THE RESOLUTION OF THE UNIVERSAL RING FOR FINITE LENGTH MODULES OF PROJECTIVE DIMENSION TWO

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Abstract. Hochster established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution $\mathbb{U}$ of the form $0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g \to 0$ such that for any commutative noetherian ring S and any resolution $\mathbb{V}$ equal to $0 \to S^e \to S^f \to S^g \to 0$, there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$. In the present paper we assume that $f = e + g$ and we find a resolution of $\mathcal{R}$ by free $\mathcal{P}$-modules, where $\mathcal{P}$ is a polynomial ring over the ring of integers. For small values of e and g our resolution is a minimal resolution of $\mathcal{R}$. For e and g both at least 5, we prove that $\mathcal{R}$ does not possess a generic minimal resolution.

Introduction.

Fix positive integers e, f, and g, with $r_1 \geq 1$ and $r_0 \geq 0$, for r_1 and r_0 defined to be $f - e$ and $g - f + e$, respectively. Hochster [15, Theorem 7.2] established the existence of a commutative noetherian ring $\mathcal{R}$ and a universal resolution

$$\mathbb{U}: \quad 0 \to \mathcal{R}^e \to \mathcal{R}^f \to \mathcal{R}^g \to 0$$

such that for any commutative noetherian ring S and any resolution

$$\mathbb{V}: \quad 0 \to S^e \to S^f \to S^g \to 0,$$

there exists a unique ring homomorphism $\mathcal{R} \to S$ with $\mathbb{V} = \mathbb{U} \otimes_{\mathcal{R}} S$. Hochster showed that the universal ring $\mathcal{R}$ is integrally closed and finitely generated as an algebra over $\mathbb{Z}$. Huneke [16] identified the generators of $\mathcal{R}$ as a $\mathbb{Z}$-algebra. These generators correspond to the entries of the two matrices from $\mathbb{U}$ and the $\binom{g}{r_1}$ multipliers from the factorization theorem of Buchsbaum and Eisenbud [8, Theorem 3.1]. Bruns [3] showed that $\mathcal{R}$ is factorial. Bruns [4] also showed that universal resolutions exist only for resolutions of length at most two. Heitmann [14] used Bruns' approach to universal resolutions in in his counterexample to the rigidity conjecture. Pragacz and Weyman [23] found the relations on the generators of $\mathcal{R}$ and used Hodge algebra techniques to prove that $k \otimes_{\mathbb{Z}} \mathcal{R}$ has rational singularities when k is a field of characteristic zero. Tchernev [25] used the theory of Gröbner bases to generalize and extend all of the above results with special interest in allowing an arbitrary base ring R_0. In particular, his method yields the following results.

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(a) The ring $R_0 \otimes_{\mathbb{Z}} \mathcal{R}$ is factorial, or Cohen-Macaulay, or Gorenstein if and only if the base ring R_0 has the same property.
(b) The ring $R_0 \otimes_{\mathbb{Z}} \mathcal{R}$ is regular if and only if the base ring R_0 is regular and $r_1 = 1$.
(c) If R_0 is a perfect field of positive characteristic, then $R_0 \otimes_{\mathbb{Z}} \mathcal{R}$ is F-regular.

When r_1 is equal to 1, then the universal ring $\mathcal{R}$ is the polynomial ring over $\mathbb{Z}$ with variables which represent entries of the second matrix from $\mathbb{U}$ together with variables which represent the Buchsbaum-Eisenbud multipliers. In particular, when $g = r_1 = 1$, then the Hilbert-Burch theorem, which classifies all resolutions of the form

$$0 \to \mathcal{R}^{f-1} \to \mathcal{R}^f \to \mathcal{R}^1 \to 0,$$

is recovered. When $e = 1$ and $r_0 = 0$, then the universal resolution looks like

$$0 \to \mathcal{R}^1 \to \mathcal{R}^f \to \mathcal{R}^{f-1} \to 0,$$

and the universal ring $\mathcal{R}$ is defined by the generic Herzog ideal of grade f [1, Def. 3.4] in the polynomial ring $\mathcal{P} = \mathbb{Z}[\{v_j\}, \{x_{ij}\}, \mathcal{B}]$, where $1 \le i \le f-1$ and $1 \le j \le f$. The minimal resolution of $\mathcal{R}$ by free $\mathcal{P}$-modules is given in [22]. Further discussion of this universal ring including its linkage history and a study of the rationality of the Poincaré series of $\mathcal{R}$-modules may be found in [1].

The present paper concerns the universal ring $\mathcal{R}$ when $r_0 = 0$. In this case, $f = e + g$ and $\mathcal{R} = \mathcal{P}/\mathcal{J}$, for $\mathcal{P}$ equal to the polynomial ring $\mathbb{Z}[\mathcal{B}, \{v_{jk}\}, \{x_{ij}\}]$, with $1 \le k \le e$, $1 \le j \le f$, and $1 \le i \le g$, where $\{\mathcal{B}\} \cup \{v_{jk}\} \cup \{x_{ij}\}$ is a list of indeterminates over $\mathbb{Z}$. We give $\mathcal{J}$ in the language of [25]. Let $\mathcal{V}$ be the $f \times e$ matrix and $\mathcal{X}$ be the $g \times f$ matrix with entries v_{jk} and x_{ij}, respectively. For each

$$\text{partition of } \{1, \dots, f\} \text{ into } A \cup \bar{A} \text{ with } |A| = e \text{ and } |\bar{A}| = g, \tag{0.1}$$

let $\nabla_{\bar{A},A}$ be the sign of the permutation which arranges the elements of $\bar{A}, A$ into increasing order, $\mathcal{V}(A)$ the submatrix of $\mathcal{V}$ consisting of the rows from A, and $\mathcal{X}(\bar{A})$ the submatrix of $\mathcal{X}$ consisting of the columns from $\bar{A}$. In this notation, the ideal which defines the universal ring $\mathcal{R}$ is

$$\mathcal{J} = I_1(\mathcal{X}\mathcal{V}) + (\{\det \mathcal{X}(\bar{A}) + \nabla_{\bar{A},A}\mathcal{B} \det \mathcal{V}(A) \mid \ A \cup \bar{A} \text{ from (0.1)}\}). \tag{0.2}$$

We produce four resolutions, $\mathbb{F}$, $\mathbb{G}$, $\mathbb{I}$, and $\mathbb{J}$, of $\mathcal{R}$ by free $\mathcal{P}$-modules. Let E, F, and G be free $\mathcal{P}$ modules of rank e, $f = e + g$, and g, respectively; and view the matrices $\mathcal{V}$ and $\mathcal{X}$ as homomorphisms of $\mathcal{P}$-modules:

$$E \xrightarrow{\mathcal{V}} F \xrightarrow{\mathcal{X}} G.$$

The resolution $\mathbb{F}$ is given in Definition 2.3. It is infinite, but fairly straightforward and coordinate free. One can view $\mathbb{F}$ as the mapping cone of the following map of complexes:

$$\begin{array}{c} \displaystyle\bigoplus_{\substack{(a,b,c,d)\\ b=a-c+e}} D_aE \otimes \textstyle\bigwedge^b F^* \otimes D_cG^* \otimes \bigwedge^d (E \otimes G^*) \\ \big\downarrow \\ \textstyle\bigwedge^\bullet (E \otimes G^*). \end{array} \tag{0.3}$$

The map $\mathcal{X}\circ\mathcal{V}\colon E\to G$ induces a map $E\otimes G^*\to R$ which gives rise to an ordinary Koszul complex

$$\cdots\to\bigwedge^d(E\otimes G^*)\to\bigwedge^{d-1}(E\otimes G^*)\to\dots\ . \tag{0.4}$$

Also, the map $\mathcal{X}\colon F\to G$ gives rise to the Koszul complex

$$\cdots\to S_cG\otimes\bigwedge^b F\to S_{c+1}G\otimes\bigwedge^{b-1}F\to\dots$$

and its dual

$$\cdots\to D_{c+1}G^*\otimes\bigwedge^{b-1}F^*\to D_cG^*\otimes\bigwedge^b F^*\to\dots; \tag{0.5}$$

the map $\mathcal{V}^*\colon F^*\to E^*$ gives rise to the Koszul complex

$$\cdots\to S_aE^*\otimes\bigwedge^b F^*\to S_{a+1}E^*\otimes\bigwedge^{b-1}F^*\to\dots$$

and its dual

$$\cdots\to D_{a+1}E\otimes\bigwedge^{f-b+1}F^*\to D_aE\otimes\bigwedge^{f-b}F^*\to\dots; \tag{0.6}$$

and the identity map on $E^*\otimes G$ gives rise to the Koszul complex

$$\cdots\to S_aE^*\otimes S_cG\otimes\bigwedge^d(E^*\otimes G)\to S_{a+1}E^*\otimes S_{c+1}G\otimes\bigwedge^{d-1}(E^*\otimes G)\to\dots$$

and its dual

$$\cdots\to D_{a+1}E\otimes D_{c+1}G^*\otimes\bigwedge^{d-1}(E\otimes G^*)\to D_aE\otimes D_cG^*\otimes\bigwedge^d(E\otimes G^*)\to\dots\ . \tag{0.7}$$

The bottom complex of (0.3) is the Koszul complex (0.4). The differential on the top complex of (0.3) involves the maps from (0.4), (0.5), (0.6), and (0.7). The map from the top complex to the bottom complex involves the minors of $\mathcal{X}$ and $\mathcal{B}$ times minors of $\mathcal{V}$. The proof, in section 3, that $\mathbb{F}$ is a resolution of $\mathcal{R}$ uses the acyclicity lemma and induction on e. When $e=1$, $\mathcal{R}$ is defined by a Herzog ideal and the resolution of $\mathcal{R}$ is already known.

The complexes $\mathbb{G}$, $\mathbb{I}$, and $\mathbb{J}$ are obtained, in section 4, by splitting off summands of $\mathbb{F}$. The splitting is induced by (0.7), as well as by an occasional appearance of $\bigwedge^0\mathcal{X}$ in the map between the two complexes in (0.3). The homology of the complex (0.7) is quite interesting and is studied in [19, 20]. The complexes $\mathbb{G}$, $\mathbb{I}$, and $\mathbb{J}$ all have length $eg+1$, which is the projective dimension of $\mathcal{R}$ as well as the grade of the defining ideal $\mathcal{J}$. The complex $\mathbb{I}$ is smaller than $\mathbb{G}$, and $\mathbb{J}$ is smaller than $\mathbb{I}$. For small values of e and g, $\mathbb{J}$ is the minimal resolution of $\mathcal{R}$. For all values of e and g, the beginning and the end of $\mathbb{J}$ agree with the beginning and the end of the minimal resolution of $\mathcal{R}$. Examples of $\mathbb{J}$ are given in section 7. Section 5 contains our proof that $\mathcal{J}$ is a generically perfect ideal of grade $eg+1$.

In Section 6 we show that if e and g are both at least 5, then $\mathcal{R}$ does not possess a generic minimal resolution; that is, there does not exist a resolution $\mathbb{X}$ of $\mathcal{R}$ by free $\mathcal{P}$-modules with the property that $\mathbb{X}\otimes_{\mathbb{Z}}k$ is the minimal resolution of $\mathcal{R}\otimes_{\mathbb{Z}}k$ by free $\mathcal{P}\otimes_{\mathbb{Z}}k$ modules for all fields k. The complex $\mathbb{F}$ is very large, but it is easy to use $\mathbb{F}$ to compute that $\operatorname{Tor}^{\mathcal{P}}_\bullet(\mathcal{R},\mathbb{Z})$ is the direct sum of the homology of complexes of the form of (0.7). The homology of these complexes is not a free abelian group; hence, $\operatorname{Tor}^{\mathcal{P}}_\bullet(\mathcal{R},\mathbb{Z})$ is not a free abelian group. The conclusion follows from a result of Roberts [24]; see, also, Hashimoto [11].

Section 8 consists of a proof that the last map in $\mathbb{J}$ is the dual of the first map in $\mathbb{J}$. We expect this to be the case since $\mathbb{J}$ is a resolution of a Gorenstein ring.

1. Preliminary results.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. An R-module M is called *perfect* if the grade of the annihilator of M is equal to the projective dimension of M. The ideal I of R is called *perfect* if R/I is a perfect R-module. An excellent reference on perfect modules is [6, Sect. 16C]. A finitely generated module M over the polynomial ring $\mathbb{Z}[X_1,\dots,X_n]$ is called *generically perfect* if M is perfect and faithfully flat as a $\mathbb{Z}$-module. The grade g perfect ideal I is called *Gorenstein* if $\operatorname{Ext}^g_R(R/I,R)\cong R/I$. For any R-module F, we let F^* denote $\operatorname{Hom}_R(F,R)$.

