CLASSIFICATION OF THE TOR−ALGEBRAS OF CODIMENSION FOUR ALMOST COMPLETE INTERSECTIONS

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Abstract. Let $(R, \mathfrak{m}, k)$ be a local ring in which 2 is a unit. Assume that every element of k has a square root in k. We classify the algebras $\mathrm{Tor}_\bullet^R(R/J, k)$ as J varies over all grade four almost complete intersection ideals in R. The analogous classification has already been found when J varies over all grade four Gorenstein ideals [21], and when J varies over all ideals of grade at most three [5, 30]. The present paper makes use of the classification, in [21], of the Tor−algebras of codimension four Gorenstein rings, as well as the (usually non-minimal) DG−algebra resolution of a codimension four almost complete intersection which is produced in [25] and [26].

Fix, for the time being, a regular local ring $(R, \mathfrak{m}, k)$. For each Cohen-Macaulay ring A of the form $A = R/I$, we consider the Tor−algebra $T_\bullet = T_\bullet(A) = \mathrm{Tor}_\bullet^R(A, k)$. A great deal of information about A is encoded in $T_\bullet(A)$. Some of the classical results along these lines are: A is regular if and only if $T_\bullet = T_0$ [27]; A is Gorenstein if and only if $T_\bullet$ is a Poincaré duality algebra [4]; A is a complete intersection if and only if $T_\bullet$ is the exterior algebra on T_1 [29, 1]. There are at least three types of modern applications of theorems which classify Tor−algebras. The major impetus for studying $T_\bullet$ is Avramov's machine for converting questions about the local ring A into questions about the algebra $T_\bullet$, provided the minimal R−free resolution of A is a DG−algebra. The algebra $T_\bullet$ is graded-commutative, instead of commutative; nonetheless, it is a much simpler object than the original ring A. In particular, $T_\bullet$ is always a finite dimensional vector space over k. Avramov's machine has been successfully applied when the codimension of A is at most three; or A is Gorenstein of codimension four; or A is one link from a complete intersection; or A is Gorenstein and two links from a complete intersection. In each case the minimal R−resolution of A is a DG−algebra [6, 17, 19, 16, 5] and the Tor−algebra $T_\bullet(A)$ has been classified [21, 30, 5]. Once the key hypotheses are established, then one is able to prove [12, 5] that the Poincaré series

$$P_A^M(z) = \sum_{i=0}^{\infty} \dim_k \mathrm{Tor}_i^A(M, k) z^i$$

is a rational function for all finitely generated A−modules M. One is also able to prove [2] that all of these rings A satisfy the Eisenbud Conjecture [8]; that is, if

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M is a finitely generated $A-$module whose Betti numbers are bounded, then the minimal resolution of M is eventually periodic of period two. See [3] for further results and problems along these lines.

Avramov's machine has been applied to Gorenstein rings of codimension four and to rings which are a "small" number of links from other "nice" rings. It is our hope that these techniques may also be applied to rings which are one link from a Gorenstein ring of codimension four, in other words to almost complete intersections of codimension four. The first step in this direction was taken in Palmer's thesis [25, 26]. Let A be a codimension four almost complete intersection. Palmer produced a DG$-$algebra resolution of A. Palmer's resolution is close to, but not always equal to, the minimal resolution of A. Palmer's work provides evidence that the minimal resolution of A is a DG$-$algebra and it is very useful in the present paper where the second step — the classification of $T_\bullet(A)$ — takes place. Palmer's work is summarized in section 3, and is applied to $T_\bullet(A)$ in section 4. (It is noteworthy that the present paper represents the first time that $T_\bullet(A)$ has been classified before the minimal resolution of A was known to be a DG$-$algebra; indeed, it is likely that the present work will help complete the project initiated in [25].)

A second application of theorems which classify Tor$-$algebras is to the Buchsbaum-Eisenbud conjecture [6] about lower bounds for Betti numbers. Charalambous, Evans, and Miller [7] have proved that if the dimension, d, of R is at most four, and M is an $R-$module of finite length, with M not equal to R modulo a regular sequence, then the Betti numbers of M satisfy $\binom{d}{i} \le \beta_i(M)$ for $0 < i < d$, and $2^d + 2^{d-1} \le \sum_{i=0}^{d} \beta_i(M)$. One of the key ingredients in their proof is the classification in [21] of $T_\bullet(A)$ for codimension four Gorenstein rings A. The classification of Tor$-$algebras contained in the present paper should lead to further progress on establishing lower bounds for Betti numbers.

Multiplicative operations in Tor$-$algebras also play some role in determining the generating set of a residual intersection. This theme is initiated in [23]. Further results along these lines will appear in subsequent papers.

The algebra $T_\bullet(A)$ has been classified when A is a codimension four Gorenstein ring [21]; and when A is a codimension three ring [30, 5]. In each case, there are at most five different families of Tor$-$algebras. Furthermore, each family is discrete, in the sense that the family members are parameterized by integers. The proofs in [21] and [5] are based on the theory of linkage. The proof in [30] comes from invariant theory. The proofs look quite different, but the ultimate linear algebra calculations are roughly equivalent. The linkage theory proof is like an induction; one must know the answer before one can prove it. For rings of codimension three, the proof in [30] preceded proof in [5]; indeed, the authors of [5] took Weyman's answer and reproved it using their linkage technique. Some further details may be found in [24]. The classification in the present paper uses the linkage style of argument. Once again the answer consists of a small number of discrete families of Tor$-$algebras; see Theorem 1.5.

The main result of the present paper is stated in section 1 and proved in section 4. Palmer's DG$-$algebra resolution $\mathbb{M}$ of a codimension four almost complete intersection is recorded in section 3. The multiplication in $\mathbb{M}$ uses the multiplication on a resolution of a codimension four Gorenstein ring. In section 2 we recall the classification of $T_\bullet(A)$ for codimension four Gorenstein rings A. In section 5 we give examples and ask questions. The remainder of the present section is a discussion of the conventions that are used throughout the paper.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. The ideal I of R is called *perfect* if the grade of I is equal to the projective dimension, $\mathrm{pd}_R(R/I)$, of the $R-$module R/I. A grade g ideal I is called a *complete intersection* if it can be generated by g generators. Complete intersection ideals are necessarily perfect. The grade g ideal I is called an *almost complete intersection* if it is a **perfect** ideal which is **not** a complete intersection and which can be generated by $g+1$ generators. The grade g ideal I is called *Gorenstein* if it is perfect and $\mathrm{Ext}^g_R(R/I,R)\cong R/I$.

Let k be a fixed field. Throughout this paper, we write

$$\text{"}S_\bullet \text{ is a graded } k-\text{algebra"} \tag{0.1}$$

to mean that $S_\bullet$ is a finite dimensional graded-commutative associative $k-$algebra of the form $S_\bullet=\bigoplus_{i=0}^n S_i$ with $S_0=k$. In particular,

$$s_is_j=(-1)^{ij}s_js_i \text{ for all } s_i\in S_i \text{ and } s_j\in S_j \quad\text{and}\quad s_is_i=0 \text{ if } s_i\in S_i \text{ and } i \text{ is odd.}$$

For example, if $(R,\mathfrak{m},k)$ is a local ring and I is an $R-$ideal of finite projective dimension, then $\mathrm{Tor}^R_\bullet(R/I,k)$ is a graded $k-$algebra in the sense of (0.1). For a more concrete example, let V be a vector space of dimension d over k. The exterior algebra

$$\bigwedge^\bullet V=\bigwedge_k^\bullet V=k\oplus V\oplus\bigwedge^2 V\oplus\bigwedge^3 V\oplus\ldots\oplus\bigwedge^d V,$$

with multiplication given by exterior product, is a graded $k-$algebra in the sense of (0.1). We use the usual conventions regarding grading. If $M=\oplus M_j$ is a graded $S_\bullet-$module, then $M(a)$ is the graded $S_\bullet-$module with the property that $M(a)_j=M_{a+j}$ and $\mathrm{Hom}_{S_\bullet}(S_\bullet(-a),M)=M(a)$. In particular, there is an isomorphism of graded $k-$vector spaces from $k(-1)^d$ to the subspace $V=\bigwedge^1 V$ of $\bigwedge^\bullet V$.

In this paper the word "trivial" is given two distinct meanings. Suppose that $S_\bullet$ is a graded $k-$algebra and W is a positively graded $S_\bullet-$module. Then the *trivial extension of* $S_\bullet$ *by* W, $S_\bullet\ltimes W$, is the graded $k-$algebra whose graded vector space structure is given by $S_\bullet\oplus W$ and whose multiplication is given by

$$(s_i,w_j)(s_k,w_\ell)=(s_is_k,s_iw_\ell+(-1)^{jk}s_kw_j)$$

for all $s_a\in S_a$ and all $w_b\in W_b$. On the other hand, we say that W is a *trivial* $S_\bullet-$*module* if $S_+W=0$. In particular, if k is viewed as an $S_\bullet-$module by way of the natural quotient map $S_\bullet\to S_\bullet/S_+=k$, then $\oplus_{i=1}^n k(-i)^{m_i}$ is a trivial $S_\bullet-$module.

Elementary results about linkage and DG−algebras may be found in [6] and [17]. In this paper, "DG−algebra" always means **associative** DG−algebra.

Section 1. The statement of the main theorem.

Let k be a fixed field. In Table 1.3 we define the graded $k-$algebras (in the sense of (0.1)) which appear in Theorem 1.5, the main theorem of the paper. Each of these algebras has the form $S_\bullet = \bigoplus_{i=0}^{4} S_i$ with $S_0 = k$ and $d_i = \dim_k S_i$. Select bases $\{x_i\}$ for S_1, $\{y_i\}$ for S_2, $\{z_i\}$ for S_3, and $\{w_i\}$ for S_4. View S_2 as the direct sum $S_2' \oplus S_1^2$. Numerical information about these algebras is collected in Table 1.4. One may combine Lemma 1.2 with Table 1.4 in order to conclude that each of the algebras of Table 1.3 represents a distinct isomorphism class of $k-$algebras, provided the parameters p, q, and r satisfy

$$0 \le p, \quad 2 \le q \le 3, \quad \text{and} \quad 2 \le r \le 5. \tag{1.1}$$

(If we had allowed q and r to take the value 1, then $\mathbf{E}^{(1)}$ would equal $\mathbf{E}[1]$ and $\mathbf{F}^{(1)}$ would equal $\mathbf{F}[1]$.)

Lemma 1.2. *If $S_\bullet$ is one of the algebras from Table 1.3, then there is a four dimensional subspace V of $S_\bullet$ with the property that $\dim V^2 = 6$ if and only if $S_\bullet$ is not equal to $\mathbf{C}^{(2)}$, $\mathbf{C}^\bigstar$, or $\mathbf{C}[p]$ for any p.*

Proof. If $S_\bullet$ is not equal to $\mathbf{C}^{(2)}$, $\mathbf{C}^\bigstar$, or $\mathbf{C}[p]$ for any p, then the subspace V of S_1 spanned by x_1, x_2, x_3, and x_4 has $\dim V^2 = 6$. On the other hand, we now suppose that $S_\bullet$ is equal to $\mathbf{C}^{(2)}$, $\mathbf{C}^\bigstar$, or $\mathbf{C}[p]$ for some p. Let x_1', x_2', x_3', x_4' be a basis for V. Select α_{ij} in k with $x_j' = \sum_{i=1}^{5} \alpha_{ij} x_i$; let $\Delta(i,j;a,b)$ and $D(a,b,c,d)$ represent the following determinants

$$\Delta(i,j;a,b) = \begin{vmatrix} \alpha_{ia} & \alpha_{ib} \\ \alpha_{ja} & \alpha_{jb} \end{vmatrix} \qquad \text{and} \qquad D(a,b,c,d) = \begin{vmatrix} \alpha_{a1} & \alpha_{a2} & \alpha_{a3} & \alpha_{a4} \\ \alpha_{b1} & \alpha_{b2} & \alpha_{b3} & \alpha_{b4} \\ \alpha_{c1} & \alpha_{c2} & \alpha_{c3} & \alpha_{c4} \\ \alpha_{d1} & \alpha_{d2} & \alpha_{d3} & \alpha_{d4} \end{vmatrix}.$$

Recall that $x_3x_4 = x_3x_5 = x_4x_5 = 0$ in $S_\bullet$. It follows that

$$x_a'x_b' = \sum_{i=1}^{2} \sum_{j=i+1}^{5} \Delta(i,j;a,b) x_i x_j \quad \text{in } S_\bullet.$$

Observe that

$$\begin{aligned} &\Delta(1,2;3,4)x_1'x_2' - \Delta(1,2;2,4)x_1'x_3' + \Delta(1,2;2,3)x_1'x_4' \\ &+\Delta(1,2;1,2)x_3'x_4' - \Delta(1,2;1,3)x_2'x_4' + \Delta(1,2;1,4)x_2'x_3' \\ &\qquad = \sum_{i=1}^{2} \sum_{j=i+1}^{5} D(1,2,i,j) x_i x_j = 0. \end{aligned}$$

There are two possibilities. If $\Delta(1,2;a,b) \ne 0$ for some pair (a,b), then $\dim V^2 \le 5$. If $\Delta(1,2;a,b) = 0$ for all (a,b), then the rank of

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{bmatrix}$$

is at most one, and V is contained in $U = (\lambda x_1 + \mu x_2, x_3, x_4, x_5)$ for some λ and μ in k. It follows that $\dim V^2 \le \dim U^2 \le 3$. $\square$

The definition of the algebras A – F*

The $k-$algebra $S_\bullet$ is graded-commutative and associative; every product of basis vectors which is not listed has been set equal to zero. The notation is explained at the beginning of the section. The parameters p, q, and r satisfy (1.1).

$S_\bullet$	d_1	d_2	d_3	d_4	$S_1\times S_1$	$S_1\times S_1\times S_1$	$S_1\times S_2'$	$S_1\times S_3$	$S_2\times S_2$
A	4	6	4	0	(a)	(a$'$)	0	0	0
B$[p]$	5	$p+7$	$2p+3$	p	(b) with $\ell=p$	(b$'$) with $\ell=2p$	(g)	(g$'$)	0
C$[p]$	5	$p+7$	$2p+3$	p	(c) with $\ell=p$	(c$'$) with $\ell=2p$	(g)	(g$'$)	0
C$^{(2)}$	5	8	7	1	(c) with $\ell=1$	(c$'$) with $\ell=4$	(h) with $j=2$	(h$'$) with $j=2$	0
C*	5	9	7	2	(c) with $\ell=2$	(c$'$) with $\ell=4$	(i)	(i$'$)	(i$'$)
D$[p]$	5	$p+8$	$2p+2$	p	(d) with $\ell=p$	(d$'$) with $\ell=2p$	(g)	(g$'$)	0
D$^{(2)}$	5	9	6	1	(d) with $\ell=1$	(d$'$) with $\ell=4$	(h) with $j=2$	(h$'$) with $j=2$	0
E$[p]$	5	$p+9$	$2p+1$	p	(e) with $\ell=p$	(e$'$) with $\ell=2p$	(g)	(g$'$)	0
E$^{(q)}$	5	10	$2q+1$	1	(e) with $\ell=1$	(e$'$) with $\ell=2q$	(h) with $j=q$	(h$'$) with $j=q$	0
F$[p]$	5	$p+10$	$2p$	p	(f) with $\ell=p$	0	(g)	(g$'$)	0
F$^{(r)}$	5	11	$2r$	1	(f) with $\ell=1$	0	(h) with $j=r$	(h$'$) with $j=r$	0
F*	5	12	10	2	(f) with $\ell=2$	0	(h) with $j=5$	(h$'$) with $j=5$	(j)

KEY:

(a) $x_1x_2=y_1$, $x_1x_3=y_2$, $x_1x_4=y_3$, $x_2x_3=y_4$, $x_2x_4=y_5$, $x_3x_4=y_6$
(a$'$) $x_1x_2x_3=z_1$, $x_1x_2x_4=z_2$, $x_1x_3x_4=z_3$, $x_2x_3x_4=z_4$
(b) $x_1x_2=y_{\ell+1}$, $x_1x_3=y_{\ell+2}$, $x_1x_4=y_{\ell+3}$, $x_1x_5=y_{\ell+4}$, $x_2x_3=y_{\ell+5}$, $x_2x_4=y_{\ell+6}$, $x_3x_4=y_{\ell+7}$
(b$'$) $x_1x_2x_3=z_{\ell+1}$, $x_1x_2x_4=z_{\ell+2}$, $x_1x_3x_4=z_{\ell+3}$
(c) $x_1x_2=y_{\ell+1}$, $x_1x_3=y_{\ell+2}$, $x_1x_4=y_{\ell+3}$, $x_1x_5=y_{\ell+4}$, $x_2x_3=y_{\ell+5}$, $x_2x_4=y_{\ell+6}$, $x_2x_5=y_{\ell+7}$
(c$'$) $x_1x_2x_3=z_{\ell+1}$, $x_1x_2x_4=z_{\ell+2}$, $x_1x_2x_5=z_{\ell+3}$
(d) $x_1x_2=y_{\ell+1}$, $x_1x_3=y_{\ell+2}$, $x_1x_4=y_{\ell+3}$, $x_1x_5=y_{\ell+4}$, $x_2x_3=y_{\ell+5}$, $x_2x_4=y_{\ell+6}$, $x_2x_5=y_{\ell+7}$, $x_3x_4=y_{\ell+8}$
(d$'$) $x_1x_2x_3=z_{\ell+1}$, $x_1x_2x_4=z_{\ell+2}$
(e) $x_1x_2=y_{\ell+1}$, $x_1x_3=y_{\ell+2}$, $x_1x_4=y_{\ell+3}$, $x_1x_5=y_{\ell+4}$, $x_2x_3=y_{\ell+5}$, $x_2x_4=y_{\ell+6}$, $x_2x_5=y_{\ell+7}$, $x_3x_4=y_{\ell+8}$, $x_3x_5=y_{\ell+9}$
(e$'$) $x_1x_2x_3=z_{\ell+1}$,
(f) $x_1x_2=y_{\ell+1}$, $x_1x_3=y_{\ell+2}$, $x_1x_4=y_{\ell+3}$, $x_1x_5=y_{\ell+4}$, $x_2x_3=y_{\ell+5}$, $x_2x_4=y_{\ell+6}$, $x_2x_5=y_{\ell+7}$, $x_3x_4=y_{\ell+8}$, $x_3x_5=y_{\ell+9}$, $x_4x_5=y_{\ell+10}$
(g) $x_1y_i=z_i$ for $1\le i\le p$,
(g$'$) $x_1z_{p+i}=w_i$ for $1\le i\le p$,
(h) $x_iy_1=z_i$ for $1\le i\le j$,
(h$'$) $x_iz_{j+i}=w_1$ for $1\le i\le j$,
(i) $x_1y_1=z_1$, $x_1y_2=z_2$, $x_2y_1=z_3$, $x_2y_2=z_4$
(i$'$) $x_1x_2y_1=w_1$, $x_1x_2y_2=w_2$,
(j) $y_1y_2=w_1$, $y_1^2=w_2$.

Table 1.3

Theorem 1.5. *Let $(R,\mathfrak{m},k)$ be a local ring in which 2 is a unit. Assume that every element of k has a square root in k. Let J be a grade four almost complete*

Numerical information about the algebras $\mathbf{A} - \mathbf{F}^{\star}$

$S_\bullet$	$\dim S_1^2$	$\dim S_1^3$	$\dim S_1S_2 - \dim S_1^3$	$\dim S_1S_3$	$\dim S_2^2$
$\mathbf{A}$	6	4	0	0	0
$\mathbf{B}[p]$	7	3	p	p	0
$\mathbf{C}[p]$	7	3	p	p	0
$\mathbf{C}^{(2)}$	7	3	2	1	0
$\mathbf{C}^{\star}$	7	3	4	2	2
$\mathbf{D}[p]$	8	2	p	p	0
$\mathbf{D}^{(2)}$	8	2	2	1	0
$\mathbf{E}[p]$	9	1	p	p	0
$\mathbf{E}^{(q)}$	9	1	q	1	0
$\mathbf{F}[p]$	10	0	p	p	0
$\mathbf{F}^{(r)}$	10	0	r	1	0
$\mathbf{F}^{\star}$	10	0	5	1	2

Table 1.4

intersection ideal in R, and let $T_\bullet$ be the graded $k-$algebra $\operatorname{Tor}_\bullet^R(R/J,k)$. Then there is a parameter p, q, or r which satisfies (1.1), an algebra $S_\bullet$ from the list

$$\mathbf{A},\ \mathbf{B}[p],\ \mathbf{C}[p],\ \mathbf{C}^{(2)},\ \mathbf{C}^{\star},\ \mathbf{D}[p],\ \mathbf{D}^{(2)},\ \mathbf{E}[p],\ \mathbf{E}^{(q)},\ \mathbf{F}[p],\ \mathbf{F}^{(r)},\ \mathbf{F}^{\star},$$

and a positively graded vector space W such that, $T_\bullet$ is isomorphic (as a graded $k-$algebra) to the trivial extension $S_\bullet \ltimes W$ of $S_\bullet$ by the trivial $S_\bullet-$module W.

NOTE: In the above theorem, the vector space W has the form $\bigoplus_{i=1}^4 k(-i)^{m_i}$ where $m_1 = 1$ if $S_\bullet = \mathbf{A}$, and $m_1 = 0$ in all other cases.

The proof of Theorem 1.5 is contained in section 4. We next record a few consequences of Theorem 1.5. If one is interested only in the subalgebra of $T_\bullet$ which is generated by T_1, then the classification of Theorem 1.5 can be made even cleaner.

Corollary 1.6. *If the notation of Theorem 1.5 is adopted, then the subalgebra $k[T_1]$ of $T_\bullet$ is isomorphic to one of the following six algebras:*

$$\mathbf{A} \ltimes k(-1), \quad \mathbf{B}[0], \quad \mathbf{C}[0], \quad \mathbf{D}[0], \quad \mathbf{E}[0], \quad \mathbf{F}[0].$$

In particular, the following numerical statements hold:

(a) $6 \le \dim T_1^2 \le 10$, *and*
(b) $\dim T_1^2 + \dim T_1^3 = 10$.

Proof. It is easy to see that if $T_\bullet$ has the form $S_\bullet \ltimes W$ (as described in Theorem 1.5) where $S_\bullet$ is $\mathbf{C}^{(2)}$, or $\mathbf{C}^{\star}$, or $\mathbf{C}[0]$ (for some p), then the subalgebra $k[T_1]$ of $T_\bullet$ is $\mathbf{C}[0]$. An analogous statement holds for all of the other algebras of Table 1.3. The numerical assertions follow from Table 1.4. $\square$

The next Corollary follows from Lemma 1.2 by way of a prime avoidance argument.

Corollary 1.7. *Adopt the notation of Theorem 1.5.* **Exactly** *one of the following statements holds:*

(a) *the subalgebra* $k[T_1]$ *of* $T_\bullet$ *is* $\mathbf{C}[0]$*; or*
(b) *there is a* **minimal** *presentation*

$$R^n \xrightarrow{d_2} R^5 \xrightarrow{[a_1,\dots,a_5]} J \tag{1.8}$$

for J *with the property that* a_1*,* a_2*,* a_3*,* a_4 *is a regular* $R-$*sequence and the first six columns of* d_2 *are*

$$\begin{bmatrix} -a_2 & -a_3 & -a_4 & 0 & 0 & 0 \\ a_1 & 0 & 0 & -a_3 & -a_4 & 0 \\ 0 & a_1 & 0 & a_2 & 0 & -a_4 \\ 0 & 0 & a_1 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad \square$$

Remarks 1.9. Some of the algebras of Table 1.3 have a compact coordinate-free representation:

(a) If V is the graded vector space $k(-1)^4$, then $\mathbf{A} \cong \bigwedge^\bullet V/\bigwedge^4 V$. In the notation of Theorem 1.5, one can show (see, for example, [25, Proposition 3.2] or [26, Proposition 4.2]) that $T_\bullet \cong \mathbf{A} \ltimes W$ if and only if there is a grade four Gorenstein ideal I and a grade four complete intersection ideal K with

$$K \subseteq \mathfrak{m}I \tag{1.10}$$

such that $J = K:I$. (The significant hypothesis in the last sentence is the one we have isolated as (1.10).)

(b) If V is the graded vector space $k(-1)^3$, then $\mathbf{B}[p]$ is isomorphic to

$$\left(\frac{\bigwedge^\bullet V}{\bigwedge^3 V} \ltimes [k(-1) \oplus k(-2)^p \oplus k(-3)^p]\right) \otimes_k \overset{\bullet}{\bigwedge} k(-1).$$

(c) The algebra $\mathbf{C}[p]$ is isomorphic to

$$\left[\left[\left(k \ltimes k(-1)^3\right) \otimes_k \overset{\bullet}{\bigwedge} k(-1)\right] \ltimes \left(k(-2)^p \oplus k(-3)^p\right)\right] \otimes_k \overset{\bullet}{\bigwedge} k(-1).$$

The algebra $\mathbf{C}^\star$ is isomorphic to

$$\left[k \ltimes \left(k(-1)^3 \oplus k(-2)^2\right)\right] \otimes_k \overset{\bullet}{\bigwedge} k(-1)^2.$$

If J' is a grade two almost complete intersection (in other words, J' is a determinantal ideal generated by the 2×2 minors of a 3×2 matrix), and J is the ideal

(J', a, b) for some elements a and b of R, with a, b a regular sequence on R/J', then $\operatorname{Tor}_\bullet^R(R/J, k)$ is isomorphic to $\mathbf{C}^\star$. (See the proof of case one in section 4.)

(d) Let $V = k(-1)^2$ and $V' = k(-1)^2$ be graded vector spaces and $S_\bullet$ be the graded $k-$algebra $\bigwedge^\bullet(V \oplus V')$. Let $\overline{S_\bullet}$ be the graded $k-$algebra and W be the $\overline{S_\bullet}-$module defined by

$$\overline{S_\bullet} = \frac{S_\bullet}{\left(\bigwedge^2 V'\right) S_+} \quad \text{and} \quad W = \frac{S_\bullet}{(V' + S_2)S_\bullet}.$$

If W^* is the $\overline{S_\bullet}-$module $\operatorname{Hom}_k(W, k)$, then

$$\begin{aligned} \mathbf{D}[0] &= \overline{S_\bullet} \ltimes W(-1), \quad \text{and} \\ \mathbf{D}^{(2)} &= \overline{S_\bullet} \ltimes \Big(W(-1) \oplus W(-2) \oplus W^*(-4)\Big). \end{aligned}$$

(e) Let V be the graded vector space $k(-1)^3$, W be the $\bigwedge^\bullet V-$module $\bigwedge^\bullet V / \bigwedge^2 V$, and W^* be the $\bigwedge^\bullet V-$module $\operatorname{Hom}_k(W, k)$. It is not difficult to see that

$$\begin{aligned} \mathbf{E}[0] &= \bigwedge^\bullet V \ltimes W(-1)^2, \quad \text{and} \\ \mathbf{E}^{(3)} &= \bigwedge^\bullet V \ltimes \Big(W(-1)^2 \oplus W(-2) \oplus W^*(-4)\Big). \end{aligned}$$

(f) If V is the graded vector space $k(-1)^5$, then $\mathbf{F}[0] \cong \bigwedge^\bullet V / \bigwedge^3 V$. Suppose that J is an ideal from Theorem 1.5 with the property that the subalgebra $k[T_1]$ of $T_\bullet$ is isomorphic to $\mathbf{F}[0]$. Let (1.8) be a **minimal** presentation of J. It follows that the basis for R^n can be chosen so that the first 10 columns of d_2 are

$$\begin{bmatrix} -a_2 & -a_3 & -a_4 & -a_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & -a_3 & -a_4 & -a_5 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & -a_4 & -a_5 & 0 \\ 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & a_3 & 0 & -a_5 \\ 0 & 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & a_3 & a_4 \end{bmatrix}.$$

Let W be the $\mathbf{F}[0]-$module $\mathbf{F}[0]/\mathbf{F}[0]^2$, and let W^* be the $\mathbf{F}[0]-$module $\operatorname{Hom}_k(W, k)$. It is not difficult to see that

$$\mathbf{F}^{(5)} \cong \mathbf{F}[0] \ltimes [W(-2) \oplus W^*(-4)].$$

Section 2. The Tor$-$algebra of a codimension four Gorenstein ring.

The classification of Tor$-$algebras for rings defined by grade four Gorenstein ideals plays a crucial role in the proof of Theorem 1.5. The following result is proved in [21] (when $\operatorname{char} k \neq 2$) and [16]. (The results in [17], [21], and [16] are stated for Gorenstein ideals in Gorenstein local rings; however, it is not difficult to check that the proofs hold for Gorenstein ideals in arbitrary local rings.) The

Tor−algebra $\operatorname{Tor}_\bullet^R(R/I,k)$ may be described intrinsically without any mention of the minimal resolution

$$\mathbb{L}: \quad 0 \to L_4 \xrightarrow{\ell_4} L_3 \xrightarrow{\ell_3} L_2 \xrightarrow{\ell_2} L_1 \xrightarrow{\ell_1} L_0 \tag{2.1}$$

of R/I. We have chosen to introduce $\mathbb{L}$ in Theorem 2.2 so that the notation in the present section coincides with the notation in section 4. We know from [17] and [16] that $\mathbb{L}$ is a DG−algebra; so, the graded k−algebras $\overline{\mathbb{L}}$ and $\operatorname{Tor}_\bullet^R(R/I,k)$ are equal. (Throughout the paper we write $\overline{}$ to mean $\_\_ \otimes_R k$ and $a \equiv b$ to mean $\overline{a} = \overline{b}$.)

Theorem 2.2. *Let $(R,\mathfrak{m},k)$ be a local ring. Assume that either every element in k has a square root in k, or else that the characteristic of k is equal to two. Let I be a grade four Gorenstein ideal in R, $\mathbb{L}$ be the minimal R−resolution of R/I, and $\overline{\mathbb{L}}$ be the graded k−algebra $\operatorname{Tor}_\bullet^R(R/I,k)$. If I is not a complete intersection, then there are bases $e_1,\dots,e_n$ for L_1; $f_1,\dots,f_{n-1},f_1',\dots,f_{n-1}'$ for L_2; $g_1,\dots,g_n$ for L_3; and h for L_4 such that the multiplication $\overline{L}_i \times \overline{L}_{4-i} \to \overline{L}_4$ is given by*

$$e_i g_j = \delta_{ij} h, \quad f_i f_j' \equiv \delta_{ij} h, \quad f_i f_j \equiv f_i' f_j' \equiv 0, \tag{2.3}$$

and the other products in $\overline{\mathbb{L}}$ are given by one of the following cases:

(a) *All products in $\overline{L}_1\overline{L}_1$ and $\overline{L}_1\overline{L}_2$ are zero.*

(b) *All products in $\overline{L}_1\overline{L}_1$ and $\overline{L}_1\overline{L}_2$ are zero except:*

$$\begin{gathered} e_1e_2 = f_1, \quad e_1e_3 = f_2, \quad e_2e_3 = f_3 \\ e_2f_1' \equiv e_3f_2' \equiv g_1, \quad -e_1f_1' \equiv e_3f_3' \equiv g_2, \quad \textit{and} \quad e_1f_2' \equiv e_2f_3' \equiv -g_3 \end{gathered} \tag{2.4}$$

(c) *There is an integer p such that $e_{p+1}e_i = f_i$, $e_if_i' \equiv g_{p+1}$, and $e_{p+1}f_i' \equiv -g_i$ for $1 \le i \le p$ and all other products in $\overline{L}_1\overline{L}_1$ and $\overline{L}_1\overline{L}_2$ are zero.* $\square$

Note: It is possible to choose the basis for $\mathbb{L}$ so that the multiplication is correct "on the nose" for $L_1 \otimes L_1 \to L_2$ and $L_1 \otimes L_3 \to L_4$, and is also correct for $\overline{L}_1 \otimes \overline{L}_2 \to \overline{L}_3$ and $\overline{L}_2 \otimes \overline{L}_2 \to \overline{L}_4$.

Remark 2.5. One consequence of the above classification is the well known fact that $\overline{L}_1^3 = 0$ when I is a grade four Gorenstein ideal which is not a complete intersection.

The proof of Theorem 1.5 requires that we understand the multiplication $V \otimes \overline{\mathbb{L}} \to \overline{\mathbb{L}}$, where V is an arbitrary subspace of $\overline{L}_1$. It is not difficult to guess all of the possibilities. For example, if the multiplication of $\overline{\mathbb{L}}$ is described in Theorem 2.2 (c), then the distinguished element $\overline{e}_{p+1}$ "may be taken" to be either in V (case (iii) below) or not in V (case (iv)). A complete proof of Corollary 2.7 (in contrast to the above heuristic argument) has two parts. We use linear algebra to find an appropriate basis of $\overline{L}_1$, and then we use the fact that $\overline{\mathbb{L}}$ is a Poincaré duality algebra to determine the rest of the multiplication in $\overline{\mathbb{L}}$. The second part of the argument is summarized in the following lemma, which appears as [21, Lemma 2.3]. (The characteristic two version of the lemma may be found at the end of [16].) The proof of Lemma 2.6, which is due to Avramov, is the only place in the present paper that the square roots of elements of k are used.

Lemma 2.6. *Let $\mathbb{L}$ be as in Theorem 2.2. If $e_1, \dots, e_n$ is any basis for L_1, h is any basis for L_4, and $f_1, \dots, f_m$ is the beginning of a basis for L_2 with $m \le n-1$ and $f_i f_j \equiv 0$ for all i and j, then there is a basis $g_1, \dots, g_n$ for L_3 and an extension of $f_1, \dots, f_m$ to a basis $f_1, \dots, f_{n-1}, f'_1, \dots, f'_{n-1}$ for L_2 such that (2.3) holds.* $\square$

Corollary 2.7. *Adopt the notation and hypotheses of Theorem 2.2. If V is a nonzero subspace of $\overline{L}_1$ of dimension t, then there are bases $\{e_i\}$ for L_1, $\{f_i, f'_i\}$ for L_2, $\{g_i\}$ for L_3 and h for L_4 such that (2.3) holds, $\overline{e}_1, \dots, \overline{e}_t$ is a basis for V, and the multiplication $V \otimes \overline{L}_1 \to \overline{L}_2$ and $V \otimes \overline{L}_2 \to \overline{L}_3$ is given by one of the following cases:*

(i) The integer t is at least 3 and the only nonzero products in $V\overline{L}_1$ and $V\overline{L}_2$ are given in (2.4).

(ii) The integer t is at least 2 and all products in $V\overline{L}_1$ and $V\overline{L}_2$ are zero, except

$$e_1e_2 = f_1, \quad e_1e_{t+1} = f_2, \quad e_2e_{t+1} = f_3$$
$$e_2f'_1 \equiv g_1, \quad -e_1f'_1 \equiv g_2, \quad \text{and} \quad e_1f'_2 \equiv e_2f'_3 \equiv -g_{t+1}$$

(iii) There are integers a and b, with $0 \le a \le t-1$ and $0 \le b$, such that the only nonzero products of basis vectors in $V\overline{L}_1$ and $V\overline{L}_2$ are

$$\begin{cases} e_1e_{1+i} = f_i, \quad e_1f'_i \equiv -g_{1+i}, \quad e_{1+i}f'_i \equiv g_1, & \text{for } 1 \le i \le a, \text{ and} \\ e_1e_{t+i} = f_{a+i}, \quad e_1f'_{a+i} \equiv -g_{t+i}, & \text{for } 1 \le i \le b. \end{cases}$$

(iv) There is an integer j, with $2 \le j \le t$, such that the only nonzero products of basis vectors in $V\overline{L}_1$ and $V\overline{L}_2$ are $e_{t+1}e_i = f_i$ and $e_if'_i \equiv g_{t+1}$ for $1 \le i \le j$.

Proof. If $\overline{\mathbb{L}}$ is described in Theorem 2.2 (a), then it is clear that $V\overline{\mathbb{L}}$ is given by (iii) with $a = b = 0$. We next suppose that $\overline{\mathbb{L}}$ is described by Theorem 2.2 (c). In this case $\overline{L}_1$ decomposes as $k\,\overline{e} \oplus U$ for some $e \in L_1$ and some $U \subseteq \overline{L}_1$ with $U^2 = 0$. There are two possibilities: either $V \subseteq U$, or else there is an element u of U such that $\overline{e} + u \in V$. If $V \subseteq U$, then we let e_{t+1} be the element e of L_1. Select elements $e_1, \dots, e_t$ of L_1 such that $\overline{e}_1, \dots, \overline{e}_t$ is a basis for V, $\overline{e}_{t+1}\overline{e}_1, \dots, \overline{e}_{t+1}\overline{e}_j$ is a basis for $\overline{e}_{t+1}V$, and $e_{t+1}e_i \equiv 0$ for $j+1 \le i \le t$. Define $f_i = e_{t+1}e_i$ in L_2 for $1 \le i \le j$. Observe that $(f_1, \dots, f_j)^2 = 0$. Complete the basis for $\mathbb{L}$ using Lemma 2.6. Observe that the multiplication $V\overline{\mathbb{L}}$ is described in (iv) (if $2 \le j$) or (iii) (with $a = 0$ and $b = j$ if $0 \le j \le 1$). If $\overline{e} + u \in V$, then let $e_1 \in L_1$ be a preimage of this element. Observe that $\overline{L}_1 = k\,\overline{e}_1 \oplus U$. Select $e_2, \dots, e_n \in L_1$ such that $\overline{e}_2, \dots, \overline{e}_n \in U$, $\overline{e}_1, \dots, \overline{e}_t$ is a basis for V, $e_1, \dots, e_n$ is a basis for L_1, $\overline{e}_1\overline{e}_2, \dots, \overline{e}_1\overline{e}_{a+1}$ is a basis for $\overline{e}_1V$, $\overline{e}_1\overline{e}_2, \dots, \overline{e}_1\overline{e}_{a+1}, \overline{e}_1\overline{e}_{t+1}, \dots, \overline{e}_1\overline{e}_{t+b}$ is a basis for $\overline{e}_1\overline{L}_1$, and $e_1e_i \equiv 0$ whenever $a+2 \le i \le t$ or $t+b+1 \le i \le n$. Define the elements $f_1, \dots, f_{a+b}$ in L_2 in the obvious manner and proceed, as in the case $V \subseteq U$, to show that $V\overline{\mathbb{L}}$ is described by (iii).

Finally, suppose that $\overline{\mathbb{L}}$ is described in Theorem 2.2 (b). In other words, we are given a decomposition $\overline{L}_1 = E \oplus U$ with $\dim E = \dim E^2 = 3$ and $U \cdot \overline{L}_1 = 0$. Consider the map $\pi \colon V \to E$ which is the composition

$$V \xrightarrow{\text{incl.}} \overline{L}_1 = E \oplus U \xrightarrow{\text{proj.}} E.$$

Let r be the rank of π. It is clear that the kernel of π is $V \cap U$; consequently, we may select $e_1, \dots, e_t$ in L_1 such that $\overline{e}_1, \dots, \overline{e}_t$ is a basis for V, and $\overline{e}_{r+1}, \dots, \overline{e}_t$ is a basis for $V \cap U$. It follows that $\pi(\overline{e}_1), \dots, \pi(\overline{e}_r)$ is a basis for $\operatorname{im} \pi$. Let $s = 3 - r$ and let $e_{t+1}, \dots, e_{t+s}$ be elements of L_1 such that $\overline{e}_{t+1}, \dots, \overline{e}_{t+s}$ are in E, and $\pi(\overline{e}_1), \dots, \pi(\overline{e}_r), \overline{e}_{t+1}, \dots, \overline{e}_{t+s}$ is a basis for E. If E' is the subspace $(\overline{e}_1, \dots, \overline{e}_r, \overline{e}_{t+1}, \dots, \overline{e}_{t+s})$ of $\overline{L}_1$, then it is clear that $\dim E' = \dim (E')^2 = 3$ and that $E' \oplus U = \overline{L}_1$. It follows that we can find $e_{t+s+1}, \dots, e_n$ in L_1 such that $\overline{e}_{t+s+1}, \dots, \overline{e}_n$ are in U, and $e_1, \dots, e_n$ is a basis of L_1. This basis has been chosen so that

$$\overline{e}_1, \dots, \overline{e}_r \in E' \cap V,\ \overline{e}_{r+1}, \dots, \overline{e}_t \in V \cap U,\ \overline{e}_{t+1}, \dots, \overline{e}_{t+s} \in E' \backslash V,\ \overline{e}_{t+s+1}, \dots, \overline{e}_n \in U \backslash V.$$

Complete the basis for $\mathbb{L}$ by using the technique of the preceding paragraph. It is now clear that $V\overline{\mathbb{L}}$ is described by (i), if $r = 3$; by (ii), if $r = 2$; by (iii) with $a = 0$ and $b = 2$, if $r = 1$; and by (iii) with $a = b = 0$, if $r = 0$. $\square$

Section 3. A DG−resolution of almost complete intersections.

Let J be a grade four almost complete intersection in the local ring $(R, \mathfrak{m}, k)$. In this section we describe Palmer's DG−algebra resolution $\mathbb{M}$ of R/J. This resolution, in general, is not the minimal resolution of R/J; nonetheless, we are able to use it in section 4 to compute the multiplication in $\operatorname{Tor}_\bullet^R(R/J, k)$.

Let K be a grade four complete intersection ideal with $K \subseteq J$ and $\mu(J/K) = 1$ (We use $\mu(M)$ to mean the minimal number of generators of the R−module M.) The ideal $I = K : J$ is known to be a grade four Gorenstein ideal. It is shown in [17] and [16] (the results in these references hold for Gorenstein ideals in arbitrary local rings) that the minimal resolution $\mathbb{L}$ of R/I is a DG−algebra. Let $\mathbb{K}$ be a Koszul complex which is the minimal resolution of R/K and let $\alpha_\bullet : \mathbb{K} \to \mathbb{L}$

$$\begin{array}{cccccccccccc}
 & 0 & \longrightarrow & K_4 & \xrightarrow{k_4} & K_3 & \xrightarrow{k_3} & K_2 & \xrightarrow{k_2} & K_1 & \xrightarrow{k_1} & K_0 \\
(3.1) & & & \Big\downarrow{\scriptstyle \alpha_4} & & \Big\downarrow{\scriptstyle \alpha_3} & & \Big\downarrow{\scriptstyle \alpha_2} & & \Big\downarrow{\scriptstyle \alpha_1} & & \Big\downarrow{\scriptstyle \alpha_0} \\
 & 0 & \longrightarrow & L_4 & \xrightarrow{\ell_4} & L_3 & \xrightarrow{\ell_3} & L_2 & \xrightarrow{\ell_2} & L_1 & \xrightarrow{\ell_1} & L_0
\end{array}$$

be a map of DG−algebras which extends the identity map $\alpha_0 : R \to R$. Fix orientation isomorphisms $[\ ] : K_4 \to R$ and $[\ ] : L_4 \to R$. A routine mapping cone argument establishes the following result.

Proposition 3.2. *Let J be a grade four almost complete intersection in the local ring $(R, \mathfrak{m}, k)$ and let K be a grade four complete intersection ideal with $K \subseteq J$ and $\mu(J/K) = 1$. Let $\mathbb{K}$ be the minimal resolution of R/K, $\mathbb{L}$ be the minimal resolution of R/I for $I = K : J$, and $\alpha_\bullet : \mathbb{K} \to \mathbb{L}$, as in (3.1), be a map of oriented DG−algebras. If $\beta_i : L_i \to K_i$ is the map defined by*

$$[\beta_i(v_i) u_{4-i}] = [v_i \alpha_{4-i}(u_{4-i})] \tag{3.3}$$

for all $u_j \in K_j$ and all $v_i \in L_i$, then

$$\mathbb{M} = \mathbb{M}(\alpha_\bullet) : \qquad 0 \to M_4 \xrightarrow{m_4} M_3 \xrightarrow{m_3} M_2 \xrightarrow{m_2} M_1 \xrightarrow{m_1} M_0$$

is a resolution of R/J, where $M_0 = R$, $M_1 = K_1 \oplus L_0$, $M_2 = K_2 \oplus L_1$, $M_3 = K_3 \oplus L_2$, $M_4 = L_3$, $m_1 = [\,k_1 \quad \beta_0\,]$,

$$m_2 = \begin{bmatrix} k_2 & -\beta_1 \\ 0 & \ell_1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} k_3 & \beta_2 \\ 0 & \ell_2 \end{bmatrix}, \quad \textit{and} \quad m_4 = \begin{bmatrix} -\beta_3 \\ \ell_3 \end{bmatrix}. \quad \square$$

NOTE: The definition of β_i makes use of the well-known perfect pairings

$$K_i \otimes K_{4-i} \to R \qquad \text{and} \qquad L_i \otimes L_{4-i} \to R,$$

which are given by $u_i \otimes u_{4-i} \mapsto [u_i u_{4-i}]$ and $v_i \otimes v_{4-i} \mapsto [v_i v_{4-i}]$. The orientation on the left side of (3.3) is the orientation on $\mathbb{K}$; whereas the orientation on the right side of (3.3) is the orientation on $\mathbb{L}$.

The next result asserts that $\mathbb{M}$ has the structure of a DG$-$algebra, provided 2 is a unit in R. A small amount of notation is needed in order to describe the multiplication in $\mathbb{M}$. Let h be the element of L_4 with $[h] = 1$ and let ε_1, ε_2, ε_3, ε_4 be a basis for K_1 with $[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4] = 1$. The result claims the existence of an $R-$module homomorphism $P\colon \bigwedge^5 L_1 \to L_2$ which satisfies a long list of properties. Two homomorphisms, $p\colon L_1 \to L_2$ and $q\colon L_2 \to L_3$, are defined in terms of P by:

$$p(v_1) = P(v_1 \wedge \alpha_1(\varepsilon_1) \wedge \alpha_1(\varepsilon_2) \wedge \alpha_1(\varepsilon_3) \wedge \alpha_1(\varepsilon_4)), \tag{3.4}$$

and $v_1 q(v_2) = v_2 p(v_1)$ for all $v_i \in L_i$.

Theorem 3.5. ([25]) *Adopt the notation of the preceding paragraph together with the notation and hypotheses of Proposition 3.2. If 2 is a unit in R, then there is a map $P\colon \bigwedge^5 L_1 \to L_2$ such that the following maps give $\mathbb{M}$ the structure of a DG$-$algebra:*

$$M_1 \otimes M_1 \to M_2 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_1' \\ v_0' \end{bmatrix} = \begin{bmatrix} u_1 u_1' \\ v_0'\alpha_1(u_1) - v_0\alpha_1(u_1') \end{bmatrix}$$

$$M_1 \otimes M_2 \to M_3 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_1 \end{bmatrix} = \begin{bmatrix} u_1 u_2 \\ v_0\alpha_2(u_2) + \alpha_1(u_1)v_1 + v_0 p(v_1) \end{bmatrix}$$

$$M_1 \otimes M_3 \to M_4 : \begin{bmatrix} u_1 \\ v_0 \end{bmatrix} \begin{bmatrix} u_3 \\ v_2 \end{bmatrix} = -[u_1 u_3]\ell_4(h) - v_0\alpha_3(u_3) + \alpha_1(u_1)v_2 - v_0 q(v_2)$$

$$M_2 \otimes M_2 \to M_4 : \begin{bmatrix} u_2 \\ v_1 \end{bmatrix} \begin{bmatrix} u_2' \\ v_1' \end{bmatrix} = -[u_2 u_2']\ell_4(h) + \alpha_2(u_2)v_1' + v_1\alpha_2(u_2') + v_1 p(v_1') + v_1' p(v_1)$$

for all $u_i, u_i' \in K_i$ and $v_i, v_i' \in L_i$. Furthermore, the map P also has the property that

$$v_1 v_1' P(v_1 \wedge v_1' \wedge \_\_)\colon \bigwedge^3 L_1 \to L_4$$

is the zero map for all $v_1, v_1' \in L_1$. $\square$

NOTE: There are two parts to the proof in [25, 26]. In the first part, a long list of properties for P is compiled such that whenever a map P satisfies all of these properties, then the above indicated multiplication gives $\mathbb{M}$ the structure of a DG$-$algebra. The one property for P that is highlighted in Theorem 3.5 is just one of the many properties from this list; however, it happens to be the only property of P that we use explicitly in section 4. The second, and much more difficult, part of the proof in [25, 26] is to prove that the desired P (a "higher order multiplication" on the resolution $\mathbb{L}$ of a codimension four Gorenstein ring) does exist.

Section 4. The proof of the main theorem.

Fix the notation and hypotheses of Theorem 1.5. If K is a grade four complete intersection ideal with $K \subseteq J$ and $\mu(J/K) = 1$, then we say that the grade four Gorenstein ideal $I = K : J$ is (*directly*) *linked to* J *by* K. For each such K, let

$$t(K) = \dim_k \left(\frac{K + \mathfrak{m}I}{\mathfrak{m}I} \right).$$

In other words, $t(K)$ is the cardinality of the largest subset of K which begins a minimal generating set for the ideal $K : J$. It is clear that $0 \le t(K) \le 4$. Our proof of Theorem 1.5 is divided into three cases:

Case 1: The ideal J is directly linked to a complete intersection.

Case 2: The ideal J is not directly linked to a complete intersection; and there exists a grade four complete intersection ideal K with $K \subseteq J$, $\mu(J/K) = 1$, and $t(K) \le 3$.

Case 3: The ideal J is not directly linked to a complete intersection; and $t(K) = 4$ for every grade four complete intersection ideal K with $K \subseteq J$ and $\mu(J/K) = 1$.

The proof of Theorem 1.5 in case 1. According to the hypothesis, there are complete intersection ideals I and K with $K \subseteq J$, $\mu(J/K) = 1$, and I linked to J by K. Let $t = t(K)$ and $s = 4 - t$. It is known (see, for example, [5, Theorem 3.2]) that there are matrices $\mathbf{a}_{1\times s}$, $\mathbf{b}_{1\times t}$ and $X_{s\times s}$ with entries in $\mathfrak{m}$ such that $J = J' + I_1(\mathbf{b})$ and the entries $b_1, \dots, b_t$ of $\mathbf{b}$ form a regular sequence on both R and R/J' where $J' = I_1(\mathbf{a}X) + I_s(X)$. (If M is a matrix with entries in R, then we use $I_\ell(M)$ to denote the ideal in R generated by the $\ell \times \ell$ minors of M.) Let $\mathbb{L}'$ be the minimal resolution of R/J' and $\mathbb{K}$ be the Koszul complex which is the minimal resolution of $R/I_1(\mathbf{b})$. Both of these resolutions are DG−algebras. (See [5, Proposition 4.4] for the multiplication on $\mathbb{L}'$.) It follows that the resolution $\mathbb{L}' \otimes_R \mathbb{K}$ of R/J is a DG−algebra; and therefore,

$$T_\bullet \cong \operatorname{Tor}^R_\bullet(R/J', k) \otimes_k \operatorname{Tor}^R_\bullet(R/I_1(\mathbf{b}), k).$$

We know that $\operatorname{Tor}^R_\bullet(R/I_1(\mathbf{b}), k)$ is the exterior algebra $\bigwedge^\bullet k(-1)^t$. Proposition 4.4 of [5] shows that

$$\operatorname{Tor}^R_\bullet(R/J', k) \cong S_\bullet \ltimes W \quad \text{where} \quad V = k(-1)^s, \quad S_\bullet = \frac{\bigwedge^\bullet V}{\bigwedge^s V},$$

and W is the trivial $S_\bullet$−module

$$\sum_{i=1}^{s} k(-i)^{\binom{s}{i-1}}.$$

The hypothesis that J is a proper ideal which is not a complete intersection ensures that $0 \le t \le 2$. It is now clear that

$$T_\bullet = \begin{cases} \mathbf{C}^\star, & \text{if } t = 2, \\ \mathbf{B}[3], & \text{if } t = 1, \text{ and} \\ \mathbf{A} \ltimes W, & \text{if } t = 0. \end{cases}$$

The proof of Theorem 1.5 in case 1 is complete.

For each choice of a grade four complete intersection ideal K with $K \subseteq J$ and J/K cyclic, we are able to use the information of sections 2 and 3 in order to calculate part of the multiplication in $T_\bullet$. To prove Theorem 1.5 in cases 2 and 3, we piece together this incomplete information in order to produce the entire multiplication table for $T_\bullet$. For the time being, let K be a fixed grade four complete intersection with $K \subseteq J$ and J/K cyclic. Let t denote $t(K)$, and let I be the Gorenstein ideal $K\colon J$. (We are finished with case 1; so we may assume that the ideal I is not a complete intersection.) Define $\mathbb{K}$, $\mathbb{L}$, and $\alpha_\bullet$ as in (3.1); β_i and $\mathbb{M} = \mathbb{M}(\alpha_\bullet)$ as in Proposition 3.2; and an algebra structure on $\mathbb{M}$ as in Theorem 3.5. We calculate multiplication in $T_\bullet$ by using the fact that $T_\bullet$ is equal to the homology algebra $H_\bullet(\overline{\mathbb{M}})$. A quick look at Proposition 3.2 shows that $T_\bullet = \bigoplus_{i=0}^4 T_i$, where $T_0 = k$, $T_1 = \overline{K}_1 \oplus \overline{L}_0$,

$$T_2 = \frac{\overline{K}_2}{\operatorname{im}\overline{\beta}_2} \oplus \overline{L}_1, \quad T_3 = \frac{\overline{K}_3}{\operatorname{im}\overline{\beta}_3} \oplus \ker\overline{\beta}_2, \quad \text{and} \quad T_4 = \ker\overline{\beta}_3. \tag{4.1}$$

Much of the multiplication on $\mathbb{M}$ becomes zero in $\overline{\mathbb{M}}$. The resolution $\mathbb{L}$ is minimal; and therefore, $\ell_4 \equiv 0$. We know from Remark 2.5 that $L_1^3 \subseteq \mathfrak{m}L_3$; thus,

$$\operatorname{im}\alpha_3 = (\operatorname{im}\alpha_1)^3 \subseteq L_1^3 \subseteq \mathfrak{m}L_3 \quad \text{and} \quad L_1 \cdot (\operatorname{im}\alpha_2) \subseteq \mathfrak{m}L_3.$$

It follows that the multiplication of Theorem 3.5 induces the following multiplication on $T_\bullet$:

(4.2)

$$T_1 \otimes T_1 \to T_2 : \begin{bmatrix} \overline{u}_1 \\ \overline{v}_0 \end{bmatrix} \begin{bmatrix} \overline{u'}_1 \\ \overline{v'}_0 \end{bmatrix} = \begin{bmatrix} \overline{u}_1\overline{u'}_1 \pmod{\operatorname{im}\overline{\beta}_2} \\ \overline{v'}_0\overline{\alpha}_1(\overline{u}_1) - \overline{v}_0\overline{\alpha}_1(\overline{u'}_1) \end{bmatrix}$$

$$T_1 \otimes T_2 \to T_3 : \begin{bmatrix} \overline{u}_1 \\ \overline{v}_0 \end{bmatrix} \begin{bmatrix} \overline{u}_2 \pmod{\operatorname{im}\overline{\beta}_2} \\ \overline{v}_1 \end{bmatrix} = \begin{bmatrix} \overline{u}_1\overline{u}_2 \pmod{\operatorname{im}\overline{\beta}_3} \\ \overline{v}_0\overline{\alpha}_2(\overline{u}_2) + \overline{\alpha}_1(\overline{u}_1)\overline{v}_1 + \overline{v}_0\overline{p}(\overline{v}_1) \end{bmatrix}$$

$$T_1 \otimes T_3 \to T_4 : \begin{bmatrix} \overline{u}_1 \\ \overline{v}_0 \end{bmatrix} \begin{bmatrix} \overline{u}_3 \pmod{\operatorname{im}\overline{\beta}_3} \\ \overline{v}_2 \end{bmatrix} = \overline{\alpha}_1(\overline{u}_1)\overline{v}_2 - \overline{v}_0\overline{q}(\overline{v}_2)$$

$$T_2 \otimes T_2 \to T_4 : \begin{bmatrix} \overline{u}_2 \pmod{\operatorname{im}\overline{\beta}_2} \\ \overline{v}_1 \end{bmatrix} \begin{bmatrix} \overline{u'}_2 \pmod{\operatorname{im}\overline{\beta}_2} \\ \overline{v'}_1 \end{bmatrix} = \overline{v}_1\overline{p}(\overline{v'}_1) + \overline{v'}_1\overline{p}(\overline{v}_1)$$

for $u_i, u'_i \in K_i$ and $v_i, v'_i \in L_i$.

Apply Corollary 2.7 to the subspace $\operatorname{im}\overline{\alpha}_1$ of $\overline{L}_1$ in the Tor$-$algebra $\overline{\mathbb{L}} = \operatorname{Tor}_\bullet^R(R/I, k)$ in order to find bases $e_1, \dots, e_n$ for L_1; $f_1, \dots, f_{n-1}, f'_1, \dots, f'_{n-1}$ for L_2; $g_1, \dots, g_n$ for L_3; and h for L_4 such that $\overline{e}_1, \dots, \overline{e}_t$ is a basis for $\operatorname{im}\overline{\alpha}$, and the multiplication $(\operatorname{im}\overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is described by one of (i)$-$(iv). In particular, there are 5 possibilities for the multiplication $(\operatorname{im}\overline{\alpha}_1) \cdot (\operatorname{im}\overline{\alpha}_1)$:

(A) all products are zero; or
(B) $\overline{e}_1\overline{e}_2 = \overline{f}_1$; or
(C) $\overline{e}_1\overline{e}_2 = \overline{f}_1$, and $\overline{e}_1\overline{e}_3 = \overline{f}_2$; or
(D) $\overline{e}_1\overline{e}_2 = \overline{f}_1$, $\overline{e}_1\overline{e}_3 = \overline{f}_2$, and $\overline{e}_2\overline{e}_3 = \overline{f}_3$; or
(E) $\overline{e}_1\overline{e}_2 = \overline{f}_1$, $\overline{e}_1\overline{e}_3 = \overline{f}_2$, and $\overline{e}_1\overline{e}_4 = \overline{f}_3$.

For each possibility we have listed the nonzero products; all other products of basis vectors are zero. In case two of our proof of Theorem 1.5, we have $t \le 3$, so possibility (E) does not occur in this case. Furthermore, Lemma 4.14 (b) shows that in case 3 the multiplication $(\operatorname{im}\overline{\alpha}_1)^2$ is described by (A); consequently there is no loss of generality if we set up our notation under the hypothesis that

(4.3) the multiplication $(\operatorname{im}\overline{\alpha}_1)^2$ is described by one of (A) – (D).

Choose a basis ε_1, ε_2, ε_3, ε_4 for K_1 such that

$$\alpha_1(\varepsilon_i) = e_i \ \text{ for } 1 \le i \le t, \qquad \alpha_1(\varepsilon_i) \equiv 0 \ \text{ for } t+1 \le i \le 4, \quad \text{and}$$

$$[\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_4] = 1. \tag{4.4}$$

(Notice that the definition of p in (3.4) appears to use a particular basis for K_1; however, every basis ε_1, ε_2, ε_3, ε_4 of K_1 which satisfies (4.4) gives rise to the exact same function p.) Now that the basis for K_1 is set, we give names to the corresponding basis elements of K_2 and K_3:

$$\varphi_1 = \varepsilon_1\varepsilon_2, \quad \varphi_2 = \varepsilon_1\varepsilon_3, \quad \varphi_3 = \varepsilon_2\varepsilon_3, \quad \varphi_1' = \varepsilon_3\varepsilon_4, \quad \varphi_2' = -\varepsilon_2\varepsilon_4, \quad \varphi_3' = \varepsilon_1\varepsilon_4,$$
$$\gamma_1 = \varepsilon_2\varepsilon_3\varepsilon_4, \quad \gamma_2 = -\varepsilon_1\varepsilon_3\varepsilon_4, \quad \gamma_3 = \varepsilon_1\varepsilon_2\varepsilon_4, \text{ and } \quad \gamma_4 = -\varepsilon_1\varepsilon_2\varepsilon_3.$$

Let $d = \operatorname{rank}\overline{\alpha}_2$. It is clear that $0 \le d \le \binom{t}{2}$. The notation has been set up, thanks to (4.3), so that

$$\alpha_2(\varphi_i) = f_i \text{ for } 1 \le i \le d, \quad \text{and} \quad \alpha_2(\varphi_i) \equiv \alpha_2(\varphi_j') \equiv 0 \text{ for } d+1 \le i \le 3 \text{ and } 1 \le j \le 3.$$

A straightforward application of (3.3) yields

$$\beta_3(g_i) \equiv \begin{cases} \gamma_i, & \text{if } 1 \le i \le t, \\ 0, & \text{if } t+1 \le i \le n, \end{cases} \qquad \beta_2(f_i') \equiv \begin{cases} \varphi_i', & \text{if } 1 \le i \le d, \\ 0 & \text{if } d+1 \le i \le n-1, \end{cases}$$

and $\beta_2(f_i) \equiv 0$ for $1 \le i \le n-1$. Thus,

(4.5)
$$\ker\overline{\beta}_3 = (\overline{g}_{t+1}, \dots, \overline{g}_n) \subseteq \overline{L}_3, \quad \ker\overline{\beta}_2 = (\overline{f'}_{d+1}, \dots, \overline{f'}_{n-1}, \overline{f}_1, \dots, \overline{f}_{n-1}) \subseteq \overline{L}_3,$$
$$\operatorname{im}\overline{\beta}_3 = (\overline{\gamma}_1, \dots, \overline{\gamma}_t) \subset \overline{K}_3, \quad \text{and} \quad \operatorname{im}\overline{\beta}_2 = (\overline{\varphi'}_1, \dots, \overline{\varphi'}_d) \subset \overline{K}_2.$$

Label the following elements of $T_\bullet$:

$$x_i = \begin{bmatrix} \overline{\varepsilon}_i \\ 0 \end{bmatrix}, \ x_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in T_1 \quad \text{for } 1 \le i \le 4;$$

$$y_i = \begin{bmatrix} \overline{\varphi'}_i \pmod{\operatorname{im}\overline{\beta}_2} \\ 0 \end{bmatrix}, \ y_{3+i} = \begin{bmatrix} \overline{\varphi}_i \pmod{\operatorname{im}\overline{\beta}_2} \\ 0 \end{bmatrix}, \ y_{6+j} = \begin{bmatrix} 0 \\ \overline{e}_j \end{bmatrix} \in T_2$$

for $1 \le i \le 3$ and $1 \le j \le n$;

$$z_i = \begin{bmatrix} \overline{\gamma}_i \pmod{\operatorname{im}\overline{\beta}_3} \\ 0 \end{bmatrix}, \ z_{4+j} = \begin{bmatrix} 0 \\ \overline{f}_j \end{bmatrix}, \ z_{n-d+3+\ell} = \begin{bmatrix} 0 \\ \overline{f'}_\ell \end{bmatrix} \in T_3$$

for $1 \le i \le 4$, $1 \le j \le n-1$, and $d+1 \le \ell \le n-1$; and

$$w_i = \overline{g}_{t+i} \in T_4 \quad \text{for } 1 \le i \le n-t.$$

(Notice that the above labeling depends on the choice of K.) We see from (4.1) and (4.5) that $y_1 = \cdots = y_d = 0$, $z_1 = \cdots = z_t = 0$. Furthermore,

$$\begin{gathered} x_1, \dots, x_5 \text{ is a basis for } T_1; \quad y_{d+1}, \dots, y_{6+n} \text{ is a basis for } T_2; \\ z_{t+1}, \dots, z_{2n+2-d} \text{ is a basis for } T_3; \text{ and } \quad w_1, \dots, w_{n-t} \text{ is a basis for } T_4. \end{gathered} \tag{4.6}$$

It is easy to see, using (4.2), that the multiplication $T_1 \otimes T_1 \to T_2$ is given by

$$\begin{gathered} x_1x_2 = y_4, \quad x_1x_3 = y_5, \quad x_2x_3 = y_6, \quad x_3x_4 = y_1, \quad x_2x_4 = -y_2, \\ x_1x_4 = y_3, \quad \text{and} \quad x_ix_5 = \begin{cases} y_{6+i}, & \text{for } 1 \le i \le t, \\ 0, & \text{for } t+1 \le i \le 4; \end{cases} \end{gathered} \tag{4.7}$$

and that the multiplication $T_1 \otimes T_1 \otimes T_1 \to T_3$ is given by

$$\begin{gathered} x_2x_3x_4 = z_1, \quad x_1x_3x_4 = -z_2, \quad x_1x_2x_4 = z_3, \quad x_1x_2x_3 = -z_4, \\ x_1x_2x_5 = \begin{cases} z_5, & \text{if } 1 \le d, \\ 0 & \text{if } d = 0, \end{cases} \qquad x_1x_3x_5 = \begin{cases} z_6, & \text{if } 2 \le d, \\ 0 & \text{if } d \le 1, \end{cases} \\ x_2x_3x_5 = \begin{cases} z_7, & \text{if } 3 \le d, \\ 0 & \text{if } d \le 2, \end{cases} \end{gathered} \tag{4.8}$$

and $x_ix_4x_5 = 0$ for all i. Furthermore, all of the products of basis vectors from

$$(x_1, \dots, x_4) \cdot (y_{t+7}, \dots, y_{n+6}) \quad \text{and} \quad (x_1, \dots, x_4) \cdot T_3 \tag{4.9}$$

are zero except

$$x_1y_{t+7} = z_6, \quad x_2y_{t+7} = z_7, \quad \text{and} \quad x_1z_{n+4} = x_2z_{n+5} = -w_1$$

when the multiplication $(\operatorname{im}\overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is described by Corollary 2.7 (ii);

$$x_1y_{6+t+i} = z_{4+d+i} \quad \text{and} \quad x_1z_{n+3+i} = -w_i \quad \text{for } 1 \le i \le b \tag{4.10}$$

when the multiplication $(\operatorname{im}\overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is described by Corollary 2.7 (iii); and

$$x_iy_{t+7} = -z_{4+i} \quad \text{and} \quad x_iz_{n+3+i} = w_1 \quad \text{for } 1 \le i \le j \tag{4.11}$$

when the multiplication $(\operatorname{im}\overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is described by Corollary 2.7 (iv). It is not possible to determine

$$x_5 \cdot (y_{t+7}, \dots, y_{n+6}), \quad x_5 \cdot T_3, \quad \text{or} \quad T_2 \cdot T_2 \tag{4.12}$$

at the present level of generality.

The proof of Theorem 1.5 in case 2. Fix a complete intersection ideal K with $K \subseteq J$, $\mu(J/K) = 1$, and $t(K) \le 3$. Use K to calculate multiplication in $T_\bullet$ as described in (4.1) and (4.2). The map p of (3.4) satisfies $p \equiv 0$ because $\operatorname{rank}\overline{\alpha}_1 = t \le 3$. The map q is defined in terms of p; hence, $q \equiv 0$. It follows that all of the products of (4.12) are zero. Combine the basis for $T_\bullet$ given in (4.6) with the multiplication from

(4.7), (4.8), and (4.9) in order to see that Table 4.13 is correct and complete, where $T_\bullet = S_\bullet \ltimes W$ for some trivial $S_\bullet$–module W. Recall that the algebras $\mathbf{A}$ – $\mathbf{F}^\star$ are defined in Table 1.3. If the multiplication $(\operatorname{im}\overline{\alpha}_1)\cdot\overline{\mathbb{L}}$ is described in part (iii) of Corollary 2.7, then the parameter a must equal d. The multiplications in part (ii) and part (iv) each require that $2 \le t$; but (ii) must have and $d = 1$, whereas (iv) requires $d = 0$.

The conclusion of the proof of Theorem 1.5 in case 2

t	d	$(\operatorname{im}\overline{\alpha}_1)\cdot\overline{\mathbb{L}}$	$k[T_1]$	$S_\bullet$
0	0	$(\operatorname{im}\overline{\alpha}_1) = 0$	$\mathbf{A} \ltimes k(-1)$	$\mathbf{A}$
1	0	(iii) with $a = 0$ and $b \ge 0$	$\mathbf{B}[0]$	$\mathbf{B}[b]$
2	0	(iii) with $a = 0$ and $b \ge 0$	$\mathbf{D}[0]$	$\mathbf{D}[b]$
2	0	(iv) with $j = 2$	$\mathbf{D}[0]$	$\mathbf{D}^{(2)}$
2	1	(ii)	$\mathbf{C}[0]$	$\mathbf{C}^{(2)}$
2	1	(iii) with $a = 1$ and $b \ge 0$	$\mathbf{C}[0]$	$\mathbf{C}[b]$
3	0	(iii) with $a = 0$ and $b \ge 0$	$\mathbf{E}[0]$	$\mathbf{E}[b]$
3	0	(iv) with $2 \le j \le 3$	$\mathbf{E}[0]$	$\mathbf{E}^{(j)}$
3	1	(ii)	$\mathbf{D}[0]$	$\mathbf{D}^{(2)}$
3	1	(iii) with $a = 1$ and $b \ge 0$	$\mathbf{D}[0]$	$\mathbf{D}[b]$
3	2	(iii) with $a = 2$ and $b \ge 0$	$\mathbf{B}[0]$	$\mathbf{B}[b]$
3	3	(i)	$\mathbf{A} \ltimes k(-1)$	$\mathbf{A}$

Table 4.13

The proof of Theorem 1.5 in case 2 is complete.

Without any further ado, we are able to identify the subalgebra $k[T_1]$ of $T_\bullet$ in case 3 of Theorem 1.5. Part (b) of the following Lemma appears to be technical; but, in particular, it yields a complete description of the minimal resolution of R/J.

Lemma 4.14. *If the notation and hypotheses for case 3 (from the beginning of the present section) are adopted, then the following statements hold:*

(a) *The algebra $k[T_1]$ is isomorphic to $\mathbf{F}[0]$.*
(b) *Let K be any grade four complete intersection ideal with $K \subseteq J$ and $\mu(J/K) = 1$. If $\mathbb{M}$ from Proposition 3.2 is the corresponding resolution of R/J, then $\beta_2 \equiv 0$, $\alpha_2 \equiv 0$, and $\operatorname{im}\beta_3 = K_3$.*

Proof. We first prove that $\dim_k T_1^2 = 10$. Let $\mathbf{a} = \{a_1, \dots, a_5\}$ be a minimal generating set of J with the property that every four element subset of $\mathbf{a}$ is a regular sequence; and let x_i' be the image of $\overline{a}_i$ under the natural isomorphism

$$\frac{J}{\mathfrak{m}J} \xrightarrow{\cong} \operatorname{Tor}_1^R(R/J, k). \tag{4.15}$$

It suffices to show that

$$\dim_k\left(\frac{T_1^2}{(x_1', \dots, \widehat{x_i'}, \dots, x_5')^2}\right) = 4 \tag{4.16}$$

for $i = 1, \dots, 5$. We establish (4.16) for $i = 5$; the other four cases follow from the symmetry of the situation. Let K be the complete intersection ideal $(a_1, \dots, a_4)$. Consider $T_\bullet$ as described in (4.1). If $\varepsilon'_1, \dots, \varepsilon'_4$ is a basis for K_1 with $k_1(\varepsilon'_i) = a_i$, then it follows that

$$x'_i = \begin{bmatrix} \overline{\varepsilon'}_i \\ 0 \end{bmatrix} \quad \text{for } 1 \le i \le 4. \qquad \text{Let} \quad x_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is not necessarily true that $x'_5 = x_5$; but we do know that $x'_5 = \lambda x_5 + x'$ for some for some unit $\lambda \in k$ and some $x' \in (x'_1, \dots, x'_4)$. The multiplication in $T_\bullet$ can be read from (4.2):

$$x'_i x'_j = \begin{bmatrix} \overline{\varepsilon'}_i \overline{\varepsilon'}_j \pmod{\operatorname{im} \overline{\beta}_2} \\ 0 \end{bmatrix} \quad \text{and} \quad x'_i x_5 = \begin{bmatrix} 0 \\ \overline{\alpha}_1(\overline{\varepsilon'}_i) \end{bmatrix}$$

for $1 \le i, j \le 4$. The hypothesis ensures that $t = 4$; so $\alpha_1(\varepsilon'_1), \dots, \alpha_1(\varepsilon'_4)$ is the beginning of a basis for L_1. We have established that $x'_1 x'_5$, $x'_2 x'_5$, $x'_3 x'_5$, and $x'_4 x'_5$ generate a four dimensional subspace of $T_1^2/(x'_1, \dots, x'_4)^2$; therefore, (4.16) holds and $\dim T_1^2 = 10$.

Furthermore, now that we know that $\dim T_1^2 = 10$, we may read the preceding paragraph from bottom to top in order to conclude that $\operatorname{im} \overline{\beta}_2 = 0$ for every resolution $\mathbb{M}$ from Proposition 3.2. It is clear that $\operatorname{rank} \overline{\alpha}_2 = \operatorname{rank} \overline{\beta}_2 = 0$, and that $\operatorname{rank} \overline{\beta}_3 = \operatorname{rank} \overline{\alpha}_1 = t = 4$; consequently, (b) has been established.

To finish the proof of (a) we must show that $T_1^3 = 0$. Once again, we use (4.2) to see that

$$x'_i x'_j x'_\ell = \begin{bmatrix} \overline{\varepsilon'}_i \overline{\varepsilon'}_j \overline{\varepsilon'}_\ell \pmod{\operatorname{im} \overline{\beta}_3} \\ 0 \end{bmatrix} \quad \text{and} \quad x'_i x'_j x_5 = \begin{bmatrix} 0 \\ \overline{\alpha}_2(\overline{\varepsilon'}_i \overline{\varepsilon'}_j) \end{bmatrix}$$

for $1 \le i, j, \ell \le 4$. The product $x'_i x'_j x'_\ell$ is equal to 0 because $\overline{\beta}_3$ is surjective; and $x'_i x'_j x_5 = 0$ because $\overline{\alpha}_2 = 0$. $\square$

We now subdivide case 3 into two further subcases:

Case 3a. There is a nonzero element $x \in T_1$ such that $xT_2 = 0$ and $xT_3 = 0$.

Case 3b. If $x \in T_1$ with $x \neq 0$, then either $xT_2 \neq 0$ or $xT_3 \neq 0$.

The proof of Theorem 1.5 in case 3A. Let a be an element of J with the property that $\overline{a}$ is sent to x under the isomorphism of (4.15), and K be a grade four complete intersection ideal such that $J = (K, a)$. Adopt the notation of the paragraph preceding (4.1) and apply Corollary 2.7 in order to pick a basis for $\mathbb{L}$ so that the multiplication in $(\operatorname{im} \overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is described by one of the cases (i) – (iv). Recall from part (b) of Lemma 4.14 that $\alpha_2 \equiv 0$; hence, the multiplication $(\operatorname{im} \overline{\alpha}_1) \cdot \overline{\mathbb{L}}$ is actually described by either (iii) with $a = 0$ or (iv). Label the elements x_i, y_i, z_i, and w_i of $T_\bullet$ exactly as was done in (4.6). (Keep in mind that $t = 4$ and $d = 0$.) Notice that $x_5 = \lambda x + x'$ for some unit $\lambda \in k$ and some $x' \in (x_1, \dots, x_4)$. We will know all of the multiplication in $T_\bullet$ once we show that $T_2 \cdot T_2 = 0$. According to (4.2) it suffices to prove that $v'_1 p(v_1) \equiv 0$ for all $v_1, v'_1 \in L_1$; and therefore, by Remark 2.5, it suffices to show that $\overline{p}(\overline{v}_1) \in \overline{L}_1^2$. Since $x' \in (x_1, \dots, x_4)$, there is an element $\varepsilon \in K_1$ such that

$$x' = \begin{bmatrix} \overline{\varepsilon} \\ 0 \end{bmatrix} \quad \text{in } T_1.$$

Recall that $xT_2 = 0$. Use (4.2) to compute that

$$\begin{bmatrix} 0 \\ \overline{\alpha}_1(\overline{\varepsilon})\overline{v}_1 \end{bmatrix} = \begin{bmatrix} \overline{\varepsilon} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ \overline{v}_1 \end{bmatrix} = x' \begin{bmatrix} 0 \\ \overline{v}_1 \end{bmatrix} = x_5 \begin{bmatrix} 0 \\ \overline{v}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{1} \end{bmatrix} \begin{bmatrix} 0 \\ \overline{v}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{p}(\overline{v}_1) \end{bmatrix} \in T_1 \cdot T_2.$$

We conclude that $\overline{p}(\overline{v}_1) = \overline{\alpha}_1(\overline{\varepsilon})\overline{v}_1 \in \overline{L}_1^2$ and $T_2^2 = 0$.

Combine Lemma 4.14 (a), together with the hypothesis $xT_2 = xT_3 = 0$ and the fact $T_2^2 = 0$, in order to see that $T_\bullet = S_\bullet \ltimes W$ for some trivial $S_\bullet-$module W where

$$S_\bullet = \begin{cases} \mathbf{F}[b] \text{ with } 0 \le b, & \text{if (4.9) is described by (4.10), and} \\ \mathbf{F}^{(j)} \text{ with } 2 \le j \le 4, & \text{if (4.9) is described by (4.11).} \end{cases}$$

The proof of Theorem 1.5 in case 3A is complete.

The remaining case (case 3B) is the most interesting case. In Lemma 4.17 we record the consequences in $T_\bullet$ of the observation that the multiplication $(\operatorname{im}\overline{\alpha}_1 \cdot \mathbb{L})$ must be described by part (iv) of Corollary 2.7. This result gives many incomplete multiplication tables for $T_\bullet$. In Lemma 4.18 we paste the incomplete multiplication tables of Lemma 4.17 together to learn all of the multiplication in $T_\bullet$, except the multiplication T_2^2. The proof of Lemma 4.20 is where the hard work takes place in case 3B with $T_2^2 \neq 0$.

Lemma 4.17. *Adopt the notation and hypotheses of case 3B. If X_1 is a four dimensional subspace of T_1, then there are elements $y_1 \in T_2$ and $w_1 \in T_4$, and there are subspaces $Y_1 \subseteq T_2$, $Z_1 \subseteq T_3$, and $Z_1' \subseteq T_3$ such that $T_2 = k\,y_1 \oplus Y_1$, $T_3 = Z_1 \oplus Z_1'$, and*

(a) $\dim(y_1 \cdot X_1) = 4$,
(b) $X_1 \cdot Y_1 = 0$,
(c) $X_1 \cdot T_3 \subseteq k\,w_1$,
(d) *the multiplication map $X_1 \otimes Z_1 \to k\,w_1$ is a perfect pairing,*
(e) $X_1 \cdot Z_1' = 0$, *and*
(f) $X_1 \cdot T_2 \subseteq Z_1'$.

Proof. Select a grade four complete intersection K with the property that the image of $\overline{K}$ under (4.15) is X_1. Use K to calculate multiplication in $T_\bullet$ as described in (4.1) and (4.2). Observe that the elements $x_1, \dots, x_4$, which are defined above (4.6), form a basis for X_1. We know from Lemma 4.14 (a) that $X_1 \cdot T_1^2 = 0$; consequently, all of the multiplication in $X_1 \cdot T_2$ and $X_1 \cdot T_3$ is given in (4.9). Recall the hypothesis that if x is a nonzero element of X_1, then either $xT_2 \neq 0$ or $xT_3 \neq 0$. It follows that the multiplication in $X_1 \cdot T_2$ and $X_1 \cdot T_3$ is described by (4.11) with $j = 4$. There is no difficulty seeing that the multiplication of (4.11), with $j = 4$, is the same as the coordinate-free description which is given in the statement of the result. $\square$

Lemma 4.18. *If the notation and hypotheses of case 3B are adopted, then there are elements $y \in T_2$ and $w \in T_4$, and there are subspaces $Y \subseteq T_2$, $Z \subseteq T_3$, and $Z' \subseteq T_3$ such that $T_2 = k\,y \oplus Y$, $T_3 = Z \oplus Z'$, and*

(a) $\dim(y \cdot T_1) = 5$,
(b) $T_1 \cdot Y = 0$,
(c) $T_1 \cdot T_3 \subseteq k\,w$,
(d) *the multiplication map* $T_1 \otimes Z \to k\,w$ *is a perfect pairing,*
(e) $T_1 \cdot Z' = 0$, *and*
(f) $T_1 \cdot T_2 \subseteq Z'$.

Before proving the above result, we notice that Lemmas 4.14 and 4.18 complete the proof in case 3B when $T_2^2 = 0$.

Corollary 4.19. *If the notation and hypotheses of case 3B are adopted and $T_2^2 = 0$, then $T_\bullet$ has the form $\mathbf{F}^{(5)} \ltimes W$ for some trivial $\mathbf{F}^{(5)}-$module W.* □

Proof of Lemma 4.18. Let X_1 and X_2 be four dimensional subspaces of T_1 with $X_1 = X_2$. Apply Lemma 4.17 to find $y_i \in T_2$, $w_i \in T_4$, $Y_i \subseteq T_2$, $Z_i \subseteq T_3$, and $Z_1' \subseteq T_3$ with $\dim(y_i \cdot X_i) = 4$, $X_i \cdot Y_i = 0$, $X_i \cdot T_3 \subseteq k\,w_i$, the multiplication map $X_i \otimes Z_i \to k\,w_i$ a perfect pairing, and $X_i \cdot Z_i' = 0$, for $i = 1$ and $i = 2$. Let $y = y_1$, $w = w_1$, and $Y = Y_1$.

(b) Let x be a nonzero element of $X_1 \cap X_2$ and let

$$(x)^\perp = \{y_0 \in T_2 \mid xy_0 = 0\}.$$

It is clear that $Y_1 = (x)^\perp = Y_2$. Furthermore, we know that $X_1 + X_2 = T_1$; therefore, $Y \cdot T_1 = 0$.

(a) It suffices to show that $\dim(y \cdot X) = 4$ for every four dimensional subspace X of T_1. The choice of X_2 is independent of our definition of y; consequently, it suffices to show that $\dim(y \cdot X_2) = 4$. But, this fact follows from the following observations which we have already established: $k\,y \oplus Y = k\,y_2 \oplus Y$, $\dim(y_2 \cdot X_2) = 4$ and $Y \cdot X_2 = 0$.

(c) Take x from the proof of (b). The hypothesis ensures that $x \cdot T_3$ is a nonzero subspace of $(w_1) \cap (w_2)$. It follows that the one dimensional subspaces (w_1) and (w_2) of T_4 are equal. Use $X_1 + X_2 = T_1$ in order to conclude that $T_1 \cdot T_3 \subseteq (w)$.

(d) and (e) Let φ be the name of the map $T_3 \to \operatorname{Hom}_k(T_1, k\,w)$ which is induced by the multiplication map $T_1 \otimes T_3 \to k\,w$, let $x_1, \dots, x_5$ be a fixed basis for T_1 and let $x_1^*, \dots, x_5^*$ be the corresponding dual basis for $\operatorname{Hom}_k(T_1, k\,w)$. Apply parts (d) and (e) of Lemma 4.17 to the subspace $(x_1, \dots, x_4)$ of T_1 in order to find a basis for T_3 for which the matrix of φ is

$$\left[\begin{array}{c|c} I & 0 \\ \hline \lambda_1 \dots \lambda_4 & \lambda_5 \dots \lambda_n \end{array}\right]$$

for some $\lambda_i \in k$. If $\lambda_5 = \dots = \lambda_n = 0$, then $x \cdot T_3 = 0$ for $x = x_5 - \sum_{i=1}^4 \lambda_i x_i$ and this contradicts Lemma 4.17 (d). Thus, $\lambda_i \neq 0$ for some i with $5 \le i \le n$ and a basis $z_1, \dots, z_n$ for T_3 may be found for which the matrix of φ is $[\,I \quad 0\,]$. Let $Z = (z_1, \dots, z_5)$ and $Z' = (z_6, \dots, z_n)$.

(f) It is immediate from Lemma 4.17 (f) that $T_1 \cdot T_1 \cdot T_2 = 0$; hence, $T_1 \cdot T_2 \subseteq Z'$.
□

Lemma 4.20. *Adopt the notation and hypotheses of case 3B with $T_2^2 \neq 0$. Let K be any complete intersection ideal with $K \subseteq J$ and J/K cyclic, $\mathbb{L}$ be the minimal resolution of $R/(K:J)$ which is shown in (2.1), and $p\colon L_1 \to L_2$ be the map of (3.4). Then there exists an integer b, with $b \geq 6$, and there exists bases $e_1, \dots, e_n$ for L_1; $f_1, \dots, f_{n-1}, f'_1, \dots, f'_{n-1}$ for L_2; $g_1, \dots, g_n$ for L_3; and h for L_4 such that*

(a) $K = (\ell_1(e_1), \dots, \ell_1(e_4))$
(b) *(2.3) holds,*
(c) *all products of basis vectors in $\overline{L}_1 \cdot \overline{L}_1$ and $\overline{L}_1 \cdot \overline{L}_2$ are zero except $e_b e_i = f_i$, $e_i f'_i \equiv g_b$, and $e_b f'_i \equiv -g_i$ for $1 \leq i \leq b-1$, and*
(d) *$p(e_b) \equiv f'_5$ and $p(e_i) \equiv 0$ for all $i \neq 5$.*

Proof. Let h be any generator for L_4. We have two ways to view the multiplication in $T_\bullet$. On the one hand, we can use the multiplication in $\overline{\mathbb{L}}$ to compute $T_\bullet \cdot T_\bullet$ as described in (4.1) and (4.2). On the other hand, Lemma 4.18 gives a complete description of all of the multiplication in $T_\bullet$, except the multiplication $T_2 \cdot T_2$. In the present proof we use the interplay between these two descriptions of $T_\bullet \cdot T_\bullet$ in order to learn about the multiplication in $\overline{\mathbb{L}}$.

Let $e_1, \dots, e_4$ be elements in L_1 with $(\ell_1(e_1), \dots, \ell_1(e_4)) = K$. The hypothesis $t = 4$, ensures that $e_1, \dots, e_4$ is the beginning of a basis for L_1. Let $\varepsilon_1, \dots, \varepsilon_4$ be the basis for K_1 which is defined by $\alpha_1(\varepsilon_i) = e_i$ for $1 \leq i \leq 4$, and let $x_1, \dots, x_5$ be the basis for T_1 which is given above (4.6). According to Lemma 4.18, we may decompose T_2 into $k\,y \oplus Y$ with

$$\dim(y \cdot T_1) = 5 \quad \text{and} \quad T_1 \cdot Y = 0. \tag{4.21}$$

We know from Lemma 4.14 (a) that $T_1^2 \subseteq Y$; consequently,

$$\begin{bmatrix} \overline{\varphi} \\ 0 \end{bmatrix} \in Y \quad \text{and} \quad \begin{bmatrix} 0 \\ \overline{e}_i \end{bmatrix} \in Y$$

for all $\varphi \in K_2$ and for all i with $1 \leq i \leq 4$. It follows that we may modify y in order to assume that

$$y = \begin{bmatrix} 0 \\ \overline{e}_0 \end{bmatrix}$$

for some $e_0 \in L_1$. It also follows that L_1 decomposes into $R\,e_0 \oplus E$ where $(e_1, \dots, e_4) \subseteq E$ and E has the property that

$$\begin{bmatrix} 0 \\ \overline{e} \end{bmatrix} \in Y$$

for all $e \in E$. When the products of (4.21) are interpreted using (4.2), we see that $e_0e_1, \dots, e_0e_4$, $p(e_0)$ is the beginning of a basis for L_2, $(\overline{e}_1, \dots, \overline{e}_4) \cdot \overline{E} = 0$, and $p(e) \equiv 0$ for all $e \in E$.

We next show that $E \cdot E \equiv 0$. We have observed that $\dim \overline{L}_1^{\,2} \geq 4$; consequently, a quick look at Theorem 2.2 shows that the multiplication in $\overline{\mathbb{L}}$ is given in multiplication table (c). In other words, there is a decomposition $\overline{L}_1 = k\,v \oplus V$ with $V^2 = 0$. The fact that $\dim \overline{e}_0 \cdot \overline{L}_1 \geq 4$ ensures that $\overline{e}_0 \notin V$; and therefore, $\overline{L}_1 = k\,\overline{e}_0 \oplus V$. It is easy to select a nonzero element $\overline{e}$ of $(\overline{e}_1, \dots, \overline{e}_4) \cap V$. Indeed, if we write

$\overline{e}_i = \lambda_i \overline{e}_0 + v_i$ with $\lambda_i \in k$ and $v_i \in V$, then either $\lambda_1 = 0$ (in which case we take $\overline{e} = \overline{e}_1$) or $\lambda_1 \neq 0$ (in which case we take $\overline{e} = \lambda_1 \overline{e}_2 - \lambda_2 \overline{e}_1$). Let $\overline{e'} = \lambda \overline{e}_0 + v$ be an arbitrary element of $\overline{E}$. We know that $(\overline{e}_1, \dots, \overline{e}_4) \cdot \overline{E} = 0$, $V^2 = 0$ and $\dim \overline{e}_0(\overline{e}_1, \dots, \overline{e}_4) = 4$. It follows from

$$0 = \overline{e'}\overline{e} = (\lambda \overline{e}_0 + v)\overline{e} = \lambda \overline{e}_0 \overline{e},$$

that $\lambda = 0$; thus $\overline{E} \subseteq V$ and $E \cdot E \equiv 0$.

We may decompose E as $(e_1, \dots, e_4) \oplus E' \oplus E''$, where

$$\dim \overline{e}_0 \left((\overline{e}_1, \dots, \overline{e}_4) \oplus \overline{E'} \right) = \dim \left((\overline{e}_1, \dots, \overline{e}_4) \oplus \overline{E'} \right) \tag{4.22}$$

and $e_0 E'' \equiv 0$. Let $b-1$ denote the dimension of the vector spaces on line (4.22). Rename e_0 by calling it e_b. Pick any basis $e_{b+1}, \cdots, e_n$ for E''.

The hypothesis $T_2^2 \neq 0$ guarantees that there are elements v_1 and v_1' in L_1 with $\overline{v}_1 \overline{p}(\overline{v'}_1) \neq 0$. We have seen that $\overline{L}_1 = k\, \overline{e}_b \oplus \ker \overline{p}$; thus, $\overline{v}_1 \overline{p}(\overline{e}_b)$ is a nonzero element of $\overline{L}_3$ for some $v_1 \in L_1$. The multiplication $\overline{L}_1 \otimes \overline{L}_3 \to \overline{L}_4$ is a perfect pairing; consequently, $\overline{p}(\overline{e}_b) \cdot \overline{L}_1^2 \neq 0$. On the other hand, we have seen that $\overline{L}_1^2 = \overline{e}_b \overline{L}_1$. Thus, $\overline{p}(\overline{e}_b)\overline{e}_b$ is a nonzero element of $\overline{L}_3$. The very last assertion in Theorem 3.5 shows that $\overline{p}(\overline{e}_b)\overline{e}_b(\overline{e}_1, \dots, \overline{e}_4) = 0$. Thus, we may select a basis $e_5, \dots, e_{b-1}$ for E' with $\overline{p}(\overline{e}_b)\overline{e}_b\overline{e}_5 = \overline{h}$ and $\overline{p}(\overline{e}_b)\overline{e}_b\overline{e}_i = 0$ for $6 \leq i \leq b-1$. Select the basis $g_1, \dots, g_n$ for L_3 with the property $e_i g_j = \delta_{ij} h$. Observe that $\overline{p}(\overline{e}_b)\overline{e}_b = -\overline{g}_5$. Label $f_i = e_b e_i$ for $1 \leq i \leq b-1$ and $f_5' = p(e_b)$ in L_2. Observe that $(f_1, \dots, f_{b-1})^2 = 0$ and $f_i f_5' \equiv \delta_{i5} h$ for $1 \leq i \leq b-1$. The proof of Lemma 2.6 (see [21] for details) allows us to extend $f_1, \dots, f_{b-1}, f_5'$ to be a basis $f_1, \dots, f_{b-1}, f_1', \dots, f_{b-1}'$ of L_2 which satisfies (2.3). It is now clear that the basis we have constructed for $\mathbb{L}$ satisfies conditions (a) – (d). $\square$

Corollary 4.23. *If the notation and hypotheses of case 3B are adopted and $T_2^2 \neq 0$, then $T_\bullet$ has the form $\mathbf{F}^\bigstar \ltimes W$ for some trivial $\mathbf{F}^\bigstar$−module W.*

Proof. Let K be any grade four complete intersection with $K \subseteq J$ and J/K cyclic. Let $\mathbb{L}$ be the minimal resolution of $R/(K\colon J)$. Fix a basis for $\mathbb{L}$ as described in Lemma 4.20. Compute multiplication in $T_\bullet$ as described in (4.1) and (4.2). Consider the basis for $T_\bullet$ which is given in (4.6). We know from Lemma 4.14 that $\dim T_1^2 = 10$ and $T_1^3 = 0$; furthermore, the individual products in T_1^2 are given in (4.7). Use (4.2) and Lemma 4.20 to compute that all products in $T_1 \cdot T_2$ are zero except

$$x_i y_{b+6} = -z_{4+i} \text{ for } 1 \leq i \leq 4 \quad \text{and} \quad x_5 y_{b+6} = z_{n+8}.$$

The map $q\colon L_2 \to L_3$ is defined below (3.4). It follows from Lemma 4.20 (d) that $q(f_5) \equiv g_b$, but $q(f_i) \equiv q(f_j') \equiv 0$ for all $i \neq 5$ and all j. It is now clear that all products in $T_1 \cdot T_3$ are zero except

$$x_i z_{n+3+i} = -w_{b-4} \text{ for } 1 \leq i \leq 4 \quad \text{and} \quad x_5 z_9 = -w_{b-4}.$$

Finally, we use (4.2) and Lemma 4.20 (d) once again to see that all products in $T_2 \cdot T_2$ are zero except

$$y_{11} y_{b+6} = w_{b-4} \quad \text{and} \quad y_{b+6} y_{b+6} = -2w_1.$$

Recall that 2 is a unit in k. There is no difficulty in verifying that $T_\bullet = \mathbf{F}^\bigstar \ltimes W$ for some trivial $\mathbf{F}^\bigstar$−module W. $\square$

The proof of Theorem 1.5 is complete.

Section 5. Examples and questions.

We begin this section by commenting on the hypotheses of Theorem 1.5. The hypothesis that k have square roots is used only in the proof of Lemma 2.6 and it is not a particularly annoying hypothesis. Indeed, if $(R', \mathfrak{m}', k')$ is an arbitrary local ring, then the technique of residue field inflation (see, for example, [9, 0_{III} 10.3.1]) yields a faithfully flat extension $(R, \mathfrak{m}, k)$ of R' for which k is closed under the square root operation. Many of the consequences of Theorem 1.5, applied to R, will descend back to R'; however, we do not know if the conclusion of Theorem 1.5 will descend to R'. The hypothesis that 2 is a unit in R is also used only sporadically. There is a very trivial division by 2 at the end of the proof of Corollary 4.23; however, if the characteristic of k had been two, then we would have calculated the second divided power $y^{(2)}$ of each element y of T_2 and in particular, we would have written $y_{b+6}^{(2)} = -w_1$ instead of $y_{b+6}^2 = -2w_1$, thereby avoiding the division by 2. The more serious use of $\operatorname{char} k \neq 2$ occurs when we appeal to Theorem 3.5. The proof of this result in [25] and [26] involves many divisions by 2. We presume (but have not proved) that these divisions can be circumvented.

We next consider the question of the existence of grade four almost complete intersection ideals with predescribed Tor−algebras.

Question 5.1. Let $S_\bullet$ be a graded k−algebra from the list in Theorem 1.5. Does there exist a grade four almost complete intersection ideal J such that

$$\operatorname{Tor}_\bullet^R(R/J, k) \cong S_\bullet \ltimes W \tag{5.2}$$

for some trivial $S_\bullet$−module W?

We are able to answer most of Question 5.1. All of the potential Tor−algebras which are listed in Theorem 2.2 for grade four Gorenstein ideals actually do exist (see [14] for Gorenstein rings whose Tor−algebras are described in Theorem 2.2 (c)); consequently, the proof in cases 1 and 2 (especially Table 4.13) can be read as an algorithm for producing an ideal J for which (5.2) holds, provided $S_\bullet$ is from the list

$$\mathbf{A},\ \mathbf{B}[p],\ \mathbf{C}[p],\ \mathbf{C}^{(2)},\ \mathbf{C}^\star,\ \mathbf{D}[p],\ \mathbf{D}^{(2)},\ \mathbf{E}[p],\ \mathbf{E}(2),\ \text{and}\ \mathbf{E}^{(3)} \quad \text{with } 0 \le p.$$

In Examples 5.6, 5.7, and 5.8 we exhibit ideals J for which (5.2) holds with $S_\bullet =$ $\mathbf{F}[0]$, $\mathbf{F}[1]$, $\mathbf{F}[2]$, $\mathbf{F}[3]$, $\mathbf{F}^{(2)}$, $\mathbf{F}^{(4)}$, and $\mathbf{F}^\star$.

A more complete classification of Tor−algebras remains elusive.

Question 5.3. Let $S_\bullet$ be a graded k−algebra from the list in Theorem 1.5. What are necessary and sufficient conditions on the vector space dimensions $\dim W_i$ in order that (5.2) hold with $W = \bigoplus_{i=0}^4 W_i$ for some grade four almost complete intersection J?

For example, the proof of Theorem 1.5 shows that if (5.2) holds with $S_\bullet = \mathbf{C}^\star$ for some grade four almost complete intersection J, then $W = 0$. (In fact, the entire resolution of R/J is known in this case.) On the other hand, every example that we have considered for which

$$\operatorname{Tor}_\bullet^R(R/J, k) \cong \mathbf{F}^\star \ltimes W, \tag{5.4}$$

also has $W = 0$. We wonder if (5.4) implies that $W = 0$; we also wonder if a structure theorem exists for the minimal resolution of R/J for those J which satisfy (5.4).

Finally, the variable of linkage class should also be thrown into the question about the classification of Tor−algebras. A number of years ago, Matthew Miller and the present author knew many Gorenstein rings of projective dimension four with $T_1^2 = 0$. None of these rings were in the linkage class of a complete intersection (licci). We conjectured, that if A is a licci Gorenstein ring of projective dimension four, then $T_1^2 \neq 0$, and we deduced a number of consequences assuming that the conjecture held. Most of the consequences of the conjecture [20] have since been proved [11]; furthermore, various attempts to gather evidence for the conjecture have netted results which are interesting in their own right [22]. In the meantime, we have shown that the conjecture itself is false. The following question remains unanswered.

Question 5.5. Suppose $T_\bullet$ is the Tor−algebra of some Cohen-Macaulay ring. Does there exist a licci ring A with $T_\bullet(A) \cong T_\bullet$?

Example 5.6. Let $Y_{1\times 5}$ be a generic matrix, $X_{5\times 5}$ be a generic alternating matrix, and R be the local ring $k[X,Y]_{(X,Y)}$. Huneke and Ulrich [10, Proposition 5.8] introduced the grade four almost complete intersection $J = I_1(YX)$. One can compute that $\operatorname{Tor}_\bullet^R(R/J,k) = \mathbf{F}^\star$. The Huneke-Ulrich almost complete intersection ideals are closely related to the Huneke-Ulrich deviation two Gorenstein ideals which have been studied rather extensively; see [15, 13, 28].

Example 5.7. Let $Y_{1\times 4}$ and $X_{4\times 3}$ be generic matrices and v be an indeterminate. Consider the local ring $R = k[X,Y,v]_{(X,Y,v)}$. Let $I = (a_1,\dots,a_7)$ be the grade four Gorenstein ideal with $a_j = \sum_{i=1}^4 y_i x_{ij}$ for $1 \le j \le 3$ and $a_{4+j} = c_j + v y_j$ for $1 \le j \le 4$, where c_j is equal to $(-1)^{j+1}$ times the determinant of X with row j removed. (The ideal I is known as a Herzog ideal; see, for example, [19], [5, Section 3], or [23, Example 7.16].) If $J = (a_1,a_4,a_5,a_6) : I$, then (5.2) holds with $S_\bullet = \mathbf{F}[2]$ and W equal to $k(-2)\oplus k(-3)^8 \oplus k(-4)$. If $J = (y_1a_3 + a_4, a_5, a_6, a_7) : I$, then (5.2) holds with $S_\bullet = \mathbf{F}[0]$ and $W = k(-2)^3 \oplus k(-3)^{12} \oplus k(-4)^3$.

Example 5.8. Let $I = (a_1,\dots,a_9)$ be the grade four Gorenstein ideal defined in [18] with $\tau = 5$, $x_{11} = 1$, and $x_{21} = x_{31} = x_{41} = x_{51} = x_{12} = x_{13} = 0$. If $J = (a_i,a_j,a_k,a_\ell){:}I$, then

$$\operatorname{Tor}_\bullet^R(R/J,k) = \begin{cases} \mathbf{F}[1] \ltimes \left(k(-2)^3 \oplus k(-3)^{12} \oplus k(-4)^3\right), & \text{if } \{i,j,k,\ell\} = \{3,5,6,7\},\\ \mathbf{F}[3] \ltimes \left(k(-2)^1 \oplus k(-3)^{8} \oplus k(-4)^1\right), & \text{if } \{i,j,k,\ell\} = \{1,5,6,7\},\\ \mathbf{F}^{(2)} \ltimes \left(k(-2)^3 \oplus k(-3)^{10} \oplus k(-4)^3\right), & \text{if } \{i,j,k,\ell\} = \{3,5,6,9\}, \text{ and}\\ \mathbf{F}^{(4)} \ltimes \left(k(-2)^3 \oplus k(-3)^{6} \oplus k(-4)^3\right), & \text{if } \{i,j,k,\ell\} = \{2,3,6,9\}.\end{cases}$$

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