\overline{B}_g^*\xrightarrow{\ \overline{m}\ }(I+K)/K\to 0, \tag{3.8}$$

where "$*$" now represents $\mathrm{Hom}_{\overline{R}}(—,\overline{R})$. (Notice that the maps $\bar{b}_g^*$ and $b_g^*\otimes 1_{\overline{R}}$ are exactly the same.) If we apply $\mathrm{Hom}(—,\overline{R})$ once more to (3.8), then the canonical identification of $\overline{B}_i$ with $\overline{B}_i^{**}$ yields a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker(\bar{b}_g) & \longrightarrow & \overline{B}_g & \xrightarrow{\bar{b}_g} & \overline{B}_{g-1} \\
 & & & & {\scriptstyle \text{eval.}}\downarrow\cong & & {\scriptstyle \text{eval.}}\downarrow\cong \\
0 & \longrightarrow & \mathrm{Hom}((I+K)/K,\overline{R}) & \xrightarrow{\overline{m}^*} & \overline{B}_g^{**} & \xrightarrow{\bar{b}_g^{**}} & \overline{B}_{g-1}^{**}.
\end{array}$$

Thus, there is an induced isomorphism $\rho\colon \ker(\bar{b}_g) \to \mathrm{Hom}((I+K)/K,\overline{R})$. Furthermore, if x is any element of $\ker(\bar{b}_g)$ and γ is any element of $\overline{B}_g^*$, then

$$\langle \rho(x), \overline{m}(\gamma)\rangle = \langle \gamma, x\rangle. \tag{3.9}$$

Since $\bar{y} = y + K$ is in $\bar{I} = (I+K)/K$, and $\bar{y}$ is regular on $\overline{R}$, it is well-known (and easy to show) that the homomorphism $\theta : \mathrm{Hom}(\bar{I},\overline{R}) \to (\bar{y}) : \bar{I}$ defined by $\theta(f) = f(\bar{y})$ is an isomorphism. Observe that (2.7):

$$(K,y)\colon(I+K) = J \tag{3.10}$$

holds under the hypotheses of (2.14). Thus, $(\bar{y}) : \bar{I} = \bar{J} = J/K$, and both isomorphisms of (3.6) have been established. Finally, if $x \in \ker(\bar{b}_g)$, then, from the definition of θ, the definition of η, the definition of $\overline{m}$ in (3.8), and (3.9), we see that

$$\theta\rho(x) = \langle \rho(x), y+K\rangle = \langle \rho(x), m(\eta)\rangle = \langle \rho(x), \overline{m}(\eta\otimes 1_{\overline{R}})\rangle = \langle \eta\otimes 1_{\overline{R}}, x\rangle.$$

It follows that the restriction of $\eta\otimes 1_{\overline{R}}$ to $\ker(b_g\otimes 1_{\overline{R}})$ is an isomorphism onto J/K.

Lemma 3.11. Retain the notation of Lemma 3.5. If T is a submodule of $\mathrm{Tor}_g(\overline{R},\overline{R})$, then the image $\lambda_m(T)$ of T in J/K does not depend on the choice of m.

Proof. Again, we may assume that R is local. Let (A,a) be a resolution of $\overline{R}$, n be an augmentation of A^* onto $(I+K)/K$, and $\alpha\colon A \to B$ be a complex map which covers the identity map on $\overline{R}$. The dual of α induces an $R-$module automorphism t of $(I+K)/K$:

$$\begin{array}{ccccccc}
\dots & \to & B_{g-1}^* & \xrightarrow{b_g^*} & B_g^* & \xrightarrow{m} & (I+K)/K \to 0 \\
 & & \downarrow {\scriptstyle \alpha_{g-1}^*} & & \downarrow {\scriptstyle \alpha_g^*} & & \downarrow {\scriptstyle t} \\
\dots & \to & A_{g-1}^* & \xrightarrow{a_g^*} & A_g^* & \xrightarrow{n} & (I+K)/K \to 0.
\end{array}$$

However, the $R-$module $\overline{R}$ is perfect of grade g and $(I+K)/K \simeq \mathrm{Ext}^g_R(\overline{R},R)$, so

$$\mathrm{Hom}_R\left(\frac{I+K}{K},\frac{I+K}{K}\right) \cong \mathrm{Hom}_R\left(\mathrm{Ext}^g_R(\overline{R},R),\mathrm{Ext}^g_R(\overline{R},R)\right) \cong \mathrm{Hom}_R(\overline{R},\overline{R}) = \overline{R}.$$

(The second isomorphism is due to the fact that if M and N are perfect $R-$modules of projective dimension g, then

$$\mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(\mathrm{Ext}^g_R(N,R),\mathrm{Ext}^g_R(M,R)).)$$

Thus, t is multiplication by some unit of R. The rest of the proof is identical to the proof of Lemma 3.3.

Corollary 3.12. Assume the hypotheses of (2.14). Let λ_m be any of the maps of Lemma 3.5. Then every choice for the map $\lambda = \lambda_m$ in (2.12) yields the same ideal H of R.

One particular choice for λ interacts especially nicely with the data used to calculate $\mathfrak{K}$ and $\mathfrak{L}$. We will make use of this λ in section 4 where we examine the relationship between $H, \mathfrak{K}$, and $\mathfrak{L}$.

Proposition 3.13. Assume the hypotheses of (2.14). Let A be generated by the regular sequence $a_1,\dots,a_g$ and let $x = \mu\kappa(\bar{a}_1 \wedge \dots \wedge \bar{a}_g)$ in $\mathrm{Tor}_g(\overline{R},\overline{R})$, where $\bar{a}_i$ is the image of a_i in K/K^2, μ is the Tor$-$algebra multiplication of (2.12), and the map $\kappa = \bigwedge^g \vartheta^{-1} \colon \bigwedge^g(K/K^2) \to \bigwedge^g \mathrm{Tor}_1(\overline{R},\overline{R})$ is induced by the isomorphism ϑ of (1.5). Then there is an isomorphism $\lambda\colon \mathrm{Tor}_g(\overline{R},\overline{R}) \to J/K$ (constructed in Lemma 3.5) so that $\lambda(x) = y + K$. Furthermore, if ρ is the isomorphism in (3.6) which corresponds to λ, then $\rho(x)\colon (I+K)/K \to \overline{R}$ is the inclusion map.

Proof. Let (K,∂) be the Koszul complex on $a_1,\dots,a_g$; (B,b) be a length g projective resolution of $\overline{R}$; and $\alpha : K \to B$ be a map of complexes which covers the natural map $R/A \to \overline{R}$. If the basis for K_1 is called $\varepsilon_1,\dots,\varepsilon_g$ with $\partial(\varepsilon_i) = a_i$, then let ε stand for the generator $\varepsilon_1 \wedge \dots \wedge \varepsilon_g$ of K_g. Observe that

(3.14) $$x = \alpha_g(\varepsilon) \otimes 1_{\overline{R}} \in \ker(b_g \otimes 1_{\overline{R}}) = \mathrm{Tor}_g(\overline{R},\overline{R}).$$

The mapping cone of α^* is a resolution of R/I, thus

$$B^*_{g-1} \oplus K^*_{g-2} \xrightarrow{d_2} B^*_g \oplus K^*_{g-1} \xrightarrow{d_1} R \to R/I \to 0$$

is exact where $d_1 = [\langle -, \alpha_g(\varepsilon)\rangle, \langle -, \partial_g(\varepsilon)\rangle]$, and

$$d_2 = \begin{bmatrix} b^*_g & 0 \\ -\alpha^*_{g-1} & \partial^*_{g-1} \end{bmatrix}.$$

The map $m\colon B_g^* \to (I+K)/K$, defined by

$$m(\gamma) = \langle \gamma, \alpha_g(\varepsilon)\rangle + K \tag{3.15}$$

for $\gamma \in B_g^*$, is an augmentation of B^* onto $I/A = (I+K)/K$. Following the proof of Lemma 3.5, we select $\eta \in B_g^*$ such that $\langle \eta, \alpha_g(\varepsilon)\rangle + K = y + K$, and we define $\lambda\colon \ker(b_g \otimes 1_{\overline{R}}) \to J/K$ by $\lambda(\xi \otimes 1_{\overline{R}}) = \langle \eta, \xi\rangle + K$ for all $\xi \in B_g$ with $\xi \otimes 1_{\overline{R}}$ in $\ker(b_g \otimes 1_{\overline{R}})$. In particular,

$$\lambda(x) = \lambda(\alpha_g(\varepsilon) \otimes 1_{\overline{R}}) = \langle \eta, \alpha_g(\varepsilon)\rangle + K = y + K.$$

If $r + K$ is an element of $(I+K)/K$, then there is an element γ of $\overline{B}_g^*$ with $\overline{m}(\gamma) = r + K$. We use (3.9), (3.14), and (3.15) to see that

$$\langle \rho(x), r + K\rangle = \langle \gamma, x\rangle = \langle \gamma, \alpha_g(\varepsilon) \otimes 1_{\overline{R}}\rangle = \overline{m}(\gamma) = r + K,$$

and the proof is complete.

4. How the Approximation Ideals are Related.

We prove part (a) of the Theorems 2.15 and 2.16 in this section. Part (a) of Theorem 2.15 can be deduced quickly once one knows that the diagram we have labeled (4.15) commutes. We show that the left side of (4.15) commutes in Lemma 4.1 and that the right side of (4.15) commutes in Lemma 4.8.

Lemma 4.1. Let L, M, and N be ideals in R with $L \subseteq M$; let z be an element of M that is regular on R/N; and let $g > 0$ be a fixed integer. Then there is a map ν such that the following diagram

$$\begin{tikzcd}
\bigwedge^g \operatorname{Tor}_1(R/L, R/N) \arrow[r, "\mu_1"] \arrow[dd, "\nu"] & \operatorname{Tor}_g(R/L, R/N) \arrow[d, "\pi_*"] \\
 & \operatorname{Tor}_g(R/M, R/N) \\
\bigwedge^{g+1} \operatorname{Tor}_1(R/M, R/(N,z)) \arrow[r, "\mu_2"] & \operatorname{Tor}_{g+1}(R/M, R/(N,z)) \arrow[u, "\delta"]
\end{tikzcd}$$

is commutative, where both maps labeled μ_i are Tor$-$algebra multiplication, π_* is induced by the natural surjection $\pi : R/L \to R/M$, and δ is the connecting homomorphism in the long exact sequence of homology induced by the short exact sequence

$$0 \to (R/N) \xrightarrow{z} (R/N) \to R/(N,z) \to 0.$$

Proof. We begin by defining ν. Let ζ be the element of $\mathrm{Tor}_1(R/M, R/(N,z))$ that is sent to $[z]$ under the isomorphism

$$\vartheta\colon \mathrm{Tor}_1(R/M, R/(N,z)) \to (M \cap (N,z))/M(N,z)$$

of (1.5). The natural maps π and $p\colon R/N \to R/(N,z)$ induce

$$p_*\pi_*\colon \mathrm{Tor}_1(R/L, R/N) \to \mathrm{Tor}_1(R/M, R/(N,z)).$$

Define ν to be the composition $[\zeta \wedge (—)] \circ \bigwedge^g(p_*\pi_*)$.

Let (B,b) be a resolution of R/L, let $\rho_1,\dots,\rho_g$ be elements of B_1 with each $\rho_i \otimes 1$ a cycle in $B \otimes (R/N)$; and let $\rho = [\rho_1 \otimes 1] \wedge \dots \wedge [\rho_g \otimes 1]$ in $\bigwedge^g \mathrm{Tor}_1(R/L, R/N)$. It suffices to show that

$$\pi_*\mu_1(\rho) = \delta\mu_2\nu(\rho). \tag{4.2}$$

Let $\mathbf{r}$ denote the sequence of elements $b_1(\rho_1),\dots,b_1(\rho_g)$ in R; let K be the Koszul complex on $\mathbf{r}$, with K_1 a free module on the basis $\{\varepsilon_1,\dots,\varepsilon_g\}$; let $u_1 : K_1 \to B_1$ be defined by $u_1(\varepsilon_i) = \rho_i$; and let $u : K \to B$ be a map of complexes which extends u_1 and the identity map $u_0\colon K_0 \to B_0$. Then $\mu_1(\rho)$ is the element of $\mathrm{Tor}_g(R/L, R/N)$ that is represented by the cycle $u_g(\varepsilon) \otimes 1$ in $B_g \otimes (R/N)$, where $\varepsilon = \varepsilon_1 \wedge \dots \wedge \varepsilon_g$ in K_g.

Let (C,c) be a resolution of R/M, and $\gamma\colon B \to C$ be a map of complexes which extends the natural map $\pi : R/L \to R/M$. Since π_* is induced by $\gamma_g \otimes 1_{R/N}$, we see that

$$\pi_*(\mu_1(\rho)) = [\gamma_g(u_g(\varepsilon)) \otimes 1] \text{ in } H_g(C \otimes (R/N)). \tag{4.3}$$

Having calculated the left side of (4.2), we turn our attention to the right side. Choose σ in C_1 with $c_1(\sigma) = z$. Since ϑ is an isomorphism and $\vartheta[\sigma \otimes 1] = [c_1(\sigma)] = [z]$, it follows that $[\sigma \otimes 1] = \zeta$. The map ν has been defined so that

$$\nu(\rho) = [\sigma \otimes 1] \wedge [\gamma_1(\rho_1) \otimes 1] \wedge \dots \wedge [\gamma_1(\rho_g) \otimes 1]$$

in $\bigwedge^{g+1}\mathrm{Tor}_1(R/M, R/(N,z))$. We calculate multiplication in Tor exactly as before. Let (K', ∂') be the Koszul complex on $c_1(\sigma), c_1(\gamma_1(\rho_1)),\dots,c_1(\gamma_1(\rho_g))$, that is, on the sequence $z, \mathbf{r}$. Consequently, we may view K as a summand of K', and in particular, take a basis for K'_1 to be $\varepsilon_1,\dots,\varepsilon_g$, together with one other element ε_0. Define $u'_1 : K'_1 \to C_1$ by $u'_1(\varepsilon_i) = \gamma_1(\rho_i)$ for $1 \le i \le g$ and $u'_1(\varepsilon_0) = \sigma$. If $u'\colon K' \to C$ is any complex map which extends u'_1 and the identity map $K'_0 \to C_0$, then

$$\mu_2\nu(\rho) = u'_{g+1}(\varepsilon') \otimes 1 \text{ in } C_{g+1} \otimes (R/(N,z)) \tag{4.4}$$

for $\varepsilon' = \varepsilon_0 \wedge \varepsilon_1 \wedge \ldots \wedge \varepsilon_g$. We may moreover insist, since K is a summand of K', that the restriction of u' to K is precisely equal to γu. Thus

$$u'_g(\varepsilon) = \gamma_g u_g(\varepsilon). \tag{4.5}$$

The connecting homomorphism

$$\delta\colon \mathrm{Tor}_{g+1}(R/M, R/(N,z)) \to \mathrm{Tor}_g(R/M, R/N)$$

is obtained by applying the snake lemma to

$$\begin{array}{ccccc} 0 \to C_{g+1} \otimes (R/N) & \longrightarrow & C_{g+1} \otimes (R/N) & \longrightarrow & C_{g+1} \otimes R/(N,z) \to 0 \\ \downarrow & & \downarrow {\scriptstyle c_{g+1}\otimes \mathrm{id}} & & \downarrow \\ 0 \to C_g \otimes (R/N) & \xrightarrow{z} & C_g \otimes (R/N) & \longrightarrow & C_g \otimes R/(N,z) \to 0. \end{array} \tag{4.6}$$

It follows from (4.4) that $\delta(\mu_2\nu(\rho))$ is represented by $\Xi \otimes 1$ in $C_g \otimes (R/N)$ for any Ξ in C_g with

$$c_{g+1}u'_{g+1}(\varepsilon') \otimes 1 = z\Xi \otimes 1 \quad \text{in} \quad C_g \otimes (R/N). \tag{4.7}$$

Since u' is a map of complexes, it follows from the definition of ε' that

$$c_{g+1}u'_{g+1}(\varepsilon') = u'_g\partial'_{g+1}(\varepsilon') \equiv u'_g(z\varepsilon) \bmod (\mathbf{r})C_g.$$

The elements $\rho_i \in B_1$ were chosen so that each $r_i \in N$. Using (4.5) we obtain $c_{g+1}u'_{g+1}(\varepsilon') \otimes 1 = z\gamma_g u_g(\varepsilon) \otimes 1$ in $C_g \otimes (R/N)$; and therefore, from (4.7) and (4.3), we conclude that

$$\delta\mu_2\nu(\rho) = [\gamma_g u_g(\varepsilon) \otimes 1] = \pi_*\mu_1(\rho)$$

in $H_g(C \otimes (R/N)) = \mathrm{Tor}_g(R/M, R/N)$.

Lemma 4.8. Under assumptions (2.14), there are isomorphisms τ and λ (constructed in Lemmas 3.2 and 3.5, respectively) so that the diagram

$$\begin{array}{ccc} \mathrm{Tor}_g(\overline{R}, \overline{R}) & \xrightarrow{\lambda} & J/K \\ {\scriptstyle \pi_*}\downarrow & & \Big| \\ \mathrm{Tor}_g(R/(I+K), \overline{R}) & & \Big| {\scriptstyle p} \\ {\scriptstyle \delta^{-1}}\downarrow & & \downarrow \\ \mathrm{Tor}_{g+1}(R/(I+K), R/(K,y)) & \xrightarrow{\tau} & J/(K,y) \end{array}$$

commutes, where p is the natural quotient map, π_* is induced by the natural quotient map $\pi\colon \overline{R} \to R/(I+K)$, and δ is the connecting homomorphism induced by the short exact sequence

$$0 \to \overline{R} \xrightarrow{y} \overline{R} \to R/(K,y) \to 0. \tag{4.9}$$

Proof. We begin by observing that δ^{-1} makes sense. Indeed, inspection of the long exact sequence of homology obtained by applying $R/(I+K) \otimes (\text{—})$ to (4.9) shows that δ is an isomorphism because y annihilates $\operatorname{Tor}_i(R/(I+K), \text{—})$ for all i and $\operatorname{pd}_R \overline{R} = g$.

Next, we produce our favorite resolutions of $\overline{R}$ and $R/(I+K)$. We have assumed that $I \cap K = A$ is generated by a regular sequence $\mathbf{a} = a_1, \ldots, a_g$. Let (L, ∂) be the Koszul complex on $\mathbf{a}$, let (F, d) be a resolution of R/I of length g; and let $\alpha\colon L \to F$ be a map of complexes which covers the identity map $\alpha_0 : L_0 = R \to F_0 = R$. We know, from the theory of linkage, that the mapping cone of the dual of α is a resolution of $\overline{R}$. After we split off α_0^*, we obtain a resolution (B, b) for $\overline{R}$ in which the modules are $B_0 = L_g^*, B_i = F_{g-i+1}^* \oplus L_{g-i}^*$ for $1 \le i \le g-1$, and $B_g = F_1^*$; and the maps are $b_1 = [\alpha_g^* \;\; \partial_g^*]$,

$$b_i = \begin{bmatrix} d_{g-i+2}^* & 0 \\ (-1)^{i+1}\alpha_{g-i+1}^* & \partial_{g-i+1}^* \end{bmatrix},$$

for $2 \le i \le g-1$, and

$$b_g = \begin{bmatrix} d_2^* \\ (-1)^{g+1}\alpha_1^* \end{bmatrix}.$$

Since $L_i = \bigwedge^i L_1$, it is well-known that exterior multiplication

$$L_i \otimes L_{g-i} \to L_g \simeq R$$

induces an isomorphism of complexes:

$$\begin{array}{ccccccc} \ldots & \to & L_i^* & \xrightarrow{\partial_{i+1}^*} & L_{i+1}^* & \to & \ldots \\ & & \downarrow \sigma_i & & \downarrow \sigma_{i+1} & & \\ \ldots & \to & L_{g-i} & \xrightarrow{\partial_{g-i}} & L_{g-i-1} & \to & \ldots \end{array}.$$

Let $\beta_i = \alpha_{g-i}\sigma_i\alpha_i^*\colon F_i^* \to F_{g-i}$. It is obvious, from the form of B, that

$$\beta_g = \alpha_0\sigma_g\alpha_g^* = \sigma_g\alpha_g^*$$

induces an isomorphism from $\operatorname{coker}(d_g^*)$ onto $K/(\mathbf{a}) = (I+K)/I$. Application of the mapping cone construction to the map of complexes

$$\begin{array}{ccccccccccc}
0 & \to & F_0^* & \xrightarrow{d_1^*} & F_1^* & \to \dots \to & F_{g-1}^* & \xrightarrow{d_g^*} & F_g^* & \to & (I+K)/I \to 0 \\
 & & \downarrow \beta_0 & & \downarrow \beta_1 & & \downarrow \beta_{g-1} & & \downarrow \beta_g & & \downarrow \text{incl.} \\
0 & \to & F_g & \xrightarrow{d_g} & F_{g-1} & \to \dots \to & F_1 & \xrightarrow{d_1} & F_0 & \to & R/I \to 0
\end{array}$$

yields a resolution (C, c) of $R/(I+K)$ in which the modules are $C_0 = F_0$,

$$C_i = F_{g+1-i}^* \oplus F_i$$

for $1 \le i \le g$, and $C_{g+1} = F_0^*$; and the maps are $c_1 = [\beta_g \;\; d_1]$,

$$c_i = \begin{bmatrix} d_{g+2-i}^* & 0 \\ (-1)^{i+1}\beta_{g-i+1} & d_i \end{bmatrix},$$

for $2 \le i \le g$, and

$$c_{g+1} = \begin{bmatrix} d_1^* \\ (-1)^g \beta_0 \end{bmatrix}.$$

The augmentation of B^* onto $(I+K)/K$ that first comes to mind is induced by d_1:

$$F_2 \oplus L_1 \xrightarrow{b_g^*} F_1 \xrightarrow{d_1} (I+K)/K \longrightarrow 0,$$

with $b_g^* = [d_2 \;\; (-1)^{g+1}\alpha_1]$. Let η be any element of F_1 with

$$d_1(\eta) = y \in I. \tag{4.10}$$

Define $\lambda\colon \ker(b_g \otimes 1_{\overline{R}}) \to J/K$ as in Lemma 3.5 by

$$\lambda(\xi \otimes 1_{\overline{R}}) = \langle \xi,\ \eta \rangle + K \tag{4.11}$$

for $\xi \in F_1^*$ with $\xi \otimes 1_{\overline{R}} \in \ker(b_g \otimes 1_{\overline{R}})$.

The augmentation of C^* onto $R/(I+K)$ that first comes to mind is the natural quotient map

$$F_1 \oplus F_g^* \xrightarrow{c_{g+1}^*} F_0 = R \xrightarrow{\text{nat.}} R/(I+K) \longrightarrow 0,$$

with $c_{g+1}^* = [d_1 \;\; (-1)^g \beta_0^*]$. (Observe that the maps β_0^* and β_g from F_g^* to $F_0 = R$ satisfy $\beta_0^* = \varepsilon\beta_g$ for ε equal to 1 or -1.) Recall from (3.10) that $J = (K, y)\colon(I+K)$. Define $\tau\colon \ker(c_{g+1} \otimes 1_{R/(K,y)}) \to J/(K, y)$ as in Lemma 3.2 by

$$\tau(\gamma \otimes 1) = \gamma(1) + (K, y) \tag{4.12}$$

for $\gamma \in F_0^* = R^*$ with $\gamma \otimes 1$ in $\ker(c_{g+1} \otimes 1_{R/(K,y)})$.

Finally, we prove that $p\lambda = \tau\delta^{-1}\pi_*$. Let $z = [\xi \otimes 1]$ be an arbitrary element of $\operatorname{Tor}_g(\overline{R}, \overline{R}) = H_g(B \otimes \overline{R}) = \ker(b_g \otimes 1_{\overline{R}})$. Recall that $B_g = F_1^*$, so $\xi \in F_1^*$. We know from (4.11) that

$$p\lambda(z) = \langle \xi, \eta \rangle + (K, y)$$

for η defined in (4.10). We now calculate $\tau\delta^{-1}\pi_*(z)$. It is easy to check that $\gamma_0 = \alpha_0\sigma_g$,

$$\gamma_i = \begin{bmatrix} \mathrm{id} & 0 \\ 0 & \alpha_i\sigma_{g-i} \end{bmatrix}$$

for $1 \le i \le g-1$, and

$$\gamma_g = \begin{bmatrix} \mathrm{id} \\ 0 \end{bmatrix}$$

defines a map of complexes $\gamma \colon B \to C$ which covers π. Then $\pi_*(z) = [\gamma_g(\xi) \otimes 1]$ in $\operatorname{Tor}_g(R/(I+K), \overline{R})$. We calculate the connecting homomorphism using a diagram which is analogous to (4.6). Since

$$\operatorname{Tor}_{g+1}(R/(I+K), R/(K,y)) = H_{g+1}(C \otimes R/(K,y)),$$

there is an element $u \in C_{g+1} = F_0^*$ such that $\delta^{-1}\pi_*(z) = [u \otimes 1]$ and

$$c_{g+1}(u) \otimes 1 = y(\gamma_g(\xi) \otimes 1) \quad \text{in} \quad C_g \otimes \overline{R}. \tag{4.13}$$

We use the definition of τ, (4.12), to conclude that

$$\tau\delta^{-1}\pi_*(z) = u(1) + (K, y). \tag{4.14}$$

On the other hand, (4.13) implies that $d_1^*(u) \otimes 1 = y\xi \otimes 1$ in $F_1^* \otimes \overline{R}$; hence, $u(1)d_1(f) - y\xi(f)$ is in K for every $f \in F_1$. In particular, if we take f to be the element η defined in (4.10), then we conclude that $u(1)y \equiv y\xi(\eta) \bmod K$. Furthermore, since y is regular on $\overline{R}$, we see that $u(1) \equiv \xi(\eta) \bmod K$. It follows from (4.14) that $\tau\delta^{-1}\pi_*(z)$ is equal to the image of $\langle \xi, \eta \rangle$ in $R/(K, y)$, and the proof is complete.

The proof of Theorem 2.15 (a). Apply Lemma 4.1 (with $L = N = K$, $M = I + K$, and $z = y$) and Lemma 4.8 in order to obtain the commutative diagram:

$$\begin{array}{ccccc} \bigwedge^g \operatorname{Tor}_1(\overline{R}, \overline{R}) & \xrightarrow{\mu} & \operatorname{Tor}_g(\overline{R}, \overline{R}) & \xrightarrow{\lambda} & J/K \\ \downarrow \nu & & \downarrow \delta^{-1}\pi^* & & \downarrow p \\ \bigwedge^{g+1} \operatorname{Tor}_1\left(\frac{R}{(I+K)}, \frac{R}{(K,y)}\right) & \xrightarrow{\mu_2} & \operatorname{Tor}_{g+1}\left(\frac{R}{(I+K)}, \frac{R}{(K,y)}\right) & \xrightarrow{\tau} & \frac{J}{(K,y)}. \end{array} \tag{4.15}$$

The ideals H and $\mathfrak{L}$ of R are defined so that

$$H/K = \operatorname{im}(\lambda\mu), \text{ and } \mathfrak{L}/(K,y) = \operatorname{im}(\tau\mu_2).$$

Furthermore, the discussion surrounding (2.10) shows that

$$\bigwedge^{g+1}\operatorname{Tor}_1(R/(I+K), R/(K,y)) = \operatorname{im}(\nu) + \ker(\mu_2);$$

thus,

$$(H,y)/(K,y) = \operatorname{im}(p\lambda\mu) = \operatorname{im}(\tau\mu_2\nu) = \operatorname{im}(\tau\mu_2) = \mathfrak{L}/(K,y).$$

The proof of Theorem 2.16 (a). We first find elements $c_1, \dots, c_{g+1}$ of R so that (2.3) and (2.4) hold for the present ring R, which is not necessarily local. Let $a_1, \dots, a_g$ be a generating set for A, $a_{g+1} = y$, (E,d) be the Koszul complex on the generating set $a_1, \dots, a_{g+1}$ of $\mathfrak{A}$, F be a length g resolution of R/I, $\alpha \colon E \to F$ be a map of complexes which covers the natural map $R/\mathfrak{A} \to R/I$, $m \colon F_g^* \to R/I$ be an augmentation of F^* onto R/I, and u be an element of F_g^* with $m(u) = 1 + I$. If we define

$$c_i = (-1)^{i+1} u\alpha_g(\varepsilon_1 \wedge \dots \wedge \widehat{\varepsilon_i} \wedge \dots \wedge \varepsilon_{g+1}) \in R, \tag{4.16}$$

then (2.3) and (2.4) follow immediately.

The ideal K is the almost complete intersection $(a_1, \dots, a_g, c_{g+1})$; so, in the notation of Proposition 3.13, the image of $\mu \colon \bigwedge^g \operatorname{Tor}_1(\overline{R}, \overline{R}) \to \operatorname{Tor}_g(\overline{R}, \overline{R})$ is generated by

$$x = \mu\kappa(\bar{a}_1 \wedge \dots \wedge \bar{a}_g),$$

together with

$$\mu\kappa(\bar{a}_1 \wedge \dots \wedge \widehat{\bar{a}_i} \wedge \dots \wedge \bar{a}_g \wedge \bar{c}_{g+1}) \text{ for } 1 \le i \le g.$$

Proposition 3.13 furnishes an isomorphism $\lambda \colon \operatorname{Tor}_g(\overline{R}, \overline{R}) \to J/K$ which satisfies $\lambda(x) = y + K$. Using (2.4) we see that

$$\begin{aligned} y\lambda\mu\kappa(\bar{a}_1 \wedge \dots \wedge \widehat{\bar{a}_i} \wedge \dots \wedge \bar{a}_g \wedge \bar{c}_{g+1}) &= -\lambda\mu\kappa(\bar{a}_1 \wedge \dots \wedge \widehat{\bar{a}_i} \wedge \dots \wedge \bar{a}_g \wedge \sum_1^g \bar{c}_j\bar{a}_j) \\ &= vc_i\lambda(x) = vc_iy + K \end{aligned}$$

for v equal to $+1$ or -1. Since y is regular on $\overline{R}$, it follows that

$$\operatorname{im}(\lambda\mu) = ((c_1, \dots, c_g, y) + K)/K.$$

The ideal H of R is defined in (2.12) so that $\operatorname{im}(\lambda\mu) = H/K$. Thus,

$$H = K + (c_1, \ldots, c_g, y).$$

On the other hand, $K = (a_1, \ldots, a_g, c_{g+1})$, $y = a_{g+1}$, and $\mathfrak{A} = (a_1, \ldots, a_{g+1})$. It follows that $H = \mathfrak{A} + (c_1, \ldots, c_{g+1})$; and hence, $H = \mathfrak{K}$ from (2.3). Finally, Theorem 2.15 (a) says that $\mathfrak{L} = (H, y)$. But y is in H; so, $H = \mathfrak{K} = \mathfrak{L}$.

5. The Identification of Coker(μ).

In this section we prove part (c) of Theorems 2.15 and 2.16.

Proof of Theorem 2.15 (c). Let Q be the total ring of quotients of $\overline{R}$ (that is, $Q = \boxed{\mathrm{S}}^{-1}\overline{R}$ where $\boxed{\mathrm{S}}$ is the multiplicative set of regular elements of $\overline{R}$), let $\rho\colon \operatorname{Tor}_g(\overline{R}, \overline{R}) \to \operatorname{Hom}((I+K)/K, \overline{R})$ be the isomorphism of Proposition 3.13, and let $\vartheta\colon \operatorname{Tor}_1(\overline{R}, \overline{R}) \to K/K^2$ be the isomorphism of (1.5). We intend to define maps $\delta : \bigwedge^g(K/K^2) \to \overline{R}$ and $\tau : \operatorname{Hom}((I+K)/K, \overline{R}) \to Q$ so that there is a commutative diagram:

$$\begin{array}{ccccccccc}
 & & \bigwedge^g \operatorname{Tor}_1(\overline{R}, \overline{R}) & \xrightarrow{\mu} & \operatorname{Tor}_g(\overline{R}, \overline{R}) & \longrightarrow & \operatorname{coker}(\mu) & \longrightarrow & 0 \\
 & & {\scriptstyle \delta(\bigwedge^g \vartheta)} \downarrow {\scriptstyle \text{onto}} & & {\scriptstyle \tau\rho} \downarrow {\scriptstyle \cong} & & & & \\
0 & \longrightarrow & \mathfrak{D}_K & \longrightarrow & \mathfrak{D}_D & \longrightarrow & \mathfrak{D}_D/\mathfrak{D}_K & \longrightarrow & 0
\end{array}$$

with exact rows. We follow the methods of Kunz [25, Lemma 3]. Let n be the dimension of R, and d be the dimension of $\overline{R}$. (It follows that $g + d = n$.) Recall that R is a power series ring over an infinite perfect field k, and that $\overline{R}$ is a reduced equidimensional ring. We use the technique of noether normalization and pick linear forms $Y_1, \ldots, Y_n$ in R such that:

(a) $R = k[[Y_1, \ldots, Y_n]]$,

(b) $k[[Y_1, \ldots, Y_d]] \hookrightarrow R/A$ is a module finite extension, and

(c) Q is a product of finite separable field extensions of the quotient field of $k[[Y_1, \ldots, Y_d]]$.

Let $S = k[[Y_1, \ldots, Y_d]]$. Observe that it follows from (b) that $S \hookrightarrow \overline{R}$ is also a module finite extension. The <u>Kähler different</u>, $\mathfrak{D}_K$, of $\overline{R}$ over S, is the Fitting ideal of the module of Kähler differentials of $\overline{R}$ over S. The <u>Dedekind different</u>, $\mathfrak{D}_D$, of $\overline{R}$ over S, is the inverse ideal of the Dedekind complementary module $\{x \in Q \mid \operatorname{tr}(x\overline{R}) \subseteq S\}$, where tr denotes the usual trace map from Q into the quotient field of S (cf. [3]).

Define $\delta\colon \bigwedge^g K/K^2 \to \overline{R}$ to be the Jacobian map

$$\delta(\bar{z}_1 \wedge \ldots \wedge \bar{z}_g) = \det\left(\frac{\partial z_i}{\partial Y_j}\right) + K$$

for $j = d+1, \ldots, d+g = n$, and elements $z_1, \ldots, z_g$ of K. It is clear that $\mathfrak{D}_K$ is the image of δ in $\overline{R}$. As always, we let $a_1, \ldots, a_g$ be the regular sequence that generates $A = I \cap K$. Recall that these elements generate K locally at its minimal prime ideals. Let

$$D = \delta(\bar{a}_1 \wedge \ldots \wedge \bar{a}_g).$$

It follows that D is not a zero-divisor on $\overline{R}$. According to Kunz, $\mathfrak{D}_K \subseteq \mathfrak{D}_D$ and $\mathfrak{D}_D$ is equal to $D((I+K)/K)^{-1}$. The inverse is computed in Q. (Kunz's blanket assumption that $\overline{R}$ be an almost complete intersection is not used in this particular lemma.) Although $\mathfrak{D}_K$ and $\mathfrak{D}_D$ individually depend on the choice of S, the $\overline{R}-$module $\mathfrak{D}_D/\mathfrak{D}_K$ does not; this is well-known, but follows in any case from the identification with $\operatorname{coker}(\mu)$.

Define $\tau : \operatorname{Hom}((I+K)/K, \overline{R}) \to Q$ by $\tau(f) = D(1_Q \otimes f)(1)$. It is clear that τ is injective and that the image of τ is precisely $\mathfrak{D}_D$. We complete the argument by showing that

$$\tau\rho\mu\kappa(\bar{z}_1 \wedge \ldots \wedge \bar{z}_g) = \delta(\bar{z}_1 \wedge \ldots \wedge \bar{z}_g) \tag{5.1}$$

for all $z_1, \ldots, z_g$ in K and for κ defined in Proposition 3.13. First, take $z_i = a_i$ for all i. We know, from the choice of ρ made in Proposition 3.13, that

$$\tau\rho\mu\kappa(\bar{a}_1 \wedge \ldots \wedge \bar{a}_g) = \tau\rho(x) = D = \delta(\bar{a}_1 \wedge \ldots \wedge \bar{a}_g).$$

Now we can take the z_i to be arbitrary elements of K. Since yz_i is in $I \cap K$ for all i, there are elements r_{ij} such that $yz_i = \sum r_{ij}a_j$. Let r be the determinant of the matrix (r_{ij}), and observe, from the product rule, that

$$y(\partial z_i/\partial Y_k) \equiv \sum_{j=1}^{g} r_{ij}(\partial a_j/\partial Y_k) \bmod K.$$

Hence,

$$\begin{aligned} y^g\tau\rho\mu\kappa(\bar{z}_1 \wedge \ldots \wedge \bar{z}_g) &= r\tau\rho\mu\kappa(\bar{a}_1 \wedge \ldots \wedge \bar{a}_g) \\ &= r\delta(\bar{a}_1 \wedge \ldots \wedge \bar{a}_g) = y^g\delta(\bar{z}_1 \wedge \ldots \wedge \bar{z}_g). \end{aligned}$$

Line (5.1) follows since y is regular on $\overline{R}$.

We turn our attention to proving Theorem 2.16 (c). There are many definitions of cotangent cohomology, but they all agree when the hypotheses coincide. We give the original definition [31, (2.3)] of $T^2(\overline{R}/R,\overline{R})$; observe that no hypotheses are placed on the commutative noetherian ring R.

Definition 5.2. Let K be an ideal in the commutative noetherian ring R, let $\overline{R} = R/K$, and let K be the Koszul complex on a generating set for K. There is a natural map $\alpha : H_1(K) \to K_1 \otimes_R \overline{R}$. (If $z \in K_1$ is a cycle and $[z]$ is the corresponding element of $H_1(K)$, then $\alpha([z]) = z \otimes 1$.) The module $T^2(\overline{R}/R,\overline{R})$ is the cokernel of

$$\alpha^* \colon \mathrm{Hom}_{\overline{R}}(K_1 \otimes \overline{R}, \overline{R}) \to \mathrm{Hom}_{\overline{R}}(H_1(K), \overline{R}).$$

Whenever the ambient ring R is clear from the context, we write $T^2(\overline{R})$ in place of $T^2(\overline{R}/R,\overline{R})$. It is not necessary for us to make a deformation theoretic interpretation of T^2; however, we remind the reader that if R is a formal power series ring over a field, then $T^2(\overline{R})$ measures the obstructions to lifting infinitesimal deformations of $\overline{R}$.

Proof of Theorem 2.16 (c). (Compare to the proof of Satz 3.1 in [13].) Recall from (2.5) that

$$K = (a_1, \dots, a_g, c_{g+1}).$$

Let $w = c_{g+1}$, and let $H_1(K)$ denote the first Koszul homology on the given generators of K. Consider the syzygetic exact sequence (see, for example, [15]):

$$H_1(K) \xrightarrow{\alpha} \overline{R} \otimes (\bigoplus_{i=1}^{g+1} Re_i) \xrightarrow{\beta} K/K^2 \longrightarrow 0 \tag{5.3}$$

in which α is the map of Definition 5.2 and

$$\beta(1_{\overline{R}} \otimes \sum_1^{g+1} r_i e_i) = (\sum_1^{g} r_i a_i + r_{g+1} w) + K^2.$$

We define a map $\pi : H_1(K) \to (I+K)/K$. If $z = \sum r_i e_i$ is a one-cycle in the Koszul complex, then let $\pi([z]) = r_{g+1} + K$. Since $(A : K)/A = (I+K)/K$, it is well-known (and easy to prove directly) that π is a well-defined isomorphism. Define $\gamma : ((I+K)/K)^* \to H_1(K)^*$ to be the dual map π^*, and consider the diagram

$$\begin{array}{ccccccc}
\bigwedge^g \mathrm{Tor}_1(\overline{R},\overline{R}) & \xrightarrow{\mu} & \mathrm{Tor}_g(\overline{R},\overline{R}) & \longrightarrow & \mathrm{coker}(\mu) & \longrightarrow & 0 \\
 & & \gamma\rho \downarrow \cong & & & & \\
(\overline{R}^{g+1})^* & \xrightarrow{\alpha^*} & H_1(K)^* & \longrightarrow & \mathrm{coker}(\alpha^*) & \longrightarrow & 0
\end{array}$$

where "$*$" means $\mathrm{Hom}(—, \overline{R})$, μ is the Tor$-$algebra multiplication of (2.12), and ρ is found in Proposition 3.13. We complete the proof by showing that

$$\mathrm{im}(\gamma\rho\mu) = \mathrm{im}(\alpha^*).$$

Let $\{f_1, \dots, f_{g+1}\}$ be the basis for $(\overline{R}^{g+1})^*$ which is dual to $\{e_1, \dots, e_{g+1}\}$, and let $z = \sum r_i e_i$ be a cycle representative for an arbitrary element $[z]$ in $H_1(K)$. Then

$$\gamma\rho\mu\kappa(\bar{a}_1 \wedge \dots \wedge \bar{a}_g)[z] = \pi^*\rho(x)[z] = \rho(x)(\bar{r}_{g+1}) = \bar{r}_{g+1} = \alpha^*(f_{g+1})[z] \tag{5.4}$$

Since z is a cycle, $\sum_1^g r_i a_i + r_{g+1} w = 0$, and on the other hand

$$\sum_1^g c_i a_i + yw = 0 \tag{5.5}$$

by (2.4) (by way of (4.16)). Therefore, $\sum_1^g (yr_i - c_i r_{g+1})a_i = 0$, from which it follows that $yr_j - c_j r_{g+1}$ is in $(a_1, \dots, \widehat{a}_j, \dots, a_g)$. In particular,

$$yr_j \equiv c_j r_{g+1} \bmod K \tag{5.6}$$

for $1 \le j \le g$. Using (5.5), (5.4), and (5.6) we obtain

$$\begin{aligned} y\gamma\rho\mu\kappa(\bar{a}_1 \wedge \dots \wedge \bar{a}_{j-1} \wedge \bar{w} \wedge \bar{a}_{j+1} \wedge \dots \wedge \bar{a}_g)[z] &= -c_j\gamma\rho\mu\kappa(\bar{a}_1 \wedge \dots \wedge \bar{a}_g)[z] \\ &= -c_j\bar{r}_{g+1} = -y\bar{r}_j = y\alpha^*(-f_j)[z]. \end{aligned}$$

Since y is regular on $\overline{R}$, we conclude that $\mathrm{im}(\alpha^*) = \mathrm{im}(\gamma\rho\mu)$.

6. The Cohen-Macaulay Property of J.

There are various results in the literature dealing with the question of when an $s-$residual intersection J of an ideal I is Cohen-Macaulay ([19, 16, 22]). These theorems require depth conditions on **all** the Koszul homology modules of I. It turns out, however, that much less is needed to describe the Cohen-Macaulayness of J when $s = \mathrm{grade}(I) + 1$; moreover, these weaker conditions are also sufficient to ensure the preservation of the residual intersection relation under specialization. This is the content of Theorem 6.1 (a), (b), (c), whereas Theorem 6.1 (d) yields a partial converse.

Theorem 6.1. Let R be a Gorenstein ring, I be an ideal in R of grade g, $\mathfrak{A} = (a_1, \dots, a_s)$ be a proper sub-ideal of I, and $J = (\mathfrak{A} : I)$ be an $s-$residual intersection. Suppose that R/I is Cohen-Macaulay and I is generically a complete intersection. If $s = g + 1$, then the following statements hold:

(a) The ideal J is unmixed of grade $g+1$. If the residual intersection is geometric, then $\mathfrak{A} = I \cap J$.

(b) If the first Koszul homology of I, $H_1(I)$, is Cohen-Macaulay, then R/J is Cohen-Macaulay.

(c) Assume that $(R, \mathfrak{M})$ is local, J is a geometric residual intersection, and $H_1(I)$ is Cohen-Macaulay. Let $\mathbf{x}$ be a sequence in $\mathfrak{M}$ which is regular on both R and R/I. Let " $'$ " denote reduction modulo $(\mathbf{x})$. If I' is generically a complete intersection and grade $(\mathfrak{A}' : I') \geq g+1$, then $J' = (\mathfrak{A}' : I')$ and $\mathbf{x}$ is a regular sequence on both $R/\mathfrak{A}$ and R/J.

(d) If R is local, I is a Gorenstein ideal, $T^2(R/I) = 0$, and R/J is Cohen-Macaulay, then the normal module of I, $\mathrm{Hom}_R(I, R/I)$, is Cohen-Macaulay.

Note. Suppose that $s \geq g+2$. It is natural to wonder if statement (b) can be generalized. In particular, it is not known if assuming I satisfies G_s and $H_i(I)$ is Cohen-Macaulay for $1 \leq i \leq s-g$ suffices to guarantee that R/J is Cohen-Macaulay.

Proof. We first reduce the proofs of parts (a) and (b) to the case where R is local. Let $\mathfrak{M}$ be any maximal ideal of R. If J is not contained in $\mathfrak{M}$, then $I_\mathfrak{M} = \mathfrak{A}_\mathfrak{M}$, and nothing is to be shown. On the other hand, if I is not contained in $\mathfrak{M}$, then $J_\mathfrak{M} = \mathfrak{A}_\mathfrak{M}$ is a complete intersection of grade $g+1$, and we are also finished. Thus, we may assume that $I + J \subseteq \mathfrak{M}$. Furthermore, if $\mathrm{grade}(I_\mathfrak{M}) > g$, then $a_1, \ldots, a_{g+1}$ form an $R_\mathfrak{M}-$regular sequence. (Indeed, if P is a prime ideal with $P \subseteq \mathfrak{M}$ and grade $P \leq g$, then $I_P = J_P = R_P$; thus, $\mathfrak{A}$ is not contained in P.) If grade $(I_\mathfrak{M}) = g+1$, then all of the assertions follow from linkage theory. If grade $(I_\mathfrak{M}) > g+1$, then all of the assertions are trivial, because, in this case, $J_\mathfrak{M} = \mathfrak{A}_\mathfrak{M}$ is again a complete intersection of grade $g+1$. Thus, we may still suppose that grade $I_\mathfrak{M} = g$. We replace R by $R_\mathfrak{M}$ and prove the theorem for a local Gorenstein ring R.

By using induction on the number of elements in the regular sequence $\mathbf{x}$, we may reduce the proof of part (c) to the case where $\mathbf{x} = x$ consists of only one element. (See the proof of Theorem 4.7 in [22] for details.) In addition, after extending the residue field, if necessary, we may assume that $a_1, \ldots, a_g$ form an $R-$regular sequence with $I_P = (a_1, \ldots, a_g)_P$ for all $P \in \mathrm{Ass}(R/I)$, and that $a'_1, \ldots, a'_g$ form an $R'-$regular sequence with $I'_{P'} = (a'_1, \ldots, a'_g)_{P'}$ for all $P' \in \mathrm{Ass}(R'/I')$. In particular, x is also regular on $R/(a_1, \ldots, a_g)$. Let $K = (a_1, \ldots, a_g) : I$ and $y = a_{g+1}$. We see that K is a geometric link of I and that y is regular on R/K. Similarly, $(a'_1, \ldots, a'_g) : I'$ is a geometric link of I', and $y' = a'_{g+1}$ is regular on

$R'/((a'_1,\ldots,a'_g):I')$. We know, from [21, Lemma 2.12], that x is regular on R/K and that the image, K', of K in R' is equal to $(a'_1,\ldots,a'_g):I'$. Set $\overline{R}=R/K$ and $\overline{R'}=R'/K'$. Recall, from the proof of (3.10), that

$$\bar{J}=(\bar{y}\overline{R}:\bar{I})\cong \operatorname{Hom}(\bar{I},\overline{R}). \tag{6.2}$$

The linkage of I' and K' is geometric; so $I'\cap K'=(a'_1,\ldots,a'_g)$. It follows, from the argument below (2.7), that $(\mathfrak{A}':I')=(K',y'):(I'+K')$; in particular

$$\mathfrak{A}':I'=\frac{(K,x,y):(I+K)}{(x)}. \tag{6.3}$$

Since I and K are geometrically linked with R Gorenstein and R/I Cohen-Macaulay, it follows, from [33], that K is generically a complete intersection, $\overline{R}$ and $\overline{R}/\bar{I}$ are both Cohen-Macaulay rings, and $\bar{I}$ is an ideal of grade one in $\overline{R}$. We see, from (6.2), that $\bar{J}$ is unmixed of grade one; hence, J is unmixed of grade $g+1$. Furthermore, if J is a geometric residual intersection, then, by (1.6), the grade of $I+J$ is at least $g+2$. It follows, since $\overline{R}$ is Cohen-Macaulay, that

$$\text{grade } (\bar{I}+\bar{J})\geq 2; \tag{6.4}$$

therefore, $\bar{I}$ and $\bar{J}$ are geometrically linked and $\bar{I}\cap\bar{J}=\bar{y}\overline{R}$. Consequently, $I\cap J$ is contained in $yR+K$. Since $y\in I$, we even conclude

$$(I\cap J)\subseteq yR+(I\cap K)=yR+(a_1,\ldots,a_g)=\mathfrak{A}.$$

The proof of part (a) is complete.

We now prove part (b). The canonical module ω of $\overline{R}$ is isomorphic to $\bar{I}$. Since I and K are linked and $H_1(I)$ is Cohen-Macaulay, it follows, from [38, Theorem 3.1], that $S_2(\omega)$ is Cohen-Macaulay. In particular, $\operatorname{Hom}(S_2(\omega),\omega)$ is Cohen-Macaulay. On the other hand, $\overline{R}$ is generically Gorenstein; and therefore,

$$\operatorname{Hom}(S_2(\omega),\omega)\cong\operatorname{Hom}(\omega\otimes\omega,\omega).$$

Thus,

$$\bar{J}\cong\operatorname{Hom}(\bar{I},\overline{R})\cong\operatorname{Hom}(\omega,\overline{R})\cong\operatorname{Hom}(\omega\otimes\omega,\omega)$$

is a Cohen-Macaulay $\overline{R}-$module; and hence, $R/J\cong\overline{R}/\bar{J}$ is a Cohen-Macaulay ring.

To prove (c), we first show that the $\overline{R}-$module $\omega\cong\bar{I}$ is cyclic locally in codimension one. Let $P\in V(K)$ with $\dim\overline{R}_{\bar{P}}=1$. We may assume that $\bar{I}\subseteq\bar{P}$. It follows, from (6.4), since J is a geometric residual intersection of I, that $\bar{J}$ is not contained in $\bar{P}$. Therefore, J is not contained in P; hence, $I_P=\mathfrak{A}_P=$

$(a_1, \ldots, a_g, y)_P$, and $\bar{I}_{\bar{P}} = \bar{y}\overline{R}_{\bar{P}}$ is cyclic. We conclude that the kernel of the natural epimorphism from $\omega \otimes \omega$ to $S_2(\omega)$ is supported in codimension at least two. Therefore

$$\mathrm{Ext}^1_{\overline{R}}(S_2(\omega), \omega) \cong \mathrm{Ext}^1_{\overline{R}}(\omega \otimes \omega, \omega).$$

On the other hand, by [14, Satz 1.2],

$$\mathrm{Ext}^1_{\overline{R}}(\omega \otimes \omega, \omega) \cong \mathrm{Ext}^1_{\overline{R}}(\omega, \overline{R}).$$

Thus,

$$0 = \mathrm{Ext}^1_{\overline{R}}(S_2(\omega), \omega) \cong \mathrm{Ext}^1_{\overline{R}}(\omega, \overline{R}) \cong \mathrm{Ext}^1_{\overline{R}}(\bar{I}, \overline{R}), \tag{6.5}$$

with the first module vanishing because $S_2(\omega)$ is a maximal Cohen-Macaulay module over $\overline{R}$. The exact sequence

$$0 \to \overline{R} \xrightarrow{x} \overline{R} \to \overline{R}/x\overline{R} \to 0$$

induces, via (6.5), an exact sequence

$$0 \to \mathrm{Hom}(\bar{I}, \overline{R}) \xrightarrow{x} \mathrm{Hom}(\bar{I}, \overline{R}) \to \mathrm{Hom}(\bar{I}, \overline{R}/x\overline{R}) \to \mathrm{Ext}^1_{\overline{R}}(\bar{I}, \overline{R}) = 0. \tag{6.6}$$

Since y is in I and y is regular on $R/(K, x)$, we see that the natural inclusion $\mathrm{Hom}((I+K)/K, R/(K,x)) \to \mathrm{Hom}(I+K, R/(K,x))$ is actually equality. By applying the isomorphism $f \to f(y)$ to (6.6) (see the discussion above (3.10)), we learn that the natural map

$$\frac{(K,y):(I+K)}{K} \to \frac{(K,x,y):(I+K)}{K+(x)}$$

is a surjection. Thus, $J + (x) = ((K,y):(I+K)) + (x) = (K,x,y):(I+K)$, and we conclude, from (6.3), that $J' = (\mathfrak{A}' : I')$. Since, moreover, by parts (a) and (b), R/J is Cohen-Macaulay and grade J = grade J', it follows that x is regular on R/J. We also conclude that x is regular on $R/\mathfrak{A}$ because $\mathfrak{A} = I \cap J$. This finishes the proof of (c).

We now prove (d). Since K is an almost complete intersection, we may choose $H_1(K) = \omega$; and therefore (6.2) becomes

$$\bar{J} \cong \mathrm{Hom}(H_1(K), \overline{R}). \tag{6.7}$$

On the other hand, $T^2(\overline{R}) \cong T^2(R/I) = 0$, by Theorem 2.19 (b); and hence, the syzygetic sequence (5.3), combined with (6.7), yields an exact sequence

$$0 \to \mathrm{Hom}(K, \overline{R}) \to \mathrm{Hom}(\overline{R}^{g+1}, \overline{R}) \to \bar{J} \to 0. \tag{6.8}$$

The ring R/J is Cohen-Macaulay by assumption; hence, $\bar{J}$ is a maximal Cohen-Macaulay module over $\overline{R}$; and therefore, $\mathrm{Hom}(K, \overline{R})$ is Cohen-Macaulay by (6.8). Since I is a perfect ideal in a local Gorenstein ring, and I and K are linked, it follows from [9] (or [8] in the case that R is regular local and contains a field) that $\mathrm{Hom}(I, R/I)$ is also Cohen-Macaulay. This finishes the proof of part (d).

7. Examples.

We apply the techniques of the previous sections in order to exhibit a set of generators for several families of **generic** residual intersections. That is, we compute the generating set for J where

(7.1) $I = (f_1, \dots, f_n)$ is an ideal of height $g > 0$ in a commutative noetherian ring R, $s \geq g$ is an integer for which I satisfies G_{s+1}, Z is an $n \times s$ matrix of indeterminates, $[a_1, \dots, a_s] = [f_1, \dots, f_n]Z$, and $J = (\mathfrak{A} : IS)$ for $\mathfrak{A} = (a_1, \dots, a_s)$ and $S = R[Z]$.

Our justification for giving examples of generic, rather than arbitrary, residual intersection is two-fold. First of all, under suitable assumptions, an arbitrary residual intersection can be deformed into a generic residual intersection. (See the proof of Theorem 5.1 in [22] for details.) Secondly, as we observe in (7.9), residual intersection behaves nicely under base change if the image is geometric. We need some general facts about generic residual intersections to show that the main theorems of this paper apply to the situation of (7.1) with $s = g+1$. (Further results about generic residual intersections may be found in [22] and [23].)

Lemma 7.2. ([22], Lemma 3.2]) In the notation of (7.1), the following statements hold.

(a) Let Q be a prime ideal of S. If either

  (i) $\mathrm{ht}(Q) \leq s-1$, or

  (ii) $\mathrm{ht}(Q) \leq s$ and $IS \subseteq Q$,

  then $(a_1, \dots, a_s)_Q = I_Q$.

(b) The ideal J is a geometric $s-$residual intersection of IS.

Lemma 7.3. If I is a perfect ideal, R is a Cohen-Macaulay ring, and $s = g+1$ in the notation of (7.1), then the hypotheses of (2.14) are satisfied for the ideal IS in the ring S.

Proof. Let $K = ((a_1, \dots, a_g) : IS)$ and $y = a_{g+1}$. It is well-known that $a_1, \dots, a_g$ forms a regular sequence; (see, for example, [17, Prop. 21]) consequently, IS and K are linked. Lemma 7.2 guarantees that the linkage is geometric and that J is a geometric $(g+1)-$residual intersection of I. The argument which appears under (2.14) shows that y is regular on R/K.

In our first few examples the ideal I is Gorenstein. Corollary 2.18, together with recipe (2.3), gives an algorithm for identifying the generators of J. The calculations which are needed to implement this algorithm are straightforward, although unpleasant to exhibit. We use a different approach. We exhibit elements $w_1, \dots, w_{g+1}$ in J of the appropriate degree and then prove that J is equal to $(w_1, \dots, w_{g+1},\ a_1, \dots, a_{g+1})$ by way of an easy linear independence argument. The technique is summarized in Proposition 7.7. We begin by explaining the phrase "the appropriate degree".

We view $R = Z[\{x_{ij}\}]$ as a graded ring in such a way that the generators $f_1, \dots, f_n$ of I all are homogeneous. (Each variable x_{ij} is assigned some positive degree, not necessarily one.) Let

$$d_1 = \max\{\deg (f_i) \mid 1 \le i \le n\}. \tag{7.4}$$

We assign a positive degree to each variable z_{ij} so that each a_i has degree (d_1+1). For all of the ideals I under consideration, there is a homogeneous resolution F of R/I in which

$$F_g = R(-d_g) \tag{7.5}$$

for some integer d_g. The elements $c_1, \dots, c_{g+1}$ of (2.2) all are homogeneous of degree

$$d = g(d_1+1) - d_g. \tag{7.6}$$

Proposition 7.7. Let I be a homogeneous Gorenstein ideal of grade $g > 0$ in a graded polynomial ring $R = Z[\{x_{ij}\}]$. Define $\mathfrak{A}$, J, and S as in (7.1) (with $s = g+1$). Assume that there is an integer d_g as defined in (7.5). Define d as in (7.6). Suppose that $w_1, \dots, w_{g+1}$ are elements of J with $\bar{w}_1, \dots, \bar{w}_{g+1}$ linearly independent in the (Z/pZ)-vector space

$$S_d/(\mathfrak{A}_d + pS_d) \tag{7.8}$$

for all prime integers p. Suppose further that I satisfies G_{g+2} and $T^2(R/I) = 0$. Then

(a) J is a geometric $(g+1)-$residual intersection of I,

(b) J is a prime ideal of height $g+1$,

(c) $J = (w_1, \dots, w_{g+1},\ a_1, \dots, a_{g+1})$,

(d) if $H_1(I)$ is Cohen-Macaulay, then J is perfect.

Note. Recall from [18] and Theorem 2.19 (a) that if I is licci and satisfies G_{g+2}, then I is strongly Cohen-Macaulay and $T^2(R/I) = 0$.

Proof. Lemma 7.2 (b) shows that (a) holds, and then Theorem 6.1 (a) implies that J is unmixed of grade $g+1$. Since $\operatorname{grade}(IS+J) \geq g+2$ by (1.6), we conclude that I is not contained in the union of the associated primes of J. It follows, as in the proof of Theorem 3.3 (vii) in [22], that J is prime. This shows (b). Assertion (d) follows from Theorem 6.1 (a) and (b) since R is regular. It remains to prove part (c). From Corollary 2.18 we see that J is equal to the ideal $(c_1, \dots, c_{g+1}) + \mathfrak{A}$ of (2.3). Let G be the ideal $(w_1, \dots, w_{g+1}) + \mathfrak{A}$ of S. The ideals J and G each are generated by $a_1, \dots, a_{g+1}$ together with $g+1$ elements of degree d. We know that $G \subseteq J$. The proof is complete when we show that $J_d \subseteq G_d$.

Fix a prime integer p. Our hypothesis guarantees that the $(Z/pZ)-$vector spaces

$$\frac{G_d + pS_d}{\mathfrak{A}_d + pS_d} \quad \text{and} \quad \frac{J_d + pS_d}{\mathfrak{A}_d + pS_d}$$

are equal. Thus, $G_d + pS_d = J_d + pS_d$. The ideal J is prime, and p is not in J (for degree reasons), so p is regular on S/J. It follows that there is an inclusion of finitely generated $Z-$modules $J_d \subseteq G_d + pJ_d$. Nakayama's Lemma yields that $(J_d)_{(p)} = (G_d)_{(p)}$ for all prime integers p, and thus $J_d = G_d$.

Before giving actual examples, we point out that the residual intersections calculated using Proposition 7.7 (d) behave very nicely under base change.

Observation 7.9. Let J be a perfect ideal of grade s in a commutative noetherian ring S. Suppose that $J = (\mathfrak{A} : I)$ is an $s-$residual intersection in S and that T is an $S-$algebra. If

(a) JT is a proper ideal of T with grade at least s, and

(b) grade $(J+I)T \geq s+1$,

then, by the "persistence of perfection", JT is a perfect ideal of grade s and $JT = (\mathfrak{A}T : IT)$ is a geometric $s-$residual intersection.

Proof. It suffices to show that $(\mathfrak{A}T : IT)_P \subseteq JT_P$ for all $P \in \operatorname{Ass}(T/JT)$. Every such prime ideal P has grade s; consequently, IT is not contained in P. Thus,

$$\mathfrak{A}T_P \subseteq JT_P \subseteq (\mathfrak{A}T_P : IT_P) = (\mathfrak{A}T_P : T_P) = \mathfrak{A}T_P.$$

Example 7.10. We first determine the generic 4−residual intersection of a grade three Gorenstein ideal. Let $n \geq 3$ be an odd integer, X be an $n \times n$ alternating matrix of indeterminates, Z be an $n \times 4$ matrix of indeterminates, and S be the ring $Z[X, Z]$. Each variable is given degree one. Let I be the ideal of S generated by the maximal order pfaffians of X. Select generators $f_1, \ldots, f_n$ for I in the usual way so that $[f_1, \ldots, f_n]X = 0$. Define $\mathfrak{A}$ and J as in (7.1). The ideal I is licci and G_∞, so Proposition 7.7 applies. The resolution of S/I is

$$0 \to S(-n) \to S(-(d_1 + 1))^n \to S(-d_1)^n \to S$$

where $d_1 = (n - 1)/2$; and therefore, the integer d of (7.6) is

$$d = \frac{3(n + 1)}{2} - n = \frac{(n + 3)}{2}.$$

Let Y be the $(n + 4) \times (n + 4)$ alternating matrix

$$Y = \begin{bmatrix} X & Z \\ -Z^T & 0 \end{bmatrix}, \tag{7.11}$$

and let w_i be the pfaffian of Y with row and column $n + i$ deleted. We will use Proposition 7.7 to show that $J = (w_1, w_2, w_3, w_4, a_1, a_2, a_3, a_4)$. It is clear that each w_i is a homogeneous polynomial of degree d.

To show that each w_i is in J we use a formula about the lower order pfaffians of an arbitrary $N \times N$ alternating matrix Y with N odd. For each index set $(i) = (i_1, \ldots, i_r)$, let $\sigma(i)$ represent the sign of the permutation which arranges $i_1, \ldots, i_r,\ j_1, \ldots, j_{N-r}$ into ascending order where

$$\{i_1, \ldots, i_r,\ j_1, \ldots, j_{N-r}\} = \{1, \ldots, N\} \text{ and } j_1 < \cdots < j_{N-r}.$$

(If some index is repeated, then $\sigma(i) = 0$.) Let $Y_{(i)}$ represent $\sigma(i)$ times the pfaffian of Y with rows and columns $i_1, \ldots, i_r$ removed. If a, b, c, d, ℓ, and m are indices, then

$$Y_{\ell ma}Y_{bcd} - Y_{\ell mb}Y_{acd} + Y_{\ell mc}Y_{abd} - Y_{\ell md}Y_{abc} = Y_\ell Y_{mabcd} - Y_m Y_{\ell abcd}. \tag{7.12}$$

(See, for example, [27, Corollary 2.1].) Fix i and j with $1 \leq i \leq n$ and $1 \leq j \leq 4$. Apply (7.12) to the matrix Y of (7.11) with $\ell = i$, $m = n+j$, $a = n+1$, $b = n+2$,

$c = n+3$, and $d = n+4$. Observe that $Y_m = \pm w_j$, $Y_{\ell abcd} = \pm f_i$, $Y_{mabcd} = 0$, and $Y_{pqr} = \pm a_s$ for $\{p,q,r,s+n\} = \{n+1, n+2, n+3, n+4\}$. It follows that $w_j f_i \in \mathfrak{A}$.

Let W be the subspace of the vector space on line (7.8) which is generated by $\overline{w}_1, \overline{w}_2, \overline{w}_3, \overline{w}_4$. If we set $x_{ij} = 0$ for $1 \le i \le 3$, then the vector space of (7.8) becomes

$$((Z/pZ)[\{x_{ij} \mid i > 3\}, Z])_d. \tag{7.13}$$

The image of W in (7.13) is $X_{123} I_3(Z')$ where Z' is the generic 3×4 matrix which consists of the top three rows of Z. It is now obvious that W has dimension four.

Using this example as a model, two of the authors [30] have found the generators of the generic $s-$residual intersection of a height 3 Gorenstein ideal for all $s \ge 4$. They have also resolved these ideals. In the notation of this example, with Z generic of size $n \times s$, the residual intersection J is generated by the pfaffians of all principal submatrices of Y containing X.

Example 7.14. We next calculate the generic $(g+1)-$residual intersection of a generic deviation two Huneke-Ulrich Gorenstein ideal of grade g. Fix an odd integer $g \ge 3$ and let m equal $g+1$. Let X be an $m \times m$ alternating matrix of indeterminates; $Y_{1\times m}$, $U_{m\times m}$, and $V_{1\times m}$ be matrices of indeterminates; and S be the ring $Z[X,Y,U,V]$. Let $\ell = [\ell_1, \dots, \ell_m]$ be the product YX and let $\mathrm{Pf}(X)$ be the pfaffian of X. The ideal $I = (\ell_1, \dots, \ell_m, \ \mathrm{Pf}(X))$ is the generic Huneke-Ulrich Gorenstein ideal of grade g. These ideals were introduced in [21, Lemma 5.12]. They were resolved in [26] and also in [24]. Recently, Srinivasan [34] has proved that the minimal resolution of the algebras defined by these ideals is a $DG-$algebra. The ideals are known to be licci, and they obviously satisfy the condition G_∞.

Let Z be the matrix

$$Z = \begin{bmatrix} U \\ V \end{bmatrix}.$$

Define $\mathfrak{A}$ and J as in (7.1) using the given generating set for I. The variables x_{ij} and y_i are all given degree one. The integer d_1 of (7.4) is

$$d_1 = \deg(\mathrm{Pf}(X)) = \frac{m}{2} = \frac{(g+1)}{2}.$$

We let each v_i have degree one and each u_{ij} have degree $d_1 - 1$. We learn from [26, Theorem 6.1] that the integer d_g of (7.5) is

$$d_g = 4d_1 - 3 = 2g - 1;$$

thus, the integer d of (7.6) is

$$d = \frac{g(g+3)}{2} - (2g-1) = \binom{g}{2} + 1.$$

If i and j are integers, then let

$$\sigma(ij) = \begin{cases} 1 & \text{if} \quad i < j \\ 0 & \text{if} \quad i = j \\ -1 & \text{if} \quad j < i. \end{cases} \tag{7.15}$$

Let X_{ij} be $\sigma(ij)$ times the pfaffian of X with rows and columns i and j removed; $U(i;b)$ be the determinant of U with row i and column b removed; Z_i be the determinant of Z with row i removed; and $Z(i,j;b)$ be $\sigma(ij)$ times the determinant of Z with rows i and j and column b removed. For each integer b with $1 \le b \le m$, let

$$w_b = \sum_{i=1}^{m} (-1)^{i+1}\, y_i\, U(i;b) + \sum_{1 \le i < j \le m} X_{ij}\, Z(i,j;b).$$

We apply Proposition 7.7 in order to show that

$$J = (a_1, \dots, a_m,\ w_1, \dots, w_m).$$

It is not difficult to see that each w_b is a homogeneous polynomial of degree d. We next prove that each w_b is in J. There is no conceptual difficulty in verifying directly that $w_b I \subseteq \mathfrak{A}$ for all b. However, it is computationally simpler to observe that it suffices to prove that $w_b \in JS_\Delta$ where Δ is the determinant of U. (Indeed, J is a prime ideal contained in $(X,Y)S$, and $\Delta \notin (X,Y)S$. One way to see that $J \subseteq (X,Y)S$ is to recall that

$$\mu(IS_{(X,Y)}) = \mu(I_{(X,Y)}) = m + 1 > \mu(\mathfrak{A}S_{(X,Y)});$$

hence, $JS_{(X,Y)} \neq S_{(X,Y)}$.) Consequently, we make our calculations in S_Δ. Let

$$\begin{aligned} [w_1', \dots, w_m'] &= [w_1, -w_2, w_3, -w_4, \dots, -w_m]\, U^T, \quad \text{and} \\ [a_1', \dots, a_m'] &= [a_1, \dots, a_m]\, U^{-1}. \end{aligned}$$

One may use the Laplace expansion for determinants as well as the Laplace expansion of Pfaffians:

$$\sum_j (-1)^j\, x_{ij}\, X_{kj} = (-1)^{i+1}\, \delta_{ik}\, \mathrm{Pf}(X)$$

in order to verify:

(a) $a'_b = \ell_b + (\frac{1}{\Delta})(-1)^b \, Z_b \, \mathrm{Pf}(X)$

(b) $w'_b = \Delta y_b + \sum_{i=1}^m \, (-1)^b \, Z_i \, X_{ib}$

(c) $\sum_{i=1}^m \, (-1)^{b+i} \, \ell_i \, X_{ib} = y_b \, \mathrm{Pf}(X)$.

Using these formulas we see that

$$\begin{aligned}\sum_{i=1}^m (-1)^{b+i} X_{ib} \Delta a'_i &= \Delta \sum_{i=1}^m (-1)^{b+i} \ell_i X_{ib} + \mathrm{Pf}(X) \sum_{i=1}^m (-1)^b Z_i X_{ib} \\ &= \mathrm{Pf}(X)(\Delta y_b + \sum_{i=1}^m (-1)^b Z_i X_{ib}) = \mathrm{Pf}(X) w'_b.\end{aligned}$$

Thus $(w_1, \ldots, w_m)(\mathrm{Pf}(X))S_\Delta \subseteq \mathfrak{A} S_\Delta$. It follows from (a) that

$$(w_1, \ldots, w_m)(IS_\Delta) \subseteq \mathfrak{A} S_\Delta;$$

and thus, $w_b \in J$ for all b.

Let W be the subspace of the vector space on line (7.8) which is generated by $\overline{w}_1, \ldots, \overline{w}_m$. If we set all x_{ij} and all v_i equal to zero, then the vector space of (7.8) becomes $(Z/p)[Y, U]_d$, and the image of W is spanned by the entries of $Y[\mathrm{adj}(U)]^T$. It is now immediate that W has dimension m.

Example 7.16. We can also use Proposition 7.7 to calculate the generic $(g+1)-$residual intersection of the Herzog ideal of grade g. Let $g \geq 3$ be an integer, $X_{g\times(g-1)}$, $Y_{1\times g}$, $t_{1\times 1}$, $U_{(g-1)\times(g+1)}$, and $V_{g\times(g+1)}$ be matrices of indeterminates, and S be the ring $Z[X, Y, t, U, V]$. For each integer i with $1 \leq i \leq g$, let

$$\Delta_i = (-1)^{i+1} \det \, (X \text{ with row } i \text{ deleted}).$$

Let Δ be the matrix $[\Delta_1, \ldots, \Delta_g]$. The ideal

$$I = I_1(YX) + I_1(\Delta + tY)$$

is called the grade g Herzog ideal. It is a Gorenstein ideal two links from a complete intersection. These ideals (with $t = 0$) were resolved in [12]; an algebra structure was put on their resolutions in [29]. It is shown in [2] that every Gorenstein ideal in a local ring which is two links from a complete intersection is a Herzog ideal. A straightforward calculation yields that the Herzog ideals satisfy the condition G_∞.

Let Z and $\mathbf{a} = [a_1, \ldots, a_{g+1}]$ be the matrices

$$Z = \begin{bmatrix} U \\ V \end{bmatrix} \quad \text{and} \quad \mathbf{a} = [YX \mid \Delta + tY]Z = YXU + \Delta V + tYV.$$

Define $\mathfrak{A}$ and J as in (7.1). If we give all $x_{ij},\ y_i$, and v_{ij} degree one; and all u_{ij} and t degree $g-2$, then

$$d_1 = g-1,\ \ d_g = 3g-4,\ \ \text{and}\ \ d = g^2 - 3g + 4.$$

Let M be the $g \times (g+1)$ matrix

$$M = XU + tV.$$

Fix integers i and j, with $1 \le i,\ j \le g+1$. Let

$$M_j = \det\ (M \text{ with column } j \text{ removed});$$

let $\sigma(ij)$ be $+1$, 0, or -1 as defined in (7.15); and let $U^{(j)}$ be the $(g+1)\times 1$ matrix whose i^{th} entry is

$$(-1)^{i+1}\sigma(ij)\ \det\ (U \text{ with columns } i \text{ and } j \text{ deleted}).$$

We will show that $J = (a_1, \ldots, a_{g+1},\ w_1, \ldots, w_{g+1})$ where w_j is the element of S defined by the equation

$$M_j - t(\Delta + tY)V\ U^{(j)} = t^2 w_j. \tag{7.17}$$

We show that w_j exists by viewing the left side of (7.17) as a polynomial in the variable t with coefficients from $Z[X, Y, U, V]$. The constant term of M_j is zero because it is the determinant of the product

$$X_{g\times(g-1)}(U \text{ with one column removed})_{(g-1)\times g}.$$

If $(XU)(i; b, j)$ represents $\sigma(bj)$ times the determinant of (XU) with row i and columns b and j deleted, then the linear coefficient in M_j is obviously equal to

$$\sum_{\substack{1\le i\le g \\ 1\le b\le g+1}} (-1)^{i+b} v_{ib}(XU)(i; b, j) = \Delta V U^{(j)}.$$

Of course, the linear coefficient of $t(\Delta + tY)VU^{(j)}$ is also equal to $\Delta VU^{(j)}$. The existence of w_j has been established.

Every entry of M is a homogeneous polynomial of degree $g-1$; consequently, M_j is a homogeneous form of degree

$$g(g-1) = d + 2(g-2).$$

It is not difficult to see that $t(\Delta + tY)VU^{(j)}$ is a form of the same degree; hence, each w_j is a form of degree d.

The ideal J is prime and is generated by $g+1$ elements of degree d_1+1 and $g+1$ elements of degree d. The element t has degree $g-2$, and $g-2$ is less than both d_1+1 and d. Thus, t is a regular element on S/J. We prove that w_j is in J by showing that $t^2 w_j \in J$. Cramer's rule shows that

$$\Delta X = 0.$$

It follows that

$$(\Delta + tY)M = t\mathbf{a}, \quad \text{and} \tag{7.18}$$

$$(\Delta + tY)X = tYX. \tag{7.19}$$

(Indeed, $(\Delta + tY)M = (\Delta + tY)(XU + tV) = t(YXU + \Delta V + tV)$.) We conclude from (7.18) that $I_g(M) \cdot I_1(\Delta + tY) \subseteq \mathfrak{A}$; and hence from (7.19) we see that $tI_g(M) \subseteq (\mathfrak{A} : I) = J$; thus

$$(M_1, \ldots, M_{g+1}) = I_g(M) \subseteq J.$$

Cramer's rule also yields that

$$UU^{(j)} = 0;$$

thus from (7.18) we see that

$$t(\Delta + tY)VU^{(j)} = (\Delta + tY)(XU + tV)U^{(j)} = (\Delta + tY)M\, U^{(j)} = t\mathbf{a}U^{(j)}.$$

In other words, $t(\Delta + tY)VU^{(j)}$ is an element of $\mathfrak{A} \subseteq J$; and the left side of (7.17) is in J.

As always we let W be the subspace of the vector space on line (7.8) generated by $\overline{w}_1, \ldots, \overline{w}_{g+1}$. If we set all $x_{ij} = 0$ and $t = 0$, then (7.8) becomes $(Z/p)[U, V, Y]_d$ and the image of W is generated by the entries of the vector

$$YV[U^{(1)} \mid U^{(2)} \mid \ldots \mid U^{(g+1)}].$$

Once again, it is clear that W has dimension $g+1$.

It is worth noting that for the next class of examples, our approximation $\mathfrak{K}$ accurately calculates the residual intersection J without these ideals being Cohen-Macaulay.

Example 7.20. Gorenstein determinantal ideals are not licci (except for complete intersections); nonetheless, they are amenable to our techniques. Let X be a generic $m \times m$ matrix, t be an integer with $1 < t < m$, B a Gorenstein ring, and $R = B[X]$. If I is the ideal $I_t(X)$ of R, then I is a Gorenstein ideal of grade $g = (m-t+1)^2$. (See [36, pp. 451–452] or [6, Corollary 8.9].) Furthermore, we show in Proposition 7.21 that $T^2(R/I) = 0$. If we form S, $\mathfrak{A}$, and J as in (7.1) with $s = g+1$, then $J = \mathfrak{K}$ by Corollary 2.18 and J is prime by Proposition 7.7 (b). However, S/J is not Cohen-Macaulay; in fact, a depth chase involving [9] and the proofs of Theorem 6.1 (d) and Proposition 7.21 yields that depth $(S/J)_{(X,Z)} = \dim\,(S/J)_{(X,Z)} - 3$.

If $t = m-1$ or if B contains the field of rational numbers, then the last shift in the $R-$resolution of R/I is known to be $d_g = m(m-t+1)$. (This formula was communicated to us by Joe Brennan. It is derived using Lascoux's description of the resolution. In the case $t = m-1$, the Gulliksen Negård complex [11] is a resolution of R/I for any commutative noetherian ring B.) We may use (7.6) in order to see that J is generated by $g+1$ forms of degree $t+1$ together with $g+1$ forms of degree $(m-t+1)(mt-t^2+1)$.

Proposition 7.21. (See also [35], [8], and [39].) Let X be a generic $m \times m$ matrix, t be an integer with $1 < t < m$, B be a commutative noetherian ring, and $R = B[X]$. If I is the ideal $I_t(X)$ in R, then $T^2(R/I) = 0$, but $(\mathrm{Hom}(I, R/I))_{(X)}$ is not Cohen-Macaulay.

Proof. Let Ω be the module of differentials of R/I over B, and consider the fundamental exact sequence (cf. [32, Theorem 58]):

$$I/I^2 \xrightarrow{\delta} \oplus\, R/I \to \Omega \to 0, \tag{7.22}$$

as well as, the syzygetic sequence (5.3):

$$H_1(I) \xrightarrow{\alpha} \oplus\, R/I \to I/I^2 \to 0. \tag{7.23}$$

If Δ is a $(t-1) \times (t-1)$ minor of X, then it is well known that $(I)_\Delta$ is generated by a regular sequence. It follows that the grade of the annihilators of the

$(R/I)-$modules $\ker(\delta)$ and $\ker(\alpha)$ is at least the grade of the ideal $I_{t-1}(X)(R/I)$, which is at least two. Thus, $\mathrm{Ext}^i_{R/I}(\ker(\delta), R/I)$ and $\mathrm{Ext}^i_{R/I}(\ker(\alpha), R/I)$ are zero for $i \le 1$; hence, (7.22), (7.23), and Definition 5.2 imply that

$$T^2(R/I) \cong \mathrm{Ext}^2_{R/I}(\Omega, R/I). \tag{7.24}$$

On the other hand, by [35, 6.8.1] (cf. also [39, Theorem 4.4.1] or [6, Theorem 15.10]),

$$\mathrm{Ext}^i_{R/I}(\Omega, R/I) = 0 \text{ for } 1 \le i \le 2. \tag{7.25}$$

It follows that $T^2(R/I) = 0$.

To see that $(\mathrm{Hom}(I, R/I))_{(X)}$ is not Cohen-Macaulay, simply compute the dual of (7.22) and use (7.25) to see that $\mathrm{Hom}(\Omega, R/I)$ is a first syzygy-module of $\mathrm{Hom}(I, R/I)$. However, by [39, Theorem 3.2] (see also [6, Remark 15.8]),

$$\mathrm{depth}(\mathrm{Hom}(\Omega, R/I))_{(X)} = \dim(R/I)_{(X)} - 1;$$

and therefore,

$$\mathrm{depth}(\mathrm{Hom}(I, R/I))_{(X)} = \dim(R/I)_{(X)} - 2.$$

We next apply our theory in the situation that I is a perfect ideal of grade 2. The generators of generic residual intersections of I were found by C. Huneke. Consequently, our proof that $J = \mathfrak{L}$ in this case, does not yield the generators for a new class of residual intersections. However, it is interesting to observe that $\mathfrak{L}$ gives rise to a minimal generating set of J.

Example 7.26. Let R be a commutative noetherian ring, X be an $n \times (n-1)$ matrix of indeterminates, Z be an $n \times s$ matrix of indeterminates, and S be the ring $R[X, Z]$. The ideal $I = I_{n-1}(X)$ is a generic perfect ideal of grade 2. It is licci and G_∞. Let

$$f_i = (-1)^{i+1}\det(X \quad \text{with row } i \text{ removed}).$$

Form $\mathfrak{A}$ and J as in (7.1). If P is the ideal $I_n(X \mid Z)$, then the proof of Theorem 4.1 in [19], combined with Observation 7.9, yields that $J = P$.

Assume, henceforth, that $s = 3$ and that R contains the field of rational numbers. We conclude from Corollary 2.17 (by way of Lemma 7.3 and Observation 7.9) that $J = \mathfrak{L}$. On the other hand, we know from (2.11) that $\mathfrak{L}$ is equal to

$$(K, y) + (\{w_{(i)} \mid (i) \text{ is an ordered 2-tuple selected from } 1, \ldots, n+1\})$$

where K is generated by $n+1$ elements. It is easily seen that one of the elements $w_{(i)}$ is already an element of (K, y). Consequently, J can be generated by $(1/2)(n+1)(n+2)$ elements. This is marvelously efficient generation since actually $J = I_n(X \mid Y)$, which is plainly minimally generated by $\binom{n+2}{n}$ elements.

8. The Minimal Number of Generators for J.

We conclude by estimating the minimal number of generators required for a $(g+1)-$residual intersection of a grade g Gorenstein ideal.

Theorem 8.1. Let I be a grade $g > 0$ Gorenstein ideal in a Gorenstein local ring $(R, \mathfrak{M})$. Suppose that I is generically a complete intersection and also that $H_1(I)$ is Cohen-Macaulay. Let $J = (\mathfrak{A} : I)$ be a $(g+1)-$residual intersection and let $t = \dim((\mathfrak{A} + \mathfrak{M}I)/\mathfrak{M}I)$. If $T^2(R/I) = 0$ and (R, I) admits a deformation $(\widetilde{R}, \widetilde{I})$ with $\widetilde{I}$ satisfying the condition G_{g+2}, then

(a) $\mu(J) \le 2g+2$,

(b) $g+1-t \le \mu(J/\mathfrak{A})$, and

(c) $2g+2-t \le \mu(J)$ if $t \le g-2$.

In particular, if $\mathfrak{A} \subseteq \mathfrak{M}I$, then $\mu(J) = 2g+2$.

Note. If the ideal I is licci and generically a complete intersection, then the other hypotheses are automatically satisfied. Indeed, $T^2(R/I) = 0$ by Theorem 2.19 (a), I is strongly Cohen-Macaulay by [18], and (R, I) admits a deformation $(\widetilde{R}, \widetilde{I})$ with $\widetilde{I}$ satisfying G_∞ by [22, Theorem 5.3].

Beginning of the proof. From Corollary 2.18 we already know that $J = \mathfrak{K}$; hence, from (2.3),

$$J = (a_1, \ldots, a_{g+1},\ c_1, \ldots, c_{g+1}),$$

so in any event $\mu(J) \le 2g+2$. Thus it suffices to prove (b) and (c). To this end, we may assume that $t \le g$ since nothing is to be shown if $t = g+1$.

We follow the procedure used in the proof of Theorem 5.1 in [22] in order to construct a suitable localization of a generic $(g+1)-$residual intersection

$$\boxed{\mathrm{J}} = (\boxed{\mathrm{A}} : \widetilde{I}\boxed{\mathrm{R}}) \subseteq \boxed{\mathrm{R}}$$

of $\widetilde{I}$. The ideal $\boxed{\mathrm{J}}$ is a geometric residual intersection by Lemma 7.2 (b) and $H_1(\widetilde{I}\boxed{\mathrm{R}})$ is still Cohen-Macaulay by [20, Lemma 2.15]. Thus, we may apply

Theorem 6.1 (c) in order to conclude, as in the proof of Theorem 5.1 in [22], that $(\boxed{\mathrm{R}},\boxed{\mathrm{J}})$, $(\boxed{\mathrm{R}}/\boxed{\mathrm{A}},\ \widetilde{I}\boxed{\mathrm{R}}/\boxed{\mathrm{A}})$, and $(\boxed{\mathrm{R}}/\boxed{\mathrm{A}},\boxed{\mathrm{J}}/\boxed{\mathrm{A}})$ are deformations of (R,J), $(R/\mathfrak{A}, I/\mathfrak{A})$, and $(R/\mathfrak{A}, J/\mathfrak{A})$ respectively. Since, moreover, $T^2(\boxed{\mathrm{R}}/\widetilde{I}\boxed{\mathrm{R}}) = 0$ by [37, Lemma 2.9], we do not change any of our assumptions or conclusions if we replace $J = (\mathfrak{A} : I)$ by $\boxed{\mathrm{J}} = (\boxed{\mathrm{A}} : \widetilde{I}\boxed{\mathrm{R}})$. Reverting to our original notation we may therefore assume that J is a **geometric** $(g+1)-$residual intersection of I, and I satisfies the condition G_{g+2}.

Before proceeding any further with the proof of Theorem 8.1 we offer two examples. In Example 8.2 we show that the bounds of Theorem 8.1 hold if I is a complete intersection; we are then able to ignore this case for the rest of the proof. Example 8.3 appears to be a special case of Theorem 8.1; however, the rest of the proof amounts to reducing the general case to this special case.

Example 8.2. Let I, in the notation of Theorem 8.1, be a complete intersection. If $\mathbf{f}$ is a $1 \times g$ matrix whose entries generate I, and X is a $g \times (g+1)$ matrix such that the entries of $\mathbf{f}X$ generate $\mathfrak{A}$, then $J = \mathfrak{A} + I_g(X)$ by [22, Example 3.4 and Theorem 4.7]. (In fact, the entire resolution of R/J may be found in [5, Theorem 4.8].) At this point it is easy to calculate the exact values:

$$\begin{array}{lll} \mu(J) = 2g+2-t & \text{if} & t \le g-2 \\ \mu(J) = g+1 & \text{if} & t = g-1 \\ \mu(J/\mathfrak{A}) = g+1-t & \text{if} & t \le g-1. \end{array}$$

(The integer t is necessarily smaller than g in this case because $\mathfrak{A}$ is a proper subideal of I and $\mu(I) = g$.) Observe that the lower bounds given in Theorem 8.1 (b) and (c) are attained in this example. Observe also, that the bound given in part (c) does not necessarily hold if $t > g-2$.

Example 8.3. Let $(S, \mathfrak{N})$ be a Gorenstein local ring, and let I be a grade $g > 0$ Gorenstein ideal of S which is not a complete intersection. Suppose that $T^2(S/I) = 0$, and I satisfies the condition G_{g+2}. Let X be a $(g-t+1) \times (g-t+1)$ matrix of indeterminates, let $\boxed{\mathrm{S}} = S[X]_{\boxed{\mathrm{N}}}$ where $\boxed{\mathrm{N}}$ is the maximal ideal $(\mathfrak{N} + (X))S[X]$ of $S[X]$, and let $\mathbf{f}$ be a $1 \times n$ matrix whose entries generate I. Suppose that the $\boxed{\mathrm{S}}-$ideal $\boxed{\mathrm{J}} = (\boxed{\mathrm{B}}\boxed{\mathrm{S}} : I\boxed{\mathrm{S}})$ is a geometric $(g+1)-$residual intersection of $I\boxed{\mathrm{S}}$, where $\boxed{\mathrm{B}}$ is the $S[X]$-ideal generated by the entries of $\boxed{\mathrm{b}}$ for

$$\boxed{\mathrm{b}} = [b_1, \ldots, b_{g+1}] = \mathbf{f}\begin{bmatrix} I_t & 0 \\ 0 & X \\ 0 & 0 \end{bmatrix},$$

where I_t is the $t \times t$ identity matrix. We will identify a generating set for $\boxed{\text{J}}$ and show that the inequalities of Theorem 8.1 hold for this example. Let F be a minimal $S-$resolution of S/I. Let (K, ∂) be the Koszul complex on the elements $f_1, \dots, f_{g+1}$ of $S[X]$. In particular, let $\varepsilon_1, \dots, \varepsilon_{g+1}$ be a basis for K_1 with $\partial_1(\varepsilon_i) = f_i$. Let $\alpha \colon K \to F \otimes S[X]$ be a comparison map that extends the identity map in degree zero. For each i, with $1 \le i \le g+1$, let

$$p_i = (-1)^{i+1} \alpha_g(\varepsilon_1 \wedge \ldots \wedge \widehat{\varepsilon_i} \wedge \ldots \wedge \varepsilon_{g+1}) \in F_g \otimes S[X] = S[X].$$

Let (E, d) be the Koszul complex on the elements $b_1, \dots, b_{g+1}$ of $S[X]$. In particular, let $e_1, \dots, e_{g+1}$ be a basis for E_1 with $d_1(e_i) = b_i$. Let $\beta_1 : E_1 \to K_1$ be the map represented by the matrix

$$Y = \begin{bmatrix} I_t & 0 \\ 0 & X \end{bmatrix},$$

and let $\beta : E \to K$ be the $DG-$algebra map induced by β_1. For each i, with $1 \le i \le g+1$, let

$$c_i = (-1)^{i+1} \alpha_g \beta_g(e_1 \wedge \ldots \wedge \widehat{e_i} \wedge \ldots \wedge e_{g+1}) \in F_g \otimes S[X] = S[X].$$

Since $\boxed{\text{J}} = \mathfrak{K}$, by Corollary 2.18, we know, from (2.3), that

$$\boxed{\text{J}} = (c_1, \dots, c_{g+1}, b_1, \dots, b_{g+1})\boxed{\text{S}}.$$

If $\mathbf{p}$ is the matrix $[p_1, \dots, p_{g+1}]$, then it is immediate that (up to sign), c_i is the determinant of the matrix obtained by deleting the i^{th} column of

$$[Y \ \mathbf{p}^T] = \left[\begin{array}{cc|c} I_t & 0 & \\ & & \mathbf{p}^T \\ 0 & X & \end{array}\right].$$

We establish inequality (b) of Theorem 8.1 for this example by showing that $\bar{c}_{t+1}, \dots, \bar{c}_{g+1}$ begins a minimal generating set for $\boxed{\text{J}}/(\boxed{\text{B}}\boxed{\text{S}})$. Let "$\frown$" denote reduction mod $I\boxed{\text{S}}$. Observe that

$$(\boxed{\text{J}}, \det X)^{\frown} = (I_{g+1}(Y|\mathbf{p}^T))^{\frown};$$

consequently, $(\boxed{\text{J}},\ \det X)^{\frown}$ is generated by the maximal order minors of the $(g-t+1) \times (g-t+2)$ matrix

$$\left[\begin{array}{c|c} & p_{t+1} \\ X & \vdots \\ & p_{g+1} \end{array}\right].$$

Since $\boxed{\mathrm{J}}$ is a geometric $(g+1)-$residual intersection of $I\boxed{\mathrm{S}}$, it follows from (1.6) that grade $(I\boxed{\mathrm{S}}+\boxed{\mathrm{J}}) \ge g+2$; and, therefore, grade $(\boxed{\mathrm{J}})^{\frown} \ge 2$ because I is a perfect ideal of grade g. In particular, grade $(\boxed{\mathrm{J}},\ \det X)^{\frown} \ge 2$, and hence $(\boxed{\mathrm{J}},\ \det X)^{\frown}$ is a determinantal ideal having generic grade, which is two. Since I is not a complete intersection, each p_j is a non-unit and it is clear that $(\det X)^{\frown},\ \widetilde{c}_{t+1},\dots,\widetilde{c}_{g+1}$ is a minimal generating set for $(\boxed{\mathrm{J}},\ \det X)^{\frown}$. It follows that

$$\widetilde{c}_{t+1},\dots,\widetilde{c}_{g+1} \tag{8.4}$$

begins a minimal generating set for $(\boxed{\mathrm{J}})^{\frown} = (\boxed{\mathrm{J}}+I\boxed{\mathrm{S}})/I\boxed{\mathrm{S}} = \boxed{\mathrm{J}}/\boxed{\mathrm{B}}\boxed{\mathrm{S}}$. (The last equality holds due to Theorem 6.1 (a).)

Assume $t \le g-2$. We establish inequality (c) of Theorem 8.1 for this example by showing that

$$b_1,\dots,b_{g+1},\ c_{t+1},\dots,c_{g+1}$$

begins a minimal generating set for $\boxed{\mathrm{J}}$. We have carefully constructed the b's and the c's to be elements of $S[X]$, so that we can take advantage of the fact that each b is a homogeneous form in $S[X]$ of degree (in (X)) at most 1 and each c is a homogeneous form of degree at least $(g-t) \ge 2$. Recall that $\boxed{\mathrm{B}}$ is the $S[X]-$ideal $(b_1,\dots,b_{g+1})$. Let J be the $S[X]-$ideal $(\boxed{\mathrm{B}}+(c_1,\dots,c_{g+1}))$. (Thus, $\boxed{\mathrm{J}} = J\boxed{\mathrm{S}}$.)

Degree considerations show that every $S[X]/\boxed{\mathrm{N}}$ relation on $\bar{b}_1,\dots,\bar{b}_{g+1}$ in $J/\boxed{\mathrm{N}}J$ is equivalent to a relation in $\boxed{\mathrm{B}}/\boxed{\mathrm{N}}\boxed{\mathrm{B}}$, and is therefore trivial (because $b_1,\dots,b_{g+1}$ is a minimal generating set for $\boxed{\mathrm{B}}$). We know from (8.4) that $\bar{c}_{t+1}$, $\dots$, $\bar{c}_{g+1}$ are linearly independent in the $S[X]/\boxed{\mathrm{N}}$ vector space $J/(\boxed{\mathrm{N}}J+\boxed{\mathrm{B}})$. We conclude that

$$\bar{b}_1,\dots,\bar{b}_{g+1},\ \bar{c}_{t+1},\dots,\bar{c}_{g+1}$$

are linearly independent in $J/\boxed{\mathrm{N}}J = \boxed{\mathrm{J}}/\boxed{\mathrm{N}}\boxed{\mathrm{J}}$.

We prove Theorem 8.1 by showing that is is legal to "replace" the given residual intersection by the residual intersection of Example 8.3. The process has three steps: we extend the original ring and form universal linear combinations of the generators of I; then we deform to a generic situation; finally, we specialize to the "semi-generic" situation of Example 8.3. The next three lemmas describe these steps. Throughout this discussion:

(8.5) I is an ideal of height $g>0$; $h_1,\dots,h_t$ is a regular sequence in I, and the ideal $I/(h_1,\dots,h_t)$ satisfies the condition G_{s-t+1} for some $s \ge g$.

Lemma 8.6. Let R be a ring. Assume the hypotheses of (8.5). Suppose that $I = (h_1, \dots, h_t,\ h_{t+1}, \dots, h_n)$. Let U be an $(n-t)\times(s-t)$ matrix of indeterminates, and let $\mathbf{f}$ be the matrix

$$\mathbf{f} = [h_1, \dots, h_n] \begin{bmatrix} I_t & 0 \\ 0 & U \end{bmatrix}.$$

Fix an integer i, with $g \le i \le s$, and a prime ideal Q of $R[U]$. If either

(a) $\operatorname{ht}(Q) \le i-1$, or

(b) $\operatorname{ht}(Q) \le i$ and $IR[U] \subseteq Q$,

then $(f_1, \dots, f_i)_Q = (IR[U])_Q$. In particular, the ideal $(f_1, \dots, f_s)/(f_1, \dots, f_t)$ of the ring $R[U]/(f_1, \dots, f_t)$ satisfies the condition G_{s-t+1}.

Proof. It suffices to do the calculation in $R/(h_1, \dots, h_t)$; consequently, we may assume that $t = 0$. In this case, the result is Lemma 7.2(a).

The next Lemma gives sufficient conditions for deforming an arbitrary residual intersection into a more general one.

Lemma 8.7. Let $J = (B : I)$ be a geometric $s-$residual intersection in the Gorenstein local ring $(S, \mathfrak{n})$. Assume that I satisfies the condition G_s, and that I is strongly Cohen-Macaulay, or $s = g+1$ and $H_1(I)$ is Cohen-Macaulay. In addition, assume the hypotheses of (8.5) and let $f_1, \dots, f_n$ be a generating set of I with $f_i = h_i$ for $1 \le i \le t$. Suppose that B is generated by the entries of $\mathbf{b}$ where

$$\mathbf{b} = [f_1, \dots, f_n] \begin{bmatrix} I_t & 0 \\ 0 & N \end{bmatrix}$$

for some $(n-t)\times(s-t)$ matrix N with entries in $\mathfrak{n}$. Let Z be an $(n-t)\times(s-t)$ matrix of indeterminates and $\mathfrak{S}$ be the ring $S[Z]_{(\mathfrak{n},(Z))}$. Let

(8.8) $\mathfrak{B}$ be the ideal of $\mathfrak{S}$ generated by the entries of

$$\mathfrak{b} = [f_1, \dots, f_n] \begin{bmatrix} I_t & 0 \\ 0 & Z \end{bmatrix},$$

and let $\mathfrak{J}$ be the ideal $(\mathfrak{B} : I\mathfrak{S})$ of $\mathfrak{S}$.

Then $\mathfrak{J}$ is a geometric $s-$residual intersection of $I\mathfrak{S}$ with respect to $\mathfrak{B}$; and

$$(\mathfrak{S}, I\mathfrak{S}),\ (\mathfrak{S}, \mathfrak{J}),\ (\mathfrak{S}/\mathfrak{B}, I\mathfrak{S}/\mathfrak{B}),\ \text{and}\ (\mathfrak{S}/\mathfrak{B}, \mathfrak{J}/\mathfrak{B})$$

are deformations of

$$(S, IS),\ (S, J),\ (S/B, IS/B),\ \text{and}\ (S/B, J/B),$$

respectively.

Proof. We know from Lemma 8.6 that $(I\mathfrak{S})_Q = \mathfrak{B}_Q$ for all prime ideals Q of $\mathfrak{S}$ with $\operatorname{ht}(Q) \le s-1$ or $\operatorname{ht}(Q) \le s$ and $I\mathfrak{S} \subseteq Q$. It follows that $\mathfrak{J} = (\mathfrak{B} : I\mathfrak{S})$ is a geometric $s-$residual intersection. If I is strongly Cohen-Macaulay, then apply [19, Theorem 3.1] to see that $\mathfrak{S}/\mathfrak{J}$ is Cohen-Macaulay, $\operatorname{depth}\mathfrak{S}/\mathfrak{B} \ge \dim \mathfrak{S} - s$, and $\operatorname{ht}(\mathfrak{J}) = s$. If $\mathbf{x}$ is the sequence of elements $\{z_{ij} - n_{ij}\}$ of $\mathfrak{S}$, then it is clear that $\mathbf{x}$ is a regular sequence on both $\mathfrak{S}$ and $\mathfrak{S}/I\mathfrak{S}$. Let " $'$ " denote reduction mod $(\mathbf{x})$. It is obvious that $\mathfrak{S}' = S$, $(I\mathfrak{S})' = I$, and $\mathfrak{B}' = B$. We conclude, from [22, Prop. 4.2] (if I is strongly Cohen-Macaulay), or, from Theorem 6.1 (c) (if $s = g+1$ and $H_1(I)$ is Cohen-Macaulay), that $\mathbf{x}$ is a regular sequence on both $\mathfrak{S}/\mathfrak{B}$ and $\mathfrak{S}/\mathfrak{J}$. We also conclude that $\mathfrak{J}' = J$. The proof is now complete.

The next result is harder to state than to prove. It gives sufficient conditions for replacing an "arbitrary" residual intersection by a "semi-generic" residual intersection like the one discussed in Example 8.3.

Lemma 8.9. Let $J = (B : I)$ be a geometric $s-$residual intersection in the Gorenstein local ring $(R, \mathfrak{M})$. Suppose that I is strongly Cohen-Macaulay, or that $s = g+1$ and $H_1(I)$ is Cohen-Macaulay. Suppose further that the conditions of (8.5) are satisfied. Extend $h_1, \dots, h_t$ to be a generating set $h_1, \dots, h_t,\ h_{t+1}, \dots h_n$ for I where $n \ge s$. Suppose that B is generated by the entries of $\mathbf{b}$, where

$$\mathbf{b} = [h_1, \dots, h_n] \begin{bmatrix} I_t & 0 \\ 0 & M \end{bmatrix}$$

for some $(n-t) \times (s-t)$ matrix M with entries in $\mathfrak{M}$. Let X be a matrix of indeterminates of shape $(s-t) \times (s-t)$. Then there exists a Gorenstein local faithfully flat extension $(S, \mathfrak{N})$ of R and a generating set $\{f_1, \dots, f_n\}$ of IS (with $f_i = h_i$ for $i \le t$) such that, if $\boxed{\text{S}}$ is the local ring $S[X]_{(\mathfrak{N}, (X))}$, $\boxed{\text{B}}$ is the ideal of $\boxed{\text{S}}$ generated by the entries of $\boxed{\text{b}}$ for

$$\boxed{\text{b}} = [f_1, \dots, f_n] \begin{bmatrix} I_t & 0 \\ 0 & X \\ 0 & 0 \end{bmatrix}, \tag{8.10}$$

and $\boxed{\text{J}}$ is the ideal $(\boxed{\text{B}} : I\boxed{\text{S}})$ of $\boxed{\text{S}}$, then $\boxed{\text{J}} = (\boxed{\text{B}} : I\boxed{\text{S}})$ is a geometric $s-$residual intersection and each pair of pairs:

(a) (S, IS) and $(\boxed{\mathrm{S}}, I\boxed{\mathrm{S}})$,

(b) (S, JS) and $(\boxed{\mathrm{S}}, \boxed{\mathrm{J}})$,

(c) $(S/BS, IS/BS)$ and $(\boxed{\mathrm{S}}/\boxed{\mathrm{B}}, I\boxed{\mathrm{S}}/\boxed{\mathrm{B}})$,

(d) $(S/BS, JS/BS)$ and $(\boxed{\mathrm{S}}/\boxed{\mathrm{B}}, \boxed{\mathrm{J}}/\boxed{\mathrm{B}})$,

has a common deformation. In particular, $\mu(J) = \mu(\boxed{\mathrm{J}})$ and $\mu(J/B) = \mu(\boxed{\mathrm{J}}/B)$.

Proof. Notice that $t \le g \le s \le n$. Let U be an $(n-t) \times (s-t)$ matrix of indeterminates and let $(S, \mathfrak{N})$ be the local ring $R[U]_{\mathfrak{M}R[U]}$. The matrix

$$E = \left[\begin{array}{c|c|c} I_t & 0 & 0 \\ \hline 0 & U & \begin{array}{c} 0 \\ \hline I_{n-s} \end{array} \end{array}\right]$$

is invertible over S and the ideal IS is generated by the entries of $\mathbf{f} = [h_1, \dots, h_n]E$. Observe that

$$\mathbf{b} = \mathbf{f}E^{-1}\begin{bmatrix} I_t & 0 \\ 0 & M \end{bmatrix} = \mathbf{f}\begin{bmatrix} I_t & 0 \\ 0 & N \end{bmatrix} \tag{8.11}$$

for some $(n-t) \times (s-t)$ matrix N with entries in $\mathfrak{N}$. Let Z be an $(n-t) \times (s-t)$ matrix of indeterminates; let $\mathfrak{S}$ be the ring $\boxed{\mathrm{S}}[Z]_{(\mathfrak{N},(X),(Z))}$; and form $\mathfrak{b}$, $\mathfrak{B}$, and $\mathfrak{J}$ as in (8.8).

The proof is completed by applying Lemma 8.7 twice. In the first application we "replace" the matrix N of (8.11) with the matrix Z and conclude that

$$(\mathfrak{S}, I\mathfrak{S}),\ (\mathfrak{S}, \mathfrak{J}),\ (\mathfrak{S}/\mathfrak{B}, I\mathfrak{S}/\mathfrak{B}),\text{ and } (\mathfrak{S}/\mathfrak{B}, \mathfrak{J}/\mathfrak{B})$$

are deformations of

$$(\boxed{\mathrm{S}}, I\boxed{\mathrm{S}}),\ (\boxed{\mathrm{S}}, J\boxed{\mathrm{S}}),\ (\boxed{\mathrm{S}}/B\boxed{\mathrm{S}}, I\boxed{\mathrm{S}}/B\boxed{\mathrm{S}}),\text{ and } (\boxed{\mathrm{S}}/B\boxed{\mathrm{S}}, J\boxed{\mathrm{S}}/B\boxed{\mathrm{S}}),$$

respectively. (The indeterminates $\{x_{ij}\}$ are harmless in the last four pairs. We mod out by these indeterminates in order to see that the last four pairs are deformations of

$$(S, IS),\ (S, JS),\ (S/BS, IS/BS),\text{ and } (S/BS, JS/BS),$$

respectively.) In the second application of Lemma 8.7 we "replace" the matrix

$$\begin{bmatrix} X \\ 0 \end{bmatrix}$$

of (8.10) with the matrix Z. Once we show that

(8.12) $$\boxed{\text{J}} = (\boxed{\text{B}} : I\boxed{\text{S}}) \text{ is a geometric } s\text{−residual intersection,}$$

then we conclude from Lemma 8.7 that

$$(\mathfrak{S}, I\mathfrak{S}),\ (\mathfrak{S}, \mathfrak{J}),\ (\mathfrak{S}/\mathfrak{B}, I\mathfrak{S}/\mathfrak{B}),\ \text{and } (\mathfrak{S}/\mathfrak{B}, \mathfrak{J}/\mathfrak{B})$$

are deformations of

$$(\boxed{\text{S}}, I\boxed{\text{S}}),\ (\boxed{\text{S}}, \boxed{\text{J}}),\ (\boxed{\text{S}}/\boxed{\text{B}}, I\boxed{\text{S}}/\boxed{\text{B}}),\ \text{and } (\boxed{\text{S}}/\boxed{\text{B}}, \boxed{\text{J}}/\boxed{\text{B}}),$$

respectively.

We conclude the proof by establishing (8.12). Apply Lemma 8.6 to the regular sequence $f_1, \dots, f_t$ in the ideal I of the ring R. We obtain two conclusions:

(8.13) $(f_1, \dots, f_s)_Q = (IS)_Q$ for all prime ideals Q of S with $\text{ht}(Q) \le s-1$, or $\text{ht}(Q) \le s$ and $IS \subseteq Q$; and

(8.14) the ideal $(f_1, \dots, f_s)/(f_1, \dots, f_t)$ of the ring $S/(f_1, \dots, f_t)$ satisfies the condition G_{s-t+1}.

Conclusion (8.14) is exactly the hypothesis we need to apply Lemma 8.6 to the regular sequence $f_1, \dots, f_t$ in the ideal $(f_1, \dots, f_s)$ of the ring S. We conclude that

(8.15) $$\boxed{\text{B}}_{\boxed{\text{Q}}} = (f_1, \dots, f_s)_{\boxed{\text{Q}}}$$

for all prime ideals $\boxed{\text{Q}}$ of $\boxed{\text{S}}$ with $\text{ht}(\boxed{\text{Q}}) \le s-1$, or $\text{ht}(\boxed{\text{Q}}) \le s$ and $(f_1, \dots, f_s) \subseteq \boxed{\text{Q}}$. Let $\boxed{\text{Q}}$ be a prime ideal of $\boxed{\text{S}}$ with $\text{ht}(\boxed{\text{Q}}) \le s-1$, or $\text{ht}(\boxed{\text{Q}}) \le s$ and $I\boxed{\text{S}} \subseteq \boxed{\text{Q}}$. We see from (8.15) that $\boxed{\text{B}}_{\boxed{\text{Q}}} = (f_1, \dots, f_s)_{\boxed{\text{Q}}}$. On the other hand, if we let $Q = \boxed{\text{Q}} \cap S$, then we may apply (8.13) to see that $(f_1, \dots, f_s)_Q = (IS)_Q$. Thus, $\boxed{\text{B}}_{\boxed{\text{Q}}} = (IS)_{\boxed{\text{Q}}}$, (8.12) is established, and the proof is complete.

The conclusion of the proof of Theorem 8.1. Recall that $J = (\mathfrak{A} : I)$ is a geometric $(g+1)$−residual intersection in the local ring $(R, \mathfrak{M})$ and

$$t = \dim((\mathfrak{A} + \mathfrak{M}I)/\mathfrak{M}I).$$

(We may make a flat extension of R, if necessary, in order to assume that the residue field of R is infinite.) Use the technique mentioned in the first paragraph of section one in order to select a generating set $b_1, \dots, b_{g+1}$ for $\mathfrak{A}$ so that

(8.16) $b_1, \dots, b_g$ is a regular sequence on R; $\bar{b}_1, \dots, \bar{b}_t$ is a basis for $(\mathfrak{A} + \mathfrak{M}I)/\mathfrak{M}I$; and $(b_1, \dots, b_g)_P = I_P$ for all prime ideals P of R with $I \subseteq P$ and $\mathrm{ht}(P) = g$.

Let $\{h_1, \dots, h_n\}$, with $n \geq g+1$, be a generating set for I which extends $h_1 = b_1$, $\dots$, $h_t = b_t$. If we add an element of $(b_1, \dots, b_t)$ to b_i for $i > t$, then the conditions of (8.16) still hold. Consequently, we may alter $b_{t+1}, \dots, b_{g+1}$ in order to assume that the conditions of (8.16) hold and

$$[b_1, \dots, b_{g+1}] = [h_1, \dots, h_n] \begin{bmatrix} I_t & 0 \\ 0 & M \end{bmatrix}$$

for some $(n-t) \times (g-t+1)$ matrix M with entries in $\mathfrak{M}$. Let $\mathbf{b}$ represent the vector $[b_1, \dots, b_{g+1}]$.

We claim that the ideal $I/(h_1, \dots, h_t)$ satisfies the condition G_{g-t+2}. Let "$\,'\,$" denote reduction modulo the regular sequence $b_1, \dots, b_t$. (Recall that $t \leq g$.) Let P' be a prime ideal in R' with $I' \subseteq P'$, $g - t \leq \dim R'_{P'} \leq g - t + 1$, and let P be the preimage of P' in R. If $\dim R'_{P'} = g - t$, then P is a minimal prime of I; hence, $I_P = (b_1, \dots, b_g)_P$, and therefore, $\mu(I'_{P'}) \leq g - t = \dim R'_{P'}$. If $\dim R'_{P'} = g - t + 1$, then, since J is a geometric residual intersection, $I_P = (b_1, \dots, b_{g+1})_P$ and therefore, $\mu(I'_{P'}) \leq g + 1 - t = \dim R'_{P'}$. Thus, I' satisfies G_{g-t+2}.

Recall that $s = g+1$ and $H_1(I)$ is Cohen-Macaulay. Now that we have verified the hypotheses of (8.5), we may apply Lemma 8.9. The residual intersection $\boxed{\mathrm{J}} = (I_1(\boxed{\mathrm{b}}) : I\boxed{\mathrm{S}})$ of Example 8.3 is obtained from the present residual intersection $J = (I_1(\mathbf{b}) : I)$ by forming a faithfully flat extension, a deformation and a specialization. In particular, $\mu(J) = \mu(\boxed{\mathrm{J}})$ and $\mu(J/I_1(\mathbf{b})) = \mu(\boxed{\mathrm{J}}/I_1(\boxed{\mathrm{b}}))$. The module $T^2(\boxed{\mathrm{S}}/I\boxed{\mathrm{S}}) = 0$, so the proof is completed by appealing to Examples 8.3 and 8.2.

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