Let F be a free R-module of finite rank. We make much use of the exterior algebra $\bigwedge^\bullet F$ and the divided power algebra $D_\bullet F$; we make some use of the tensor algebra $T_\bullet F$. In particular, $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$ are modules over one another. Indeed, if $\alpha_i\in\bigwedge^i F^*$ and $b_j\in\bigwedge^j F$, then

$$\alpha_i(b_j)\in\bigwedge\nolimits^{j-i}F\quad\text{and}\quad b_j(\alpha_i)\in\bigwedge\nolimits^{i-j}F^*. \tag{1.1}$$

(We view $\bigwedge^i F$ and D_iF to be meaningful for every integer i; in particular, these modules are zero whenever i is negative.) The exterior and divided power algebras A come equipped with co-multiplication $\Delta\colon A\to A\otimes A$. The following facts are well known; see [9, section 1], [10, Appendix], and [17, section 1].

Proposition 1.2. *Let F be a free module of rank f over a commutative noetherian ring R and let $b_r\in\bigwedge^r F$, $b'_p\in\bigwedge^p F$, and $\alpha_q\in\bigwedge^q F^*$.*

(a) *If $p=f$, then $[b_r(\alpha_q)](b'_p)=b_r\wedge\alpha_q(b'_p)$.*

(b) *If $X\colon F\to G$ is a homomorphism of free $R-$modules and $\gamma_{s+r}\in\bigwedge^{s+r}G^*$, then* $(\bigwedge^s X^*)\left[((\bigwedge^r X)(b_r))\,(\gamma_{s+r})\right]=b_r\left[\left(\bigwedge^{s+r}X^*\right)(\gamma_{s+r})\right]$.

Notation 1.3. Let E and G be R-modules. If $A_k\in D_kE$ and $\gamma_k\in\bigwedge^k G^*$, then we write $A_k\bowtie\gamma_k$ for the image of $A_k\otimes\gamma_k$ under the natural map

$$D_kE\otimes\bigwedge\nolimits^k G^*\to\bigwedge\nolimits^k(E\otimes G^*). \tag{1.4}$$

In a similar manner, if $\varepsilon_k\in\bigwedge^k E$ and $B_k\in D_kG^*$, then we write $\varepsilon_k\bowtie B_k$ for the image $\varepsilon_k\otimes B_k$ under the natural map $\bigwedge^k E\otimes D_kG^*\to\bigwedge^k(E\otimes G^*)$.

Remark. To understand (1.4), first, consider the composition

$$D_kE\otimes T_kG^*\xrightarrow{\Delta\otimes 1}T_kE\otimes T_kG^*\xrightarrow{\psi}\bigwedge\nolimits^k(E\otimes G^*),$$

where

$$\psi\left((\varepsilon_1^{[1]}\otimes\dots\otimes\varepsilon_1^{[k]})\otimes(\gamma_1^{[1]}\otimes\dots\otimes\gamma_1^{[k]})\right)=(\varepsilon_1^{[1]}\otimes\gamma_1^{[1]})\wedge\dots\wedge(\varepsilon_1^{[k]}\otimes\gamma_1^{[k]}).$$

Observe that the composition factors through $D_kE\otimes\bigwedge^k G^*$.

Indexing Convention 1.5. In sections 1 through 7, when we refer to an element of an exterior algebra or a divided power algebra, the subscript of the element usually tells the component of the element. For example, γ_k is in $\bigwedge^k G^*$. If we have many elements of the same component, we name the elements by using square brackets in a superscript. For example, we would say, "Consider $\gamma_1^{[1]}, \gamma_1^{[2]}, \dots, \gamma_1^{[k]}$ in G^*." Round brackets in the superscript mean divided power; that is, if $\varepsilon_1 \in E$, then $\varepsilon_1^{(k)}$ is the k^{th} divided power of ε_1 in $D_k E$.

Definition 1.6. If $Y\colon E \to G$ is a homomorphism of free R-modules of finite rank, then let $\check{Y}$ be the element of $(E \otimes G^*)^*$ which corresponds to Y under the adjoint isomorphism. In other words, $\check{Y}(\varepsilon \otimes \gamma) = [Y(\varepsilon)](\gamma)$. In light of (1.1), we view $\check{Y}$ as a differential on the exterior algebra $\bigwedge^\bullet(E \otimes G^*)$.

Remark. If one thinks of Y as a matrix and takes ε and γ to be basis elements of E and G^*, respectively, then $\check{Y}(\varepsilon \otimes \gamma)$ is the corresponding entry of Y. The differential graded algebra $\left(\bigwedge^\bullet(E \otimes G^*), \check{Y}\right)$ is the "Koszul complex" associated to the entries of a matrix representation of Y.

Observation 1.7. *If $Y\colon E \to G$, then*

$$\check{Y}(\varepsilon_k \bowtie \gamma_1^{(k)}) = \Big([Y^*(\gamma_1)](\varepsilon_k)\Big) \bowtie \gamma_1^{(k-1)}$$

for all $\varepsilon_k \in \bigwedge^k E$ and $\gamma_1 \in G^*$.

Lemma 1.8. *Suppose R is a polynomial ring over the ring of integers, E and M are free $R-$modules, and $\varphi\colon D_r E \to M$ is an $R-$module homomorphism. If $\varphi(\varepsilon_1^{(r)}) = 0$ for all $\varepsilon_1 \in E$, then φ is identically zero.*

Remarks. In the above lemma, if $R \to \bar{R}$ is any base change, then $\varphi \otimes 1_{\bar{R}}$ is also identically zero. On the other hand, the lemma would be false if R were allowed to have torsion. Indeed, if $R = \mathbb{Z}/(2)$, $E = Rx \oplus Ry$ has rank 2, and M has rank 1, then $\varphi\colon D_3 E \to R$, given by $\varphi(x^{(3)}) = \varphi(y^{(3)}) = 0$ and $\varphi(xy^{(2)}) = \varphi(x^{(2)}x) = 1$, defines an R-module homomorphism with $\varphi(\varepsilon_1^{(3)}) = 0$ for all $\varepsilon_1 \in E$, but φ is not identically zero.

Proof. Every element of $D_r E$ is a linear combination of elements of the form

$$\varepsilon_1^{(a_1)} \cdots \varepsilon_s^{(a_s)} \tag{1.9}$$

for some positive integers s, and $a_1, \dots, a_s$, with $a_1 + \dots + a_s = r$, and elements $\varepsilon_1, \dots, \varepsilon_s$ in E. (In this argument, we do not follow Indexing Convention 1.5.) We show that $D_r(E) \subseteq \ker \varphi$ by induction on s. The case $s = 1$ is the original hypothesis. Suppose that all elements of the form (1.9) are in $\ker \varphi$ for some s. Fix the element $Y = \varepsilon_1^{(a_1)} \cdots \varepsilon_s^{(a_s)} \varepsilon_{s+1}^{(a_{s+1})}$ of $D_r E$. Let $a = a_s + a_{s+1}$, and X be the element $\varepsilon_1^{(a_1)} \cdots \varepsilon_{s-1}^{(a_{s-1})}$ of $D_{r-a} E$. The induction hypothesis ensures that for each integer n, $X(\varepsilon_s + n\varepsilon_{s+1})^{(a)}$ is in $\ker \varphi$. We see that $X(\varepsilon_s + n\varepsilon_{s+1})^{(a)}$ is equal to the product

$$[1 \quad n \quad n^2 \quad \dots \quad n^a] \begin{bmatrix} X\varepsilon_s^{(a)}\varepsilon_{s+1}^{(0)} \\ X\varepsilon_s^{(a-1)}\varepsilon_{s+1}^{(1)} \\ X\varepsilon_s^{(a-2)}\varepsilon_{s+1}^{(2)} \\ \vdots \\ X\varepsilon_s^{(0)}\varepsilon_{s+1}^{(a)} \end{bmatrix}.$$

The row vector in the above product is a row from a Vandermonde matrix. A matrix argument produces a non-zero integer N so that $NX\varepsilon_s^{(a-i)}\varepsilon_{s+1}^{(i)} \in \ker\varphi$, for all i, with $0 \le i \le a$. It follows that $N\varphi(Y) = 0$ in the free abelian group M; so, $\varphi(Y) = 0$. $\square$

Lemma 1.10. *Let $(\mathbb{E}, \boldsymbol{d})$ be a complex of finitely generated projective R-modules, and $\mathfrak{P}$ be a partially ordered set. Suppose that, as a graded module, $\mathbb{E} = \bigoplus_{p\in\mathfrak{P}} \mathbb{E}^{[p]}$ and that the differential $\boldsymbol{d}$ is monotone with respect to the $\mathfrak{P}$ grading. Let $\boldsymbol{\partial}$ be the homogeneous part of $\boldsymbol{d}$ of degree zero with respect to $\mathfrak{P}$. Fix an integer j.*

(a) *If $\mathrm{H}_j(\mathbb{E}, \boldsymbol{\partial}) = 0$, then $\mathrm{H}_j(\mathbb{E}, \boldsymbol{d}) = 0$.*

(b) *If $\mathrm{H}_j(\mathbb{E}, \boldsymbol{\partial}) = 0$ and $\operatorname{im}\boldsymbol{\partial}_j$ is a summand of $\mathbb{E}_{j-1}$, then $\operatorname{im}\boldsymbol{d}_j$ is also a summand of $\mathbb{E}_{j-1}$.*

Remark. In the above statement, "the differential $\boldsymbol{d}$ is monotone" means that, for each integer i, $\boldsymbol{d}(\mathbb{E}_i^{[p]})$ is contained in $\sum \mathbb{E}_{i-1}^{[\pi]}$. The sum is taken over all $\pi \in \mathfrak{P}$, with $\pi \le p$ if $\boldsymbol{d}$ is monotone non-increasing; and the sum is taken over all $\pi \in \mathfrak{P}$, with $p \le \pi$ if $\boldsymbol{d}$ is monotone non-decreasing. The map $\boldsymbol{\partial}_i \colon \mathbb{E}_i^{[p]} \to \mathbb{E}_{i-1}^{[p]}$ is defined to be the composition

$$\mathbb{E}_i^{[p]} \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\text{proj}} \mathbb{E}_{i-1}^{[p]}.$$

The hypothesis that $\boldsymbol{d}$ is monotone ensures that $(\mathbb{E}, \boldsymbol{\partial})$ is a complex.

Proof. For the sake of concreteness, we assume that $\boldsymbol{d}$ is non-increasing. Let x be a non-zero j-cycle of $\mathbb{E}$. Consider $x = \sum x^{[p]}$, with $x^{[p]} \in \mathbb{E}^{[p]}$, and let

$$U(x) = \{p \in \mathfrak{P} \mid x^{[\pi]} = 0 \text{ for all } \pi \in \mathfrak{P} \text{ with } p \le \pi\}.$$

Let p_0 be a maximal element of the support of x. It is clear that $\boldsymbol{\partial}(x^{[p_0]}) = 0$. It follows that there exists $y \in \mathbb{E}^{[p_0]}$ with $\boldsymbol{\partial}(y) = x^{[p_0]}$. We see that $U(x) \subsetneq U(x - \boldsymbol{d}y)$. The proof of (a) is completed by induction. We prove (b). Let $\mathbb{E}'_{j-1}$ be a direct sum complement of $\operatorname{im}\boldsymbol{\partial}_j$ in $\mathbb{E}_{j-1}$. Assertion (a) may be applied to

$$\overline{\mathbb{E}}: \quad \mathbb{E}_{j+1} \to \mathbb{E}_j \to \frac{\mathbb{E}_{j-1}}{\mathbb{E}'_{j-1}} \to 0.$$

We are given that $(\overline{\mathbb{E}}, \boldsymbol{\partial})$ is exact. We conclude that $(\overline{\mathbb{E}}, \boldsymbol{d})$ is exact. It follows readily that $\mathbb{E}_{j-1} = \operatorname{im}\boldsymbol{d}_j \oplus \mathbb{E}'_{j-1}$. $\square$

Terminology 1.11. Let $(\mathbb{E}, \boldsymbol{d})$ be a complex. Suppose that $\mathbb{E}_i = \mathbb{K}_i \oplus \mathbb{L}_i$ for each module $\mathbb{E}_i$ of $\mathbb{E}$. If $\boldsymbol{k}_i \colon \mathbb{K}_i \to \mathbb{K}_{i-1}$ is defined by

$$\mathbb{K}_i \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\text{proj}} \mathbb{K}_{i-1},$$

then we refer to

$$\dots \to \mathbb{K}_i \xrightarrow{\boldsymbol{k}_i} \mathbb{K}_{i-1} \to \dots$$

as a *strand* of the complex $(\mathbb{E}, \boldsymbol{d})$. In particular, for each p in the poset $\mathfrak{P}$ of Lemma 1.10, we refer to the subcomplex $(\mathbb{E}^{[p]}, \boldsymbol{\partial})$ of $(\mathbb{E}, \boldsymbol{\partial})$ as a *homogeneous strand* of $(\mathbb{E}, \boldsymbol{d})$.

Convention 1.12. Usually, we impose the *inverse lexicographic order* on $\mathbb{Z} \times \mathbb{Z}$. In other words, $(P', Q') \le_{\text{il}} (P, Q)$, if $Q' \le Q$, and if $Q' = Q$, then $P' \le P$. Occasionally, we consider the *natural partial order*: $(P', Q') \le_{\text{np}} (P, Q)$, if $P' \le P$ and $Q' \le Q$. It is clear that

$$(P', Q') \le_{\text{np}} (P, Q) \implies (P', Q') \le_{\text{il}} (P, Q).$$

Both of these orders are employed in [2].

Convention 1.13. If F is a free module of rank f, then we orient F by fixing basis elements $\omega_F \in \bigwedge^f F$ and $\omega_{F^*} \in \bigwedge^f F^*$, which are compatible in the sense that $\omega_F(\omega_{F^*}) = 1$.

Convention 1.14. For each statement "S", we write $\chi(S)$ to mean "only if statement S is true". In numerical situations,

$$\chi(\mathrm{S}) = \begin{cases} 1, & \text{if S is true, and} \\ 0, & \text{if S is false.} \end{cases}$$

In particular, $\chi(i = j)$ has the same value as the Kronecker delta δ_{ij}. In a set theoretic situation,

$$A \cup \chi(S) B \text{ means } \begin{cases} A \cup B, & \text{if S is true, and} \\ A, & \text{if S is false,} \end{cases}$$

for sets A and B. In a module situation,

$$A \oplus \chi(S) B \text{ means } \begin{cases} A \oplus B, & \text{if S is true, and} \\ A, & \text{if S is false,} \end{cases}$$

for modules A and B.

Terminology 1.15. The complex $\mathbb{E}$ is *acyclic* if $\mathbb{E}_i = 0$ for all $i < 0$ and the homology $\mathrm{H}_i(\mathbb{E}) = 0$ for all $i > 0$. The complex $\mathbb{E}$ is a *free resolution* of the module M if $\mathbb{E}$ is an acyclic complex of free modules and $\mathrm{H}_0(\mathbb{E}) = M$.

2. The complex $\mathbb{F}$.

Data 2.1. Let R be a commutative noetherian ring and let e, f, and g be positive integers which satisfy $f = e + g$. The complex $\mathbb{F}$ is built from data $(\mathfrak{b}, V, X)$ where $\mathfrak{b}$ is an element of R, and V and X are R-module homomorphisms:

$$E \xrightarrow{V} F \xrightarrow{X} G,$$

with E, F and G free R-modules of rank e, f, and g, respectively. For integers a, b, c, and d, define

$$A(a,b,c,d) = D_aE \otimes \bigwedge^b F^* \otimes D_cG^* \otimes \bigwedge^d (E \otimes G^*)$$

and $B(d) = \bigwedge^d (E \otimes G^*)$.

Remark. If $V = [v_{jk}]$ and $X = [x_{ij}]$ are matrices, with $1 \le k \le e$, $1 \le j \le f$, and $1 \le i \le g$, and R is the polynomial ring $R_0[\{\mathfrak{b}\} \cup \{v_{jk}\} \cup \{x_{ij}\}]$, where $\{\mathfrak{b}\} \cup \{v_{jk}\} \cup \{x_{ij}\}$ is a list of indeterminates over a commutative noetherian ring R_0, then we say that the data of 2.1 is generic.

Convention 2.2. Suppose that the ring R is graded, the element $\mathfrak{b}$ is homogeneous, the homomorphism $X\colon F \to G$ is homogeneous of degree d_x, and the homomorphism $V\colon E \to F$ is homogeneous of degree d_v. If these degrees satisfy $\deg \mathfrak{b} = gd_x - ed_v$, then we say that the data $(\mathfrak{b}, V, X)$ satisfies the grading convention.

Remark. Tchernev takes $(\deg \mathfrak{b}, d_v, d_x) = (g, g, e+1)$; and thereby satisfies the grading convention.

Definition 2.3. Let $(\mathfrak{b}, V, X)$ be the data of 2.1. For each integer i, the module $\mathbb{F}_i$ in the complex $(\mathbb{F}, \boldsymbol{d})$ is

$$\mathbb{F}_i = B(i) \oplus \bigoplus_{(a,b,c,d)} A(a,b,c,d),$$

where the parameters satisfy $i = a + c + d + 1$ and $b = a - c + e$. The differential $\boldsymbol{d}\colon \mathbb{F}_i \to \mathbb{F}_{i-1}$ is defined as follows. If $z_d \in B(d)$, then

$$\boldsymbol{d}(z_d) = (X \check{\circ} V)(z_d) \in B(d-1), \text{ and}$$

if $x = \varepsilon_1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes z_d \in A(a,b,c,d)$, then $\boldsymbol{d}(x)$ is equal to

$$\begin{cases} \varepsilon_1^{(a-1)} \otimes [V(\varepsilon_1)](\alpha_b) \otimes \gamma_1^{(c)} \otimes z_d \in A(a-1,b-1,c,d) \\ -\varepsilon_1^{(a)} \otimes X^*(\gamma_1) \wedge \alpha_b \otimes \gamma_1^{(c-1)} \otimes z_d \in A(a,b+1,c-1,d) \\ +(-1)^{a+c}\varepsilon_1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes (X \check{\circ} V)(z_d) \in A(a,b,c,d-1) \\ +(-1)^{a+c}\varepsilon_1^{(a-1)} \otimes \alpha_b \otimes \gamma_1^{(c-1)} \otimes (\varepsilon_1 \otimes \gamma_1) \wedge z_d \in A(a-1,b,c-1,d+1) \\ +(-1)^{a+d}\chi(c=0)\varepsilon_1^{(a)} \bowtie \left[(\bigwedge^{f-b} X)(\alpha_b[\omega_F])\right] (\omega_{G^*}) \wedge z_d \in B(a+d) \\ +(-1)^d\chi(a=0)\mathfrak{b} \cdot [(\bigwedge^b V^*)(\alpha_b)](\omega_E) \bowtie \gamma_1^{(c)} \wedge z_d \in B(c+d). \end{cases}$$

Remarks. (a) When we want to emphasize the data $(\mathfrak{b}, V, X)$ which was used in the construction of $\mathbb{F}$, we write $\mathbb{F}(\mathfrak{b}, V, X)$.
(b) Our notation is explained in 1.3, 1.6, 1.13, and 1.14. The map from

$$A(a,b,c,d) \to A(a-1,b-1,c,d)$$

is the composition

$$\begin{tikzcd} D_aE \otimes \bigwedge^b F^* \otimes M \arrow[r, "\Delta\otimes 1\otimes 1"] & D_{a-1}E \otimes E \otimes \bigwedge^b F^* \otimes M \arrow[r, "1\otimes V\otimes 1\otimes 1"] & D_{a-1}E \otimes F \otimes \bigwedge^b F^* \otimes M \arrow[d, "1\otimes(1.1)\otimes 1"'] \\ & & D_{a-1}E \otimes \bigwedge^{b-1} F^* \otimes M, \end{tikzcd}$$

where $M = D_cG^* \otimes \bigwedge^d(E \otimes G^*)$. The maps from $A(a,b,c,d)$ to $A(a,b+1,c-1,d)$ and $A(a-1,b,c-1,d+1)$ also involve co-multiplication in a divided power algebra.

Remark 2.4. If the data of 2.1 satisfies the grading convention of 2.2, then the complex $\mathbb{F}$ is homogeneous provided
(a) the shift for $A(a,b,c,d)$ is $(a+d)d_v + (g+c+d)d_x$, and
(b) the shift for $B(d)$ is $dd_v + dd_x$.

Proposition 2.5. *The maps and modules* $(\mathbb{F}, \boldsymbol{d})$ *of Definition 2.3 form a complex*

Proof. In light of Lemma 1.8 it suffices to show that $\boldsymbol{d} \circ \boldsymbol{d}(x) = 0$ for

$$x = \varepsilon_1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes z_d \in A(a,b,c,d).$$

The calculation is routine. We pick out a couple of high points. Use Observation 1.7 to see that the $B(a+d-1)$ component of $\boldsymbol{d} \circ \boldsymbol{d}(x)$ is zero, when $c = 0$, and that the $B(c+d-1)$ component of $\boldsymbol{d} \circ \boldsymbol{d}(x)$ is zero, when $a = 0$. In the $B(c+d)$ component of $\boldsymbol{d} \circ \boldsymbol{d}(x)$, when $a = 1$, we use Proposition 1.2 (b) and (a) to see that

$$\begin{aligned} &[(\textstyle\bigwedge^{b-1} V^*)([V(\varepsilon_1)](\alpha_b))](\omega_E) \bowtie \gamma_1^{(c)} = \left(\varepsilon_1[(\textstyle\bigwedge^b V^*)(\alpha_b)]\right)(\omega_E) \bowtie \gamma_1^{(c)} \\ &= \left(\varepsilon_1 \wedge [(\textstyle\bigwedge^b V^*)(\alpha_b)](\omega_E)\right) \bowtie \gamma_1^{(c)} = (\varepsilon_1 \otimes \gamma_1) \wedge \left([(\textstyle\bigwedge^b V^*)(\alpha_b)](\omega_E) \bowtie \gamma_1^{(c-1)}\right). \end{aligned}$$

The same type of argument gives

$$\left[(\textstyle\bigwedge^{f-b-1} X)([X^*(\gamma_1) \wedge \alpha_b][\omega_F])\right](\omega_{G^*}) = \gamma_1 \wedge \left((\textstyle\bigwedge^{f-b} X)(\alpha_b[\omega_F])\right)(\omega_{G^*}),$$

which is the key to seeing that the $B(a+d)$ component of $\boldsymbol{d} \circ \boldsymbol{d}(x)$ is equal to zero when $c = 1$. $\square$

Definition 2.6. Let $(\mathfrak{b}, V, X)$ be the data of 2.1. Define λ to be the element

$$(-1)^{eg}[(\textstyle\bigwedge^g X^*)(\omega_{G^*})](\omega_F) + \mathfrak{b}(\textstyle\bigwedge^e V)(\omega_E)$$

of $\bigwedge^e F$ and define J to be the image of the map

$$[\,\lambda \quad (X \check{\circ} V)\,] : \textstyle\bigwedge^e F^* \oplus (E \otimes G^*) \to R.$$

Observation 2.7. *If* $(\mathbb{F}, \boldsymbol{d})$ *is the complex of 2.3 and* J *is the ideal of Definition 2.6, then the homology* $\mathrm{H}_0(\mathbb{F})$ *is equal to* R/J*. Furthermore, if* $(\mathbb{F}, \boldsymbol{d})$ *is formed using the polynomial ring* $\mathcal{P}$ *and the data* $(\mathcal{B}, \mathcal{V}, \mathcal{X})$ *of (0.2), then the homology* $\mathrm{H}_0(\mathbb{F})$ *is equal to the universal ring* $\mathcal{R} = \mathcal{P}/\mathcal{J}$*.*

Proof. The beginning of $\mathbb{F}$ is $\mathbb{F}_1 \to \mathbb{F}_0 \to 0$, with

$$\mathbb{F}_1 = A(0,e,0,0) \oplus B(1) = \textstyle\bigwedge^e F^* \oplus (E \otimes G^*) \quad \text{and} \quad \mathbb{F}_0 = B(0) = R.$$

The map $E \otimes G^* \to R$ is $(X \mathbin{\check{\circ}} V)$, and the element $\alpha_e \in \bigwedge^e F^*$ is sent to

$$[(\textstyle\bigwedge^g X)(\alpha_e(\omega_F)]\,(\omega_{G^*}) + \mathfrak{b}\,[(\textstyle\bigwedge^e V^*)(\alpha_e)]\,(\omega_E) = \lambda(\alpha_e) \in R.$$

The first assertion is established. The homomorphisms $\mathcal{X}$ and $\mathcal{V}$, of the second assertion, are represented by matrices. Let α_e be a basis vector in $\bigwedge^e F^*$. The element $[(\bigwedge^e \mathcal{V}^*)(\alpha_e)]\,(\omega_E)$ of R is the determinant of the submatrix of $\mathcal{V}$ determined by the e rows picked out by α_e. The element $\alpha_e(\omega_F)$ of $\bigwedge^g F$ picks out the complementary columns of $\mathcal{X}$ with the correct sign, and $[(\bigwedge^g \mathcal{X})(\alpha_e(\omega_F)]\,(\omega_{G^*})$ is the (signed) determinant of this submatrix of $\mathcal{X}$. $\square$

In Theorem 3.1 we prove that $\mathbb{F}$ is acyclic whenever the data is generic. However, $\mathbb{F}$ is far from a minimal resolution. On the other hand, it is possible to isolate the part of $\mathbb{F}$ in which the splitting occurs. To do this, we partition $\mathbb{F}$ into strands. Our definition of the strands is motivated by Remark 2.4.

Definition 2.8. Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of Definition 2.3 and let P and Q be integers. The module $A(a,b,c,d)$ from $\mathbb{F}$ is in the strand $S(P,Q)$ if $P = a+d$ and $Q = c+d$. The module $B(d)$ is in $S(P,Q)$ if $P = d$ and $Q = d-g$. Let $\boldsymbol{\partial} : S(P,Q) \to S(P,Q)$ be the composition

$$S(P,Q) \xrightarrow{\text{incl}} \mathbb{F} \xrightarrow{\boldsymbol{d}} \mathbb{F} \xrightarrow{\text{proj}} S(P,Q).$$

Observation 2.9.
(a) *If the strand* $S(P,Q)$ *is non-zero, then* $0 \le P$ *and* $-g \le Q - P \le e$*.*
(b) *As a module,* $\mathbb{F} = \bigoplus_{(P,Q)} S(P,Q)$*.*
(c) *The differential* $\boldsymbol{d}$ *is non-increasing with respect to the inverse lexicographic order on* $\{(P,Q)\}$*, see 1.12.*
(d) *Each strand* $S(P,Q)$ *is a complex with differential* $\boldsymbol{\partial}$*.*

Proof. The first assertion holds because if $A(a,b,c,d)$ is a non-zero summand of $S(P,Q)$, then $0 \le b \le f$ and $b = a - c + e = P - Q + e$. Assertion (b) is obvious. If $P = d$ and $Q = d - g$, then $B(d) \subset S(P,Q)$ and $\boldsymbol{d}(B(d)) \subset S(P-1, Q-1)$. If $a + d = P$ and $c + d = Q$, then $A(a,b,c,d) \subset S(P,Q)$ and

$$\boldsymbol{d}(A(a,b,c,d)) \subset \begin{cases} S(P-1,Q) \oplus S(P,Q-1) \oplus S(P-1,Q-1) \oplus S(P,Q) \\ \oplus\, \chi(c=0) S(P, Q+a-g) \oplus \chi(a=0) S(P+c, Q-g). \end{cases}$$

We have already seen that if $c = 0$, then $a - g \le 0$. Thus,

$$\boldsymbol{d}(S(P,Q)) \subset \bigoplus_{(P',Q')} S(P',Q'),$$

as (P',Q') varies over all pairs of integers with $Q' < Q$; or else, $Q' = Q$ and $P' \le P$. Assertion (c) has been established; and (d) follows from (c). $\square$

Definition 2.10. The complex $(\mathbb{F}, \boldsymbol{\partial})$ is defined to be $\bigoplus_{(P,Q)}(S(P,Q), \boldsymbol{\partial})$.

Example 2.11. Fix integers P and Q. Let $b = P - Q + e$. If $P \le Q$, then $S(P,Q)$ is

$$0 \to A(P,b,Q,0) \xrightarrow{\boldsymbol{\partial}} A(P-1,b,Q-1,1) \xrightarrow{\boldsymbol{\partial}} \dots \xrightarrow{\boldsymbol{\partial}} A(0,b,Q-P,P) \to 0.$$

If $Q \le P$, then $S(P,Q)$ is

$$\begin{gathered} 0 \to A(P,b,Q,0) \xrightarrow{\boldsymbol{\partial}} A(P-1,b,Q-1,1) \xrightarrow{\boldsymbol{\partial}} \dots \\ \xrightarrow{\boldsymbol{\partial}} A(P-Q,b,0,Q) \xrightarrow{\boldsymbol{\partial}} \chi(Q = P-g)B(P) \to 0. \end{gathered}$$

In each case, the module $A(a,b,c,d)$ has position $i = a + c + d + 1$ and is denoted by $S_i(P,Q)$. The module $B(P)$ is equal to $S_P(P, P-g)$.

Let $|S(P,Q)|$ denote the absolute value of the Euler characteristic of the strand $S(P,Q)$. We see that $|S(P,Q)|$ is equal to

$$\begin{cases} \left| \sum\limits_{i=0}^{P} (-1)^i \operatorname{rank} A(i, P-Q+e, Q-P+i, P-i) \right| & \text{if } P \le Q, \text{ and} \\ \left| \chi(Q = P-g)\binom{eg}{P} - \sum\limits_{i=0}^{Q} (-1)^i \operatorname{rank} A(P-Q+i, P-Q+e, i, Q-i) \right| & \text{if } Q \le P. \end{cases}$$

Theorem 2.12. *The complex $(\mathbb{F}, \boldsymbol{d})$ of Definition 2.3 is acyclic when the data is generic and $e = 1$.*

Proof. In Definition 2.14 and Proposition 2.15, we produce $q\colon \mathbb{P} \to \mathbb{P}'$, which is a map of acyclic complexes. We define a map of complexes $\varphi\colon \mathbb{F} \to \mathbb{P}$ in Proposition 2.16. It is clear that $q \circ \phi\colon \mathbb{F} \to \mathbb{P}'$ is surjective. In Proposition 2.18 we identify the kernel of $q \circ \varphi$ as $M + \boldsymbol{d}M$. Lemma 2.20 gives an isomorphism of complexes $\Theta\colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{D})$ which carries $M + \boldsymbol{d}(M)$ to $M + \boldsymbol{D}(M)$. This lemma also shows that $M + \boldsymbol{D}(M)$ is split exact. It follows that $M + \boldsymbol{d}(M) = \ker(q \circ \varphi)$ is split exact; and therefore, $\mathrm{H}_i(\mathbb{F}) = \mathrm{H}_i(\mathbb{P}')$ for all i; thus, $\mathbb{F}$ is acyclic. $\square$

Data 2.13. Let R be a commutative noetherian ring, g be a positive integer, and $f = g + 1$. The complex $\mathbb{P}$ is built from data $(\mathfrak{b}, v, X)$, where $\mathfrak{b}$ is an element of R, $X\colon F \to G$ is an R-module homomorphism, with F and G free R-modules of rank f and g, respectively, and v is an element of F.

Remark. If we think as the data of 2.13 as matrices $v = [v_{j1}]$ and $X = [x_{ij}]$, with $1 \le j \le f$ and $1 \le i \le g$, and R is the polynomial ring $R_0[\{\mathfrak{b}\} \cup \{v_{j1}\} \cup \{x_{ij}\}]$, where $\{\mathfrak{b}\} \cup \{v_{j1}\} \cup \{x_{ij}\}$ is a list of indeterminates over a ring R_0, then we say that the data of 2.13 is generic.

Definition 2.14. Adopt the data $(\mathfrak{b}, v, X)$ of 2.13. The complex $(\mathbb{P}, \boldsymbol{d})$, has modules $\mathbb{P}_i = \bigwedge^i G^* \oplus \bigwedge^i F^* \oplus \bigwedge^{i-1} G^*$ and differential

$$\boldsymbol{d}_i = \begin{bmatrix} X(v) & B_i & (-1)^{i+1}\mathfrak{b} \\ 0 & -v & A_{i-1} \\ 0 & 0 & X(v) \end{bmatrix},$$

where the maps $A_i\colon \bigwedge^i G^* \to \bigwedge^i F^*$ and $B_i\colon \bigwedge^i F^* \to \bigwedge^{i-1} G^*$ are given by $A_i(\gamma_i) = \bigwedge^i X^*(\gamma_i)$ and $B_i(\alpha_i) = [(\bigwedge^{f-i} X)(\alpha_i[\omega_F])](\omega_{G^*})$. The complex $(\mathbb{P}', \boldsymbol{d}')$ is the same as the complex $(\mathbb{P}, \boldsymbol{d})$, except

$$\mathbb{P}'_1 = \bigwedge^1 G^* \oplus \bigwedge^1 F^*, \quad \mathbb{P}'_0 = R, \quad \boldsymbol{d}'_2 = \begin{bmatrix} X(v) & B_2 & -\mathfrak{b} \\ 0 & -v & A_1 \end{bmatrix},$$

and $\boldsymbol{d}'_1 = [\, X(v) \quad B_1 + \mathfrak{b}v \,]$. The map of complexes $q\colon \mathbb{P} \to \mathbb{P}'$ be given by q_i is the identity map for $2 \le i$,

$$q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \text{and} \quad q_0 = [\, 1 \quad -\mathfrak{b} \,].$$

Proposition 2.15. *If the data $(\mathfrak{b}, v, X)$ of 2.13 is generic, then the complexes $\mathbb{P}$ and $\mathbb{P}'$ are acyclic.*

Proof. One may prove this directly or see [22, Prop. 1.2] or [21, Theorem 1.3]. □

Proposition 2.16. *Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of Definition 2.3 constructed from data $(\mathfrak{b}, V, X)$ with $e = 1$, and let $\mathbb{P}$ be the complex of Definition 2.14 constructed from $(\mathfrak{b}, v, X)$, where $v = V(1) \in F$. If the function $\varphi\colon \mathbb{F} \to \mathbb{P}$ is defined by*

$$\varphi(\gamma_d) = \begin{bmatrix} \gamma_d \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \varphi(x) = \begin{bmatrix} 0 \\ \chi(0 \le a)\chi(0 = c)(-1)^a \alpha_b \wedge (\bigwedge^d X^*)(\gamma_d) \\ \chi(0 = a)(-1)^c (\bigwedge^b v)(\alpha_b) \cdot \gamma_1^{(c)} \wedge \gamma_d \end{bmatrix},$$

for $\gamma_d \in B(d)$ and $x = 1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes \gamma_d \in A(a, b, c, d)$, then φ is a map of complexes.

Proof. We pick out one high point of this calculation. Fix a, b, c, and d with $i = a + c + d + 1$ and $b = a - c + 1$. Let

$$y = (d_i \circ \varphi_i - \varphi_{i-1} \circ d_i)(x) \in \mathbb{P}_{i-1} = \bigwedge^{i-1} G^* \oplus \bigwedge^{i-1} F^* \oplus \bigwedge^{i-2} G^*.$$

When $c = 0$, the $\bigwedge^{i-1} G^*$ component of y is zero because

$$\begin{aligned} B_i\left(\alpha_b \wedge (\textstyle\bigwedge^d X^*)(\gamma_d)\right) &= \left((\textstyle\bigwedge^{f-i} X)\left[\left(\alpha_b \wedge (\textstyle\bigwedge^d X^*)(\gamma_d)\right)(\omega_F)\right]\right)(\omega_{G^*}) \\ &= (-1)^{bd}\left((\textstyle\bigwedge^{f-i} X)\left[\left((\textstyle\bigwedge^d X^*)(\gamma_d)\right)(\alpha_b(\omega_F))\right]\right)(\omega_{G^*}) \\ &= (-1)^{bd}\gamma_d \wedge \left[(\textstyle\bigwedge^{f-b} X)(\alpha_b(\omega_F))\right](\omega_{G^*}). \end{aligned}$$

The final equality follows from Proposition 1.2 (b) and (a). □

Lemma 2.17. *Let $(\mathbb{F}, \boldsymbol{d})$ be the complex of Definition 2.3 constructed from data $(\mathfrak{b}, V, X)$ with $e = 1$. If $1 \le i$, then there are submodules $K(0,0,1,i-1)$ and $L(0,0,1,i-1)$ of $A(0,0,1,i-1)$ and a homomorphism $s_i\colon \bigwedge^i G^* \to A(0,0,1,i-1)$ such that*

(a) $A(0,0,1,i-1) = K(0,0,1,i-1) \oplus L(0,0,1,i-1)$,

(b) *$K(0,0,1,i-1)$ is the image of $\boldsymbol{\partial}\colon A(1,0,2,i-2) \to A(0,0,1,i-1)$ (see Definition 2.8),*
(c) *exterior multiplication carries $L(0,0,1,i-1)$ isomorphically onto $\bigwedge^i G^*$, and*
(d) *s_i is a splitting of the exterior multiplication map $\mu\colon A(0,0,1,i-1) \to \bigwedge^i G^*$.*

Proof. The module $A(0,0,1,i-1)$ is equal to $G^* \otimes \bigwedge^{i-1} G^*$. The map μ of (d) is surjective; hence, there exists $s_i\colon \bigwedge^i G^* \to A(0,0,1,i-1)$ with $\mu \circ s_i$ equal to the identity map on $\bigwedge^i G^*$. Let $L(0,0,1,i-1) = \operatorname{im} s_i$ and $K(0,0,1,i-1) = \ker \mu$. The decomposition $A(0,0,1,i-1) = \ker \mu \oplus \operatorname{im} s_i$ gives (a) and (b). To complete the proof, recall that the Eagon-Northcott complex

$$\cdots \to D_2G^* \otimes \bigwedge^{i-2} G^* \to D_1G^* \otimes \bigwedge^{i-1} G^* \to D_0G^* \otimes \bigwedge^{i} G^* \to 0$$

is split exact for $1 \le i$. (See, for example, [5, Def. 1.8].) The map

$$\boldsymbol{\partial}\colon A(1,0,2,i-2) \to A(0,0,1,i-1)$$

is exactly the same as the Eagon-Northcott map

$$D_2G^* \otimes \bigwedge^{i-2} G^* \to D_1G^* \otimes \bigwedge^{i-1} G^*. \quad \square$$

Proposition 2.18. *Adopt the notation and hypotheses of Proposition 2.16. Let M be the submodule $M = \bigoplus A(a,b,c,d)$ of $\mathbb{F}$, where the sum is taken over all 4-tuples (a,b,c,d), with $1 \le a$ and $1 \le c$. Then* $\ker(q \circ \varphi) = M + \boldsymbol{d}(M)$.

Proof. It is clear that $M + \boldsymbol{d}(M) \subseteq \ker(q \circ \varphi)$. It is not difficult to see that $\ker(q \circ \varphi)_i = 0$ for $0 \le i \le 1$, and $M_i = 0$ for $2 \le i$. Henceforth, we take $2 \le i$. We next show that

$$\mathbb{F}_i = M_i + \boldsymbol{d}(M_{i+1}) + B(i) + A(i-1,i,0,0) + L(0,0,1,i-2), \tag{2.19}$$

where $L(0,0,1,i-2)$ is defined in Lemma 2.17. First of all, it is easy to see that

$$\mathbb{F}_i = M_i + B(i) + A(0,1,0,i-1) + A(0,0,1,i-2) + \sum_{a=1}^{i-1} A(a,a+1,0,i-1-a).$$

Indeed, if $A(a,b,c,d)$ is a summand of $\mathbb{F}_i$, but is not a summand of M_i, then either $a = 0$ or $c = 0$. If $a = 0$, then either $c = 0$ (in which case $(a,b,c,d) = (0,1,0,i-1)$) or $c = 1$ (in which case $(a,b,c,d) = (0,0,1,i-2)$). If $c = 0$ and $1 \le a$, then $A(a,b,c,d)$ is a summand of $\sum_{a=1}^{i-1} A(a,a+1,0,i-1-a)$. Apply $\boldsymbol{d}$ to $A(1,1,1,i-2)$, which is a summand of M_{i+1}, to see that

$$A(0,1,0,i-1) \subseteq \boldsymbol{d}(M_{i+1}) + M_i + A(0,0,1,i-2) + A(1,2,0,i-2).$$

Apply $\boldsymbol{d}$ to $A(1,0,2,i-3)$, which is a summand of M_{i+1}, to see that

$$A(0,0,1,i-2) \subseteq L(0,0,1,i-2) + \boldsymbol{d}(M_{i+1}) + M_i.$$

If $1 \le a \le i-2$, then apply $\boldsymbol{d}$ to $A(a+1,a+1,1,i-2-a)$, which is a summand of M_{i+1}, to see that

$$A(a,a+1,0,i-1-a) \subseteq M_i + \boldsymbol{d}(M_{i+1}) + A(a+1,a+2,0,i-2-a).$$

Let

$$P_i = \frac{B(i) \oplus A(i-1,i,0,0) \oplus L(0,0,1,i-2)}{[B(i) \oplus A(i-1,i,0,0) \oplus L(0,0,1,i-2)] \cap [M_i + \boldsymbol{d}(M_{i+1})]}.$$

Now that (2.19) is established, we know that $\mathbb{F}_i/[M_i+\boldsymbol{d}(M_{i+1})] \cong P_i$. On the other hand, the composition

$$\mathbb{P}_i \cong B(i) \oplus A(i-1,i,0,0) \oplus L(0,0,1,i-2) \xrightarrow{\text{nat}} P_i \cong \frac{\mathbb{F}_i}{M_i + \boldsymbol{d}(M_{i+1})} \xrightarrow{\varphi_i} \mathbb{P}_i$$

is an isomorphism, where nat is the natural quotient map. It follows that

$$[B(i) \oplus A(i-1,i,0,0) \oplus L(0,0,1,i-2)] \cap [M_i + \boldsymbol{d}(M_{i+1})] = 0,$$

and $\ker(q \circ \varphi) = M + \boldsymbol{d}(M)$. $\square$

Lemma 2.20. *Adopt the notation and hypotheses of Proposition 2.18. There exists a differential $\boldsymbol{D}$ on $\mathbb{F}$ and a module automorphism Θ of $\mathbb{F}$, such that*
(a) $(\mathbb{F}, \boldsymbol{D})$ *is a complex,*
(b) $\Theta\colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{D})$ *is an isomorphism of complexes,*
(c) Θ *acts like the identity map on* M,
(d) $\Theta(M+\boldsymbol{d}(M)) = M + \boldsymbol{D}(M)$, *and*
(e) $M + \boldsymbol{D}(M)$ *is split exact.*

Proof. Recall s_d and $K(0,0,1,d)$ from Lemma 2.17. We define $\Theta\colon \mathbb{F} \to \mathbb{F}$. The map Θ acts like the identity on each $B(i)$. If

$$x = 1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes \gamma_d \in A(a,b,c,d),$$

then $\Theta(x)$ is equal to

$$\begin{cases} +1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes \gamma_d \in A(a,b,c,d) \\ +\chi(c=0)\chi(1 \le d)(-1)^d 1^{(a+d)} \otimes \alpha_b \wedge \bigwedge^d X^*(\gamma_d) \otimes \gamma_1^{(c)} \otimes 1 \\ \qquad \in A(a+d,b+d,c,0) \\ -\chi(a=0)\chi(c=0) 1^{(a)} \otimes v(\alpha_b) \otimes s_d(\gamma_d) \in A(a,b-1,c+1,d-1). \end{cases}$$

Let $\boldsymbol{D}_i = \Theta_{i-1} \circ \boldsymbol{d}_i \circ \Theta_i^{-1}$. Assertions (a), (b), and (c) are established. Assertion (d) follows from (b) and (c). If the element x, from the above display, is in M_i, then a straightforward calculation shows that $\boldsymbol{D}_i(x)$ is equal to

$$\begin{cases} +\chi(2 \le a) 1^{(a-1)} \otimes v(\alpha_b) \otimes \gamma_1^{(c)} \otimes \gamma_d \in A(a-1,b-1,c,d) \\ +\chi(a=1)\chi(c=1) 1^{(a-1)} \otimes v(\alpha_b) \otimes (\mathrm{id} - s \circ \mu)(\gamma_1 \otimes \gamma_d) \in K(0,0,1,d) \\ -[\chi(2 \le c) + \chi(c=1)\chi(1 \le d)] 1^{(a)} \otimes X^*(\gamma_1) \wedge \alpha_b \otimes \gamma_1^{(c-1)} \otimes \gamma_d \\ \qquad \in A(a,b+1,c-1,d) \\ +(-1)^{a+c} 1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes [X(v)](\gamma_d) \in A(a,b,c,d-1) \\ +(-1)^{a+c} 1^{(a-1)} \otimes \alpha_b \otimes \gamma_1^{(c-1)} \otimes \gamma_1 \wedge \gamma_d \in A(a-1,b,c-1,d+1). \end{cases}$$

For each i, with $2 \le i$, let

$$\mathbb{S}_i = M_i \oplus K(0,0,1,i-2) \oplus \sum_{(a,d)} A(a,a+1,0,d),$$

where the sum is taken over all pairs (a,d) with $0 \le a$, $1 \le d$, and $a+d+1 = i$. Observe that $\boldsymbol{D}(M_i) \subseteq \mathbb{S}_{i-1}$. The only term which causes any effort is the term in $A = A(a-1,b,c-1,d+1)$. If $a-1 = 0$, then $b = a-c+e$ forces $c \le 2$. On the other hand, A is zero unless $1 \le c$. If $c = 1$, then $A = A(0,1,0,d+1) \subseteq \mathbb{S}_{i-1}$. If $c = 2$, then $A = A(0,0,1,d+1)$, but it is clear that

$$(-1)^{a+c} 1^{(a-1)} \otimes \alpha_b \otimes \gamma_1^{(c-1)} \otimes \gamma_1 \wedge \gamma_d \in K(0,0,1,d+1) \subseteq \mathbb{S}_{i-1}.$$

If $1 \le a-1$, then either A is in $M_{i-1} \subseteq \mathbb{S}_{i-1}$ or $c = 1$, in which case, we still have $A = A(a-1,a,0,d+1) \subseteq \mathbb{S}_{i-1}$.

Let $\mathbb{S} = \bigoplus_{2 \le i} \mathbb{S}_i$. We have shown that $\boldsymbol{D}(M_i) \subseteq \mathbb{S}_{i-1}$; hence $M + \boldsymbol{D}(M) \subseteq \mathbb{S}$. We complete the proof by showing that $(\mathbb{S}, \boldsymbol{D})$ is split exact and $\mathbb{S} \subseteq M + \boldsymbol{D}(M)$. In a manner analogous to Definition 2.8, we partition $\mathbb{S}$ into strands $\bigoplus_{(P,Q)} \bar{S}(P,Q)$, where the sum varies over all pairs (P,Q) with $1 \le P$ and $1 \le Q \le P+1$. For parameters P and Q, the summand $X(a,b,c,d)$ of $\mathbb{S}$ is in $\bar{S}(P,Q)$ if $P = a+d$ and $Q = c+d$, where $X = A$ for all (a,b,c,d), except $(0,0,1,d)$, and $X = K$ for $(a,b,c,d) = (0,0,1,d)$. Observe that every summand of $\mathbb{S}$ lives in exactly one strand. Our calculation of $\boldsymbol{D}(x)$, for $x \in M_i$, shows that $\boldsymbol{D}$ is a non-increasing function with respect to the inverse lexicographic order on the parameters $\{(P,Q)\}$. Furthermore, the restriction of the homogeneous part of $\boldsymbol{D}$ to $\bar{S}(P,Q)$ is

$$0 \to A(P,b,Q,0) \to A(P-1,b,Q-1,1) \to \cdots \to A(P-Q,b,0,Q) \to 0,$$

if $Q \le P$; and

$$0 \to A(P,b,Q,0) \to A(P-1,b,Q-1,1) \to \cdots \to K(0,b,1,Q-1) \to 0,$$

if $Q = P+1$, where $b = P-Q+e$. These strands are exact because the Eagon Northcott complexes

$$0 \to D_Q G^* \otimes \textstyle\bigwedge^0 G^* \to D_{Q-1} G^* \otimes \bigwedge^1 G^* \to \cdots \to D_0 G^* \otimes \bigwedge^Q G^* \to 0, \text{ and}$$
$$0 \to D_Q G^* \otimes \textstyle\bigwedge^0 G^* \to D_{Q-1} G^* \otimes \bigwedge^1 G^* \to \cdots \to K(0,0,1,Q-1) \to 0$$

are split exact since $1 \le Q$. Lemma 1.10 completes the proof. $\square$

3. The complex $\mathbb{F}$ is acyclic.

Theorem 3.1. *If the data $(\mathfrak{b}, V, X)$ of 2.1 is generic, then the complex $(\mathbb{F}, \boldsymbol{d})$ of Definition 2.3 is acyclic.*

Proof. Corollary 4.9 ensures the existence of a subcomplex $\mathbb{G}$ of $\mathbb{F}$ such that $\mathbb{G}$ consists of free modules, $\mathbb{G}$ has length $eg+1$ and $\mathbb{F}/\mathbb{G}$ is split exact. According to the acyclicity lemma [9, Cor. 4.2], it suffices to show that $\mathbb{G}_P$ is acyclic for all prime ideals P of R with $\operatorname{grade} P < eg+1$. Thus, it suffices to show that $\mathbb{F}_P$ is acyclic for all prime ideals P of R with $\operatorname{grade} P < eg+1$. The ideal $I_1(V)$ has grade

$ef \geq eg+1$. If v is an entry of a matrix representation of V, then Lemma 3.2 shows that $\mathbb{F}_v$ is isomorphic to the complex $\mathbb{F}$ of Lemma 3.3, which is built with generic data over the ring $R_0[v^{-1}]$. The complex $\mathbb{F}$ of Lemma 3.3 has the same homology as $\mathbb{F}' \otimes \mathbb{K}$, where $\mathbb{F}'$ is made from generic data, with $e-1$ in place of e, and $\mathbb{K}$ is the Koszul complex associated to the sequence of g new indeterminates. Induction on e completes the result. The base case is Theorem 2.12. $\square$

Lemma 3.2. *Form the complex* $(\mathbb{F}, \boldsymbol{d})$ *using the data* $(\mathfrak{b}, V, X)$ *of 2.1. Let* θ *and* τ *be automorphisms of* F *and* E*, respectively. Form* $(\mathbb{F}, \boldsymbol{d}')$ *using* $(u'\mathfrak{b}, \theta \circ V, X \circ \theta^{-1})$ *and form* $(\mathbb{F}, \boldsymbol{d}'')$ *using* $(u''\mathfrak{b}, V \circ \tau, X)$*, where* $u' = ((\bigwedge^f \theta^{-1})[\omega_F])(\omega_{F^*})$ *and* u'' *is* $[(\bigwedge^e \tau^{-1})(\omega_E)](\omega_{E^*})$*. Then the complexes* $(\mathbb{F}, \boldsymbol{d})$*,* $(\mathbb{F}, \boldsymbol{d}')$*, and* $(\mathbb{F}, \boldsymbol{d}'')$ *are isomorphic to one another.*

Proof. Define $\varphi' \colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{d}')$ and $\varphi'' \colon (\mathbb{F}, \boldsymbol{d}) \to (\mathbb{F}, \boldsymbol{d}'')$, by $\varphi'(z_d) = u' z_d$ in $B(d)$, $\varphi''(z_d) = (\bigwedge^d(\tau^{-1} \otimes 1))(z_d) \in B(d)$,

$$\begin{aligned}
&\varphi'(x) = \varepsilon_1^{(a)} \otimes (\textstyle\bigwedge^b \theta^{*-1})(\alpha_b) \otimes \gamma_1^{(c)} \otimes z_d \in A(a,b,c,d), \text{ and} \\
&\varphi''(x) = (D_a \tau^{-1})(\varepsilon_1^{(a)}) \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes (\textstyle\bigwedge^d(\tau^{-1} \otimes 1))(z_d) \in A(a,b,c,d),
\end{aligned}$$

for $z_d \in B(d)$, and $x = \varepsilon_1^{(a)} \otimes \alpha_b \otimes \gamma_1^{(c)} \otimes z_d \in A(a,b,c,d)$. It is not difficult to see that φ' and φ'' both are isomorphisms of complexes. $\square$

Lemma 3.3. *Let* $(\mathfrak{b}, V, X)$ *be the data of 2.1. Suppose that* $E = E' \oplus E''$ *and* $F = F' \oplus F''$*, with* $E'' = R\varepsilon$ *and* $F'' = Rf$*. Suppose further that*

$$V = \begin{bmatrix} V' & 0 \\ 0 & V'' \end{bmatrix}, \quad \textit{and} \quad X = [\, X' \quad X'' \,], \quad \textit{where}$$

$$E'' \xrightarrow{V''} F'', \quad E' \xrightarrow{V'} F' \xrightarrow{X'} G, \quad \textit{and} \quad F'' \xrightarrow{X''} G$$

are R*-module homomorphisms, and* $V''(\varepsilon) = f$*. Form the complexes* $(\mathbb{F}, \boldsymbol{d})$ *and* $(\mathbb{F}', \boldsymbol{d}')$ *using the data* $(\mathfrak{b}, V, X)$ *and* $(\mathfrak{b}, V', X')$*, respectively. Let* $x'' = X''(f) \in G$*. Then there exists a split exact complex* $\mathbb{L}$ *and a short exact sequence of complexes:*

$$0 \to (\mathbb{F}', \boldsymbol{d}') \otimes (\textstyle\bigwedge^\bullet G^*, x'') \to \mathbb{F} \to \mathbb{L} \to 0.$$

Proof. Take $\omega_E = \varepsilon \wedge \omega_{E'}$ and $\omega_F = f \wedge \omega_{F'}$. Let α be the element of F^* with $\alpha(F') = 0$ and $\alpha(f) = 1$, $A(a_1, b_1, c, d_1; a_2, b_2, d_2)$ equal

$$D_{a_1} E' \otimes D_{a_2} E'' \otimes \textstyle\bigwedge^{b_1} F'^* \otimes \bigwedge^{b_2} F''^* \otimes D_c G^* \otimes \bigwedge^{d_1}(E' \otimes G^*) \otimes \bigwedge^{d_2}(E'' \otimes G^*),$$

and $B(d_1; d_2) = \bigwedge^{d_1}(E' \otimes G^*) \otimes \bigwedge^{d_2}(E'' \otimes G^*)$. We see that

$$A(a,b,c,d) = \sum A(a_1, b_1, c, d_1; a_2, b_2, d_2) \quad \text{and} \quad B(d) = \sum B(d_1; d_2),$$

where the first sum varies over all tuples $(a_1, a_2, b_1, b_2, d_1, d_2)$ with $a_1 + a_2 = a$, $b_1 + b_2 = b$, and $d_1 + d_2 = d$, and the second sum varies over all tuples (d_1, d_2) with

$d_1+d_2=d$. The complex $(\mathbb{F},d)$ is built using the modules $A(a,b,c,d)$ and $B(d)$. The complex $(\mathbb{F}',\boldsymbol{d}')$ is built using the modules

$$A'(a,b,c,d)=A(a,b,c,d;0,0,0) \quad \text{and} \quad B'(d)=B(d;0).$$

The differential in the complex $\Big((\mathbb{F}',\boldsymbol{d}')\otimes(\bigwedge^\bullet G^*,x''),D\Big)$ is given by

$$D(a\otimes b)=\boldsymbol{d}'(a)\otimes b+(-1)^{|a|+1}a\otimes x''(b).$$

Define $\varphi:\Big((\mathbb{F}',\boldsymbol{d}')\otimes(\bigwedge^\bullet G^*,x''),D\Big)\to(\mathbb{F},\boldsymbol{d})$ by

$$\varphi\Big(z_{d_1}\otimes\gamma_{d_2}\in B'(d_1)\otimes{\textstyle\bigwedge^{d_2}}G^*\Big)=(-1)^{d_2}z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\in B(d_1;d_2),$$

and $\varphi\Big(\varepsilon_1^{(a_1)}\otimes\alpha_{b_1}\otimes\gamma_1^{(c)}\otimes z_{d_1}\otimes\gamma_{d_2}\in A'(a_1,b_1,c,d_1)\otimes\bigwedge^{d_2}G^*\Big)$ is equal to

$$\varepsilon_1^{(a_1)}\otimes\alpha_{b_1}\wedge\alpha\otimes\gamma_1^{(c)}\otimes z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\in A(a_1,b_1,c,d_1;0,1,d_2).$$

It is clear that φ is injective. We see that φ is a map of complexes, because $\boldsymbol{d}\circ\varphi$ and $\varphi\circ D$ both carry the element $z_{d_1}\otimes\gamma_{d_2}$ of $B'(d_1)\otimes\bigwedge^{d_2}G^*$ to

$$(-1)^{d_2}(X'\check{\circ}V')(z_{d_1})\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})+(-1)^{d_1+d_2}z_{d_1}\wedge\Big(\varepsilon^{(d_2-1)}\bowtie x''(\gamma_{d_2})\Big),$$

and carry the element $\varepsilon_1^{(a_1)}\otimes\alpha_{b_1}\otimes\gamma_1^{(c)}\otimes z_{d_1}\otimes\gamma_{d_2}$ of $A'(a_1,b_1,c,d_1)\otimes\bigwedge^{d_2}G^*$ to

$$\begin{cases}\ \ \varepsilon_1^{(a_1-1)}\otimes[V'(\varepsilon_1)](\alpha_{b_1})\wedge\alpha\otimes\gamma_1^{(c)}\otimes z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\\ -\varepsilon_1^{(a_1)}\otimes X'^*(\gamma_1)\wedge\alpha_{b_1}\wedge\alpha\otimes\gamma_1^{(c-1)}\otimes z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\\ +(-1)^{a_1+c}\varepsilon_1^{(a_1)}\otimes\alpha_{b_1}\wedge\alpha\otimes\gamma_1^{(c)}\otimes(X'\check{\circ}V')(z_{d_1})\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\\ +(-1)^{a_1+c+d_1}\varepsilon_1^{(a_1)}\otimes\alpha_{b_1}\wedge\alpha\otimes\gamma_1^{(c)}\otimes z_{d_1}\wedge(\varepsilon^{(d_2-1)}\bowtie x''(\gamma_{d_2}))\\ +(-1)^{a_1+c}\varepsilon_1^{(a_1-1)}\otimes\alpha_{b_1}\wedge\alpha\otimes\gamma_1^{(c-1)}\otimes(\varepsilon_1\otimes\gamma_1)\wedge z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\\ +(-1)^{a_1+d_1+d_2}\delta_{c0}\varepsilon_1^{(a_1)}\bowtie[(\bigwedge^{f-b_1-1}X')(\alpha_{b_1}[\omega_{F'}])](\omega_{G^*})\wedge z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2})\\ +(-1)^{d_1+d_2}\chi(a_1=0)\mathfrak{b}\cdot[(\bigwedge^{b_1}V'^*)(\alpha_{b_1})](\omega_{E'})\bowtie\gamma_1^{(c)}\wedge z_{d_1}\wedge(\varepsilon^{(d_2)}\bowtie\gamma_{d_2}).\end{cases}$$

The cokernel of φ consists of all $A(a_1,b_1,c,d_1;a_2,b_2,d_2)$ such that either $0<a_2$ and $1=b_2$; or $0\le a_2$ and $0=b_2$. Apply Lemma 1.10 to see that the cokernel of φ is exact. Decompose $\operatorname{coker}\varphi$ into a direct sum of strands $\mathcal{S}(P,Q)$, where

$$A(a_1,b_1,c,d_1;a_2,b_2,d_2)\in\mathcal{S}(P,Q) \text{ if } P=a_2-b_2+d_2 \quad\text{and}\quad Q=a_1+c+d_1.$$

Impose the inverse lexicographic order of 1.12 on $\{(P,Q)\}$. It is easy to see that the map $\boldsymbol{d}$ is non-increasing on $\operatorname{coker}\varphi$ and each strand $\mathcal{S}(P,Q)$ is the split exact sequence

$$\bigoplus 0\to A(a_1,b_1,c,d_1;a_2,1,d_2)\xrightarrow{\cong}A(a_1,b_1,c,d_1;a_2-1,0,d_2)\to 0,$$

where the sum varies over all tuples $(a_1,b_1,c,d_1;a_2,1,d_2)$ with $a_2+d_2=P+1$, $a_1+c+d_1=Q$, and $1\le a_2$. $\square$

4. We split summands from $\mathbb{F}$.

In this section we produce three complexes of free modules; each of these complexes is homologically equivalent to the complex $(\mathbb{F},\boldsymbol{d})$ of Definition 2.3. Each of the new complexes has length $eg+1$; and therefore is much smaller than $\mathbb{F}$, which has infinite length. The complex $\mathbb{G}$ of Corollary 4.9 is a submodule of $\mathbb{F}$; thus, the differential on $\mathbb{G}$ is merely the restriction of $\boldsymbol{d}$ to $\mathbb{G}$. The complex $\mathbb{I}$ of Theorem 4.10 is much smaller than $\mathbb{G}$, and the complex $\mathbb{J}$ of Theorem 4.15 is much smaller than $\mathbb{I}$. The proof of 4.15 incorporates the proof of 4.10 and then splits off more summands from $\mathbb{F}$. We offer both stopping points because the description of $\mathbb{I}$ is cleaner than the description of $\mathbb{J}$, and surely $\mathbb{I}$ is sufficient for many purposes. All three complexes are graded, provided $\mathbb{F}$ is graded in the sense of Convention 2.2. The complexes $\mathbb{I}$ and $\mathbb{J}$ are created using various unspecified homotopies, as described in Observation 4.3; and therefore, the differentials are too complicated to record in a meaningful manner.

Observation 4.3 is a fine tuning of Lemma 1.10. It is more complicated to state than it is to prove. It provides a framework for splitting off part of a complex without changing the homology. After the notation is set, then the hypothesis is that various homogeneous strands of the complex $\mathbb{E}$ have been identified and each of these strands contains a splittable substrand. The conclusion is that, in the original non-homogeneous complex $\mathbb{E}$, each splittable substrand may be replaced by its homology, at the expense of complicating the differential.

Definition 4.1. We say that the complex $\mathbb{L}$ is *splittable* if $\mathbb{L}$ is the direct sum of two subcomplexes $\mathbb{L}'$ and $\mathbb{L}''$, with $\mathbb{L}'$ split exact, and the differential on $\mathbb{L}''$ identically zero.

Proposition 4.2. *If $\mathbb{L}$ is a bounded complex of projective modules, then $\mathbb{L}$ is splittable if and only if $\mathrm{H}_j(\mathbb{L})$ is projective for all j.*

Proof. The proof is not difficult; see, for example, [19, Prop. 3.2].

Observation 4.3. *Let $(\mathbb{E},\boldsymbol{d})$ be a complex of finitely generated projective R-modules, and $\mathfrak{P}$ be a partially ordered set. Suppose that each module $\mathbb{E}_i$ from $\mathbb{E}$ decomposes into a direct sum $\bigoplus_{p\in\mathfrak{P}}\mathbb{E}_i^{[p]}$, and that the differential $\boldsymbol{d}$ is non-increasing with respect to the $\mathfrak{P}$ grading. Let $\boldsymbol{\partial}$ be the homogeneous part of $\boldsymbol{d}$ of degree zero with respect to $\mathfrak{P}$. Suppose that each module $\mathbb{E}_i^{[p]}$ decomposes into $\mathbb{L}_i^{[p]}\oplus\mathbb{K}_i^{[p]}$. Let $\mathbb{L}_i=\bigoplus_p\mathbb{L}_i^{[p]}$, $\mathbb{K}_i=\bigoplus_p\mathbb{K}_i^{[p]}$, and $\mathbb{L}=\bigoplus_i\mathbb{L}_i$. View $\mathbb{L}$ as a substrand of $(\mathbb{E},\boldsymbol{\partial})$, in the sense of 1.11. If $\mathbb{L}$ is a splittable complex, then there exists a split exact subcomplex $(\mathbb{N},\boldsymbol{d})$ of $(\mathbb{E},\boldsymbol{d})$ such that $\mathbb{N}$ is a direct summand of $\mathbb{E}$ as a module, and $(\mathbb{E}/\mathbb{N})_i\cong\mathrm{H}_i(\mathbb{L})\oplus\mathbb{K}_i$.*

Proof. The hypothesis guarantees that $\mathbb{L}$ decomposes into the direct sum of two subcomplexes $\mathbb{P}\oplus\mathbb{Q}$, where $\mathbb{Q}$ is split exact and $\mathbb{P}\cong\mathrm{H}(\mathbb{L})$. For each i, let $\mathbb{Q}_i$ equal $A_i\oplus B_i$, where B_i is equal to the image of $\mathbb{Q}_{i+1}$ in $\mathbb{L}$. We see that the differential in $\mathbb{L}$ carries A_i isomorphically onto B_{i-1}. Observe that $\mathbb{E}_i=A_i\oplus B_i\oplus\mathbb{P}_i\oplus\mathbb{K}_i$ and that the composition

$$A_i\xrightarrow{\text{incl}}\mathbb{E}_i\xrightarrow{\boldsymbol{d}_i}\mathbb{E}_{i-1}\xrightarrow{\text{proj}}B_{i-1}\tag{4.4}$$

is an isomorphism for each i. The second assertion holds because $\boldsymbol{d}_i$ is non-increasing and the homogeneous part of (4.4) is an isomorphism. Define $\mathbb{N}$ to be $\bigoplus_i \mathbb{N}_i$ and $\mathbb{M}$ to be $\bigoplus_i \mathbb{M}_i$, with $\mathbb{N}_i = A_i + \boldsymbol{d}_{i+1}(A_{i+1})$ and $\mathbb{M}_i = \mathbb{P}_i \oplus \mathbb{K}_i$. Use the decomposition $\mathbb{E}_i = A_i \oplus B_i \oplus \mathbb{M}_i$ to produce the projection maps

$$\pi_i^B \colon \mathbb{E}_i \to B_i \quad \text{and} \quad \pi_i^{\mathbb{M}} \colon \mathbb{E}_i \to \mathbb{M}_i.$$

Let $\theta_{i-1} \colon B_{i-1} \to A_i$ be the inverse of the map of (4.4); $\psi_i \colon \mathbb{E}_i \to \mathbb{M}_i$ be

$$\psi_i = \pi_i^{\mathbb{M}} \circ (1 - \boldsymbol{d}_{i+1} \circ \theta_i \circ \pi_i^B);$$

and $m_i \colon \mathbb{M}_i \to \mathbb{M}_{i-1}$ be the composition

$$\mathbb{M}_i \xrightarrow{\text{incl}} \mathbb{E}_i \xrightarrow{\boldsymbol{d}_i} \mathbb{E}_{i-1} \xrightarrow{\psi_{i-1}} \mathbb{M}_{i-1}.$$

A straightforward calculation (see, for example, [18, Prop. 7.2] or [17, Prop. 3.14]) shows that

$$0 \to (\mathbb{N}, \boldsymbol{d}|_{\mathbb{N}}) \xrightarrow{\text{incl}} (\mathbb{E}, \boldsymbol{d}) \xrightarrow{\psi} (\mathbb{M}, m) \to 0$$

is short exact sequence of complexes, and that $\mathbb{N}$ fulfills all of the requirements. We notice, for future reference, that as a module, $\mathbb{M}$ is isomorphic to $\bigoplus_{p \in \mathfrak{P}} \mathbb{M}^{[p]}$ and that the differential m on $\mathbb{M}$ is non-increasing with respect to the $\mathfrak{P}$ grading. □

For the rest of this section the complexes $(\mathbb{F}, \boldsymbol{d})$ and $(\mathbb{F}, \boldsymbol{\partial})$ of Definitions 2.3 and 2.10 are fixed. Let

$$\boldsymbol{\alpha} = (e-1)(g-1). \tag{4.5}$$

We define complexes $(\mathbb{P}(a_0, c_0, d_0), \boldsymbol{\partial})$ and $(\mathbb{E}, \boldsymbol{\partial})$. Each of the new complexes is a quotient of $(\mathbb{F}, \boldsymbol{\partial})$ under the natural quotient map. In particular, $\mathbb{P}_i$ and $\mathbb{E}_i$ are defined for all integers i. If we don't specify a value for one of these modules, then the module is automatically equal to zero. Furthermore, if the parameters a, c, and d of the module $A(a, b, c, d)$ are known, then the parameter b is automatically equal to $a - c + e$. The position of the module $A(a, b, c, d)$ is $a + c + d + 1$ in every complex which contains it. Let

$$\bar{A}(a, b, c, d) = \frac{A(a, b, c, d)}{\boldsymbol{\partial}(A(a+1, b, c+1, d-1))} \quad \text{and} \quad \bar{B}(d) = \frac{B(d)}{\boldsymbol{\partial}(A(g, f, 0, d-g))}.$$

Definition 4.6. If a_0, c_0, and d_0 are integers, with a_0 and c_0 non-negative, then let $(\mathbb{P}(a_0, c_0, d_0), \boldsymbol{\partial})$ be the complex

$$0 \to A(a_0{+}d_0, b, c_0{+}d_0, 0) \xrightarrow{\boldsymbol{\partial}} \dots \xrightarrow{\boldsymbol{\partial}} A(a_0{+}1, b, c_0{+}1, d_0{-}1) \xrightarrow{\boldsymbol{\partial}} A(a_0, b, c_0, d_0) \to 0,$$

where $b = a_0 - c_0 + e$. The absolute value of the Euler characteristic of $\mathbb{P}(a_0, c_0, d_0)$ is

$$|\mathbb{P}(a_0, c_0, d_0)| = \left| \sum_{i=0}^{d_0} (-1)^i \operatorname{rank} A(a_0 + i, b, c_0 + i, d_0 - i) \right|.$$

Remarks. (a) If $d_0 < 0$, then $\mathbb{P}(a_0, c_0, d_0)$ is the zero complex.
(b) If $P = a_0 + d_0$ and $Q = c_0 + d_0$, then the complex $\mathbb{P}(a_0, c_0, d_0)$ is a quotient of the homogeneous strand $(S(P,Q), \boldsymbol{\partial})$ of Observation 2.9. Furthermore, if

(i) $a_0 = 0$ and $0 \le c_0 \le e$; or
(ii) $c_0 = 0$ and $0 \le a_0 \le g - 1$; or
(iii) $c_0 = 0$, $a_0 = g$, and $eg + 1 \le P$,

then $\mathbb{P}(a_0, c_0, d_0) = S(P,Q)$.
(c) The homogeneous strands $(S(P,Q), \boldsymbol{\partial})$ have been studied extensively, under a slightly different name, in [19]. The exact connection between the two notations is

$$S(P,Q) = \begin{cases} \mathbb{M}(P,Q) \otimes \bigwedge^b F^*, & \text{if } -e \le P - Q \le g-1, \text{ and} \\ \widetilde{\mathbb{M}}(P,Q) \otimes \bigwedge^b F^*, & \text{if } P = Q + g, \end{cases}$$

for $b = P - Q + e$. The differential $\boldsymbol{\partial}$ of $S(P,Q)$ is equal to the tensor product of the differential of $\mathbb{M}(P,Q)$ or $\widetilde{\mathbb{M}}(P,Q)$ with the identity map on $\bigwedge^b F^*$.

Theorem 4.7.
(a) *Assume* $1 - e \le P - Q \le g - 1$. *If either* $eg - g + 1 \le Q$ *or* $eg - e + 1 \le P$, *then* $S(P,Q)$ *is split exact.*
(b) *If* $Q = e + P$, *then* $S(P,Q)$ *has free homology which is equal to* $\bar{A}(0,0,e,P)$; *furthermore, if* $eg - e + 1 \le P$, *then* $\bar{A}(0,0,e,P) = 0$.
(c) *If* $g + Q = P$, *then* $S(P,Q)$ *has free homology which is equal to* $\bar{B}(P)$; *furthermore, if* $eg - e + 1 \le P$, *then* $\bar{B}(P) = 0$.
(d) *Fix integers* a, c, *and* d. *Assume that* $1 - e \le a - c \le g - 1$. *If* $g - 1 \le a$ *or* $e - 1 \le c$, *then the complex* $\mathbb{P}(a,c,d)$ *has free homology equal to* $\bar{A}(a,b,c,d)$.
(e) *Fix integers* a, c, *and* d, *with* $0 \le a$ *and* $0 \le c$. *Let* $P = a + d$ *and* $Q = c + d$. *If the pair* (P,Q) *satisfies any one of the hypotheses* (a), (b), *or* (c), *then* $\mathbb{P}(a,c,d)$ *has free homology equal to* $\bar{A}(a,b,c,d)$.

Proof. Assertions (a) – (d) are [19, Cor. 5.1]. We may assume that the base ring in (e) is $\mathbb{Z}$. The complex $\mathbb{P}$ is a truncation of the splittable complex $S(P,Q)$. The homology of $S(P,Q)$ is concentrated at the right most non-zero position. $\square$

Lemma 4.8. *If* $(\mathbb{E}, \boldsymbol{\partial})$ *is the complex* $0 \to \bigoplus A(a,b,c,d) \to 0$, *where the parameters satisfy* $eg \le a + c + d$, *then* $\mathbb{E}$ *is a splittable complex and* $\mathrm{H}(\mathbb{E})$ *is equal to the free module* $\bigoplus \bar{A}(a,b,c,d)$, *where the parameters satisfy* $eg = a + c + d$.

Proof. Observe that $\mathbb{E}$ is equal to the direct sum

$$\bigoplus_{eg = a_0 + c_0 + d_0} \mathbb{P}(a_0, c_0, d_0) \oplus \bigoplus_{\substack{eg+1 \le a_0 + c_0 + d_0 \\ 0 = a_0 c_0}} \mathbb{P}(a_0, c_0, d_0),$$

with the parameters a_0, c_0, and d_0 all non-negative. Indeed, let $X = A(a,b,c,d)$ be a summand of $\mathbb{E}$. If $a \le c$, then X is a summand of

$$\begin{cases} \mathbb{P}(0, c-a, d+a), & \text{if } eg + 1 \le c + d, \text{ and} \\ \mathbb{P}(eg - c - d, eg - a - d, a + c + 2d - eg) & \text{if } c + d \le eg. \end{cases}$$

If $c<a$, then X is a summand of

$$\begin{cases} \mathbb{P}(a-c,0,d+c), & \text{if } eg+1\le a+d, \text{ and}\\ \mathbb{P}(eg-c-d,eg-a-d,a+c+2d-eg), & \text{if } a+d\le eg.\end{cases}$$

Let $\mathbb{P}$ be the strand $\mathbb{P}(a_0,c_0,d_0)$ of $\mathbb{E}$, and let $P=a_0+d_0$. If $a_0+c_0+d_0=eg$, then, according to Theorem 4.7, $\mathbb{P}$ has free homology equal to $\bar{A}(a_0,b,c_0,d_0)$. Indeed, if $a_0=g+c_0$ or $c_0=a_0+e$, then apply part (e), by way of (b) or (c). If $1-g\le c_0-a_0\le e-1$, then apply (d) if $g-1\le a_0$, or apply (e), by way of (a), if $a_0\le g-2$. If $eg+1\le a_0+c_0+d_0$ and $a_0c_0=0$, then Theorem 4.7 yields that $\mathbb{P}$ is split exact. If $a_0=g+c_0$, then c_0 must be zero; hence, $eg+1\le P$ and part (c) applies. If $c_0=a_0+e_0$, then a_0 must be zero; hence, $eg-e+1\le P$ and (b) applies. If $1-g\le c_0-a_0\le e-1$, then $c_0\le e-1$; hence, $eg-e+1\le P$ and (a) applies. □

Corollary 4.9. *Let $(\mathbb{F},\boldsymbol{d})$ be the complex of Definition 2.3. If $A(a,b,c,d)$ is a summand of $\mathbb{F}$ with $a+c+d=eg$, then there exists a free submodule $A'(a,b,c,d)$ of $A(a,b,c,d)$ so that*

$$A(a,b,c,d)=A'(a,b,c,d)\oplus\boldsymbol{\partial}A(a+1,b,c+1,d-1).$$

If $\mathbb{G}$ is the submodule

$$\bigoplus_{a+c+d=eg}A'(a,b,c,d)\oplus\bigoplus_{a+c+d\le eg-1}A(a,b,c,d)\oplus\bigoplus_{i}B(i)$$

of $\mathbb{F}$, then $\mathrm{H}_i(\mathbb{G},\boldsymbol{d})=\mathrm{H}_i(\mathbb{F},\boldsymbol{d})$ for all integers i.

Proof. Lemma 4.8 guarantees the existence of $A'(a,b,c,d)$. Apply Lemma 1.10 to see that $\mathbb{G}$ and $\mathbb{F}$ have the same homology. □

Theorem 4.10. *Let $\mathbb{F}$ be the complex of Definition 2.3. There exists a split exact subcomplex $\mathbb{N}$ of $\mathbb{F}$, such that the quotient complex $\mathbb{I}=\mathbb{F}/\mathbb{N}$ is isomorphic, as a graded module, to*

$$\bigoplus_{d\le eg-e}\bar{B}(d)\oplus\bigoplus_{d\le eg-e}\bar{A}(0,0,e,d)\oplus\bigoplus_{V_1}\bar{A}(a,b,c,d)\oplus\bigoplus_{Z_2}A(a,b,c,d),$$

where V_1 and Z_2 are the following sets of triples of non-negative integers

$$\begin{aligned}V_1=&\{(a,c,d)\mid d\le\boldsymbol{\alpha},\quad a\le g-1,\quad c\le e-1,\ \text{and}\ (g-1-a)(e-1-c)=0\}\ \text{and}\\ Z_2=&\{(a,c,d)\mid a\le g-2,\quad c\le e-2,\quad a+d\le eg-e,\quad \text{and}\quad c+d\le eg-g\}.\end{aligned}$$

Each of the modules $\bar{B}(d)$ and $\bar{A}(a,b,c,d)$, which appears above, is a free module. The complex $\mathbb{I}$ has length $eg+1$.

Proof. The module $A(a,b,c,d)$ is a summand of $\mathbb{F}$ only if $b=a-c+e$. This module is non-zero only if $0\le b\le f$; which is equivalent to $-e\le a-c\le g$. Let

$$U=\{(a,c,d)\mid 0\le a,c,d,\ \text{and}\ -e\le a-c\le g\}. \tag{4.11}$$

We see that
$$\mathbb{F}=\bigoplus_{d} B(d)\oplus\bigoplus_{U} A(a,b,c,d).$$
It is obvious that U is the disjoint union $Z_1\cup Z_2\cup Z_3\cup Z_4\cup Z_5$, where
$$\begin{aligned}
Z_1&=\left\{(a,c,d)\in U\,\middle|\,\begin{matrix}1-e\le a-c\le g-1,\ a+d\le eg-e,\ c+d\le eg-g,\\ \text{and at least one of } g-1\le a \text{ and/or } e-1\le c\end{matrix}\right\},\\
Z_3&=\{(a,c,d)\in U\mid c-a=-g\},\\
Z_4&=\{(a,c,d)\in U\mid c-a=e\},\ \text{and}\\
Z_5&=\left\{(a,c,d)\in U\,\middle|\,\begin{matrix}1-e\le a-c\le g-1,\ \text{and at least one of}\\ eg-e+1\le a+d \text{ and/or } eg-g+1\le c+d\end{matrix}\right\}.
\end{aligned}$$
Furthermore, if $(a,c,d)\in Z_1$, then there exists $(a_0,c_0,d_0)\in V_1$ and a non-negative integer δ, such that $(a,c,d)=(a_0+\delta,c_0+\delta,d_0-\delta)$. Indeed, take
$$\delta=\begin{cases} a-g+1, & \text{if } c-a\le e-g\\ c-e+1, & \text{if } e-g+1\le c-a.\end{cases}$$
Decompose $\mathbb{F}$ into $\mathbb{K}\oplus\mathbb{L}$ where $\mathbb{K}=\bigoplus_{Z_2} A(a,b,c,d)$ and
$$\mathbb{L}=\bigoplus_{d} B(d)\oplus\bigoplus_{U\setminus Z_2} A(a,b,c,d).$$

We use Theorem 4.7 to show that $(\mathbb{L},\boldsymbol{\partial})$ is splittable. Observation 4.3 completes the proof. Observe that $\bigoplus_{Z_5} A(a,b,c,d)$ is the direct sum of strands $S(P,Q)$ with $1-e\le P-Q\le g-1$ and at least one of the inequalities $eg-e+1\le P$ and/or $eg-g+1\le Q$ holds. Each such strand is split exact. The sum $\bigoplus_{Z_4}$A(a,b,c,d) is equal to the direct sum of strands $S(P,Q)$ with $Q=P+e$. Each such strand has free homology equal to $\bar{A}(0,0,e,P)$; furthermore, this homology is zero unless $P\le eg-e$. The sum $\bigoplus_{Z_3} A(a,b,c,d)\oplus\bigoplus_i B(i)$ is the direct sum of strands $S(P,Q)$ with $P=Q+g$. Such a strand has free homology equal to $\bar{B}(P)$; furthermore, this homology is zero unless $P\le eg-e$. The sum $\bigoplus_{Z_1} A(a,b,c,d)$ is the direct sum of the complexes $\mathbb{P}(a_0,c_0,d_0)$, as (a_0,c_0,d_0) vary over the elements of V_1. Each such complex has free homology equal to $\bar{A}(a_0,b,c_0,d_0)$. $\square$

Examples 4.12. (a) If $g=1$, then the ideal J of Definition 2.6 is the order ideal of $\lambda(\omega_{F^*})\in F^*$, and $\mathbb{I}$ is the Koszul complex associated to $\lambda(\omega_{F^*})$. Indeed, $\mathbb{I}$ is
$$\bar{B}(0)\oplus\bar{A}(0,0,e,0)\oplus\bigoplus_{c\le e-1}\bar{A}(0,e-c,c,0),$$
with $\bar{B}(0)=R$ in position 0, $\bar{A}(0,0,e,0)=R$ in position f, and $\bar{A}(0,e-c,c,0)$ equal to $\bigwedge^{e-c}F^*\cong\bigwedge^{c+1}F$ in position $c+1$.
(b) If $e=1$, then $\mathbb{I}$ is the complex $\mathbb{P}'$ of Definition 2.14 (after the isomorphism $B_f\colon \bigwedge^f F^*\to\bigwedge^{f-1}G^*$ has been split off). Indeed, $\mathbb{I}$ is
$$\bigoplus_{d\le g-1}\bar{B}(d)\oplus\bigoplus_{d\le g-1}\bar{A}(0,0,1,d)\oplus\bigoplus_{a\le g-1}\bar{A}(a,a+1,0,0),$$
with $\bar{B}(d)=\bigwedge^d G^*$ in position d, $\bar{A}(0,0,1,d)\cong\bigwedge^{d+1}G^*$ in position $d+2$, and $\bar{A}(a,a+1,0,0)=\bigwedge^{a+1}F^*$ in position $a+1$.

Convention 4.13. In the next result we have integers a, c, d, P, and Q with $1-e\le P-Q\le g-1$ and $a+d=P$ and $c+d=Q$. The *homology of $S(P,Q)$ at spot* (a,c,d), is denoted $\mathrm{H}_S(a,c,d)$, and is equal to the homology of

$$A(a+1,b,c+1,d-1)\xrightarrow{\boldsymbol{\partial}} A(a,b,c,d)\xrightarrow{\boldsymbol{\partial}} A(a-1,b,c-1,d+1),$$

where $b=a-c+e$. Of course, this is equal to $\mathrm{H}_i(S(P,Q))$ for $i=a+c+d+1$. Furthermore, we have

$$\mathrm{H}_S(a,c,d)=\mathrm{H}_{\mathcal{M}}(a,c,d)\otimes \textstyle\bigwedge^b F^*,$$

where $\mathrm{H}_{\mathcal{M}}(a,c,d)$ is the homology module of [19].

The following result is [19, Thm. 5.7].

Theorem 4.14. *Let $S=S(P,Q)$. Assume $1-e\le P-Q\le g-1$.*
(0) *If (P,Q) is equal to $(1,1)$ or $(eg-e-1,eg-g-1)$, then S is split exact.*
(1) *If $2\le P=Q\le 4$, then S has free homology concentrated at spot $(1,1,P-1)$.*
(2) *If $2\le eg-e-P=eg-g-Q\le 4$, then S has free homology concentrated at spot $(g-2,e-2,P-g+2)$.*
(3) *If $2Q-1\le P$, then S has free homology concentrated at spot $(P-Q,0,Q)$.*
(4) *If $2P-1\le Q$, then S has free homology concentrated at spot $(0,Q-P,P)$.*
(5) *If $eg-2g+e-1\le 2Q-P$, then S has free homology concentrated at spot $(P-Q-1+e,e-1,Q+1-e)$.*
(6) *If $eg-2e+g-1\le 2P-Q$, then S have free homology concentrated at spot $(g-1,Q-P+g-1,P-g+1)$.*
(7) *If $(P,Q)=(3,4)$ or $(eg-e-3,eg-g-4)$, then S have free homology. The non-zero homology modules have rank*

$$\binom{g}{4}\binom{e+2}{3}\binom{e+g}{e-1}\ \textit{at spot}\ (0,1,3)\ \textit{or}\ (g-1,e-2,\boldsymbol{\alpha}-3)\ \textit{and}$$

$$\frac{g}{2}\binom{e}{3}\binom{g+1}{3}\binom{e+g}{e-1}\ \textit{at spot}\ (1,2,2)\ \textit{or}\ (g-2,e-3,\boldsymbol{\alpha}-2).$$

(8) *If $(P,Q)=(4,3)$ or $(eg-e-4,eg-g-3)$, then S has free homology. The non-zero homology modules have rank*

$$\binom{e}{4}\binom{g+2}{3}\binom{e+g}{e+1}\ \textit{at spot}\ (1,0,3)\ \textit{or}\ (g-2,e-1,\boldsymbol{\alpha}-3)\ \textit{and}$$

$$\frac{e}{2}\binom{g}{3}\binom{e+1}{3}\binom{e+g}{e+1}\ \textit{at spot}\ (2,1,2)\ \textit{or}\ (g-3,e-2,\boldsymbol{\alpha}-2).$$

Note. Complexes (1)–(6) have free homology concentrated in one position. For such a complex, the rank of the homology is easy to compute, because the complex and the homology of the complex have the same Euler characteristic.

Theorem 4.15. *Let $\mathbb{F}$ be the complex of Definition 2.3. Let $\mathfrak{S}$ be the set of pairs of integers (P,Q) which satisfy all of the following conditions:*

$$\begin{array}{l} 1-e \le P-Q \le g-1, \\ 4 \le P \le eg-e-4, \\ 4 \le Q \le eg-g-4, \\ 2 \le 2P-Q \le eg+g-2e-2, \\ 2 \le 2Q-P \le eg+e-2g-2, \\ (P,Q) \ne (4,4), \text{ and} \\ (P,Q) \ne (eg-e-4, eg-g-4). \end{array}$$

Let W_0 and W_2 be the following sets of triples of non-negative integers

$$W_0 = \{(a,c,d) \mid (a+d,c+d) \in \mathfrak{S},\ a \le g-1,\ c \le e-1,\ (g-1-a)(e-1-c)=0\}, \text{ and}$$

$$W_2 = \{(a,c,d) \mid (a+d,c+d) \in \mathfrak{S},\ a \le g-2,\ c \le e-2\}.$$

Let $\boldsymbol{T} = T \setminus \{(0,0,1),(g-1,e-1,\boldsymbol{\alpha}-1)\}$, where T is the set of all triples (a,c,d) of non-negative integers $\bigcup\limits_{i=1}^{8} T_i$, with

$$\begin{aligned} T_1 &= \{(1,1,d) \mid 1 \le d \le 3,\ d \le \boldsymbol{\alpha}\}, \\ T_2 &= \{(g-2,e-2,d) \mid \boldsymbol{\alpha}-3 \le d \le \boldsymbol{\alpha}-1\}, \\ T_3 &= \{(a,0,d) \mid a \le g-1,\ d \le a+1,\ a+d \le eg-e,\ d \le eg-g\}, \\ T_4 &= \{(0,c,d) \mid c \le e-1,\ d \le c+1,\ c+d \le eg-g,\ d \le eg-e\}, \\ T_5 &= \{(a,e-1,d) \mid a \le d-\boldsymbol{\alpha}+g,\ d \le \boldsymbol{\alpha},\ a \le g-1\}, \\ T_6 &= \{(g-1,c,d) \mid c \le d+e-\boldsymbol{\alpha},\ d \le \boldsymbol{\alpha},\ c \le e-1\}, \\ T_7 &= \chi(2 \le g)\chi(4 \le eg-g)\{(0,1,3),(1,2,2)\} \\ &\quad \cup \{(g-1-\ell, e-2-\ell, \boldsymbol{\alpha}-3+\ell) \mid 0 \le \ell \le 1\}, \\ T_8 &= \chi(4 \le eg-e)\chi(2 \le e)\{(1,0,3),(2,1,2)\} \\ &\quad \cup \{(g-2-\ell, e-1-\ell, \boldsymbol{\alpha}-3+\ell) \mid 0 \le \ell \le 1\}. \end{aligned}$$

Then, there exists a split exact subcomplex $\mathbb{N}$ of $\mathbb{F}$, such that quotient complex $\mathbb{J} = \mathbb{F}/\mathbb{N}$ is isomorphic, as a graded module, to

$$\bigoplus_{d \le eg-e} \bar{B}(d) \oplus \bigoplus_{d \le eg-e} \bar{A}(0,0,e,d) \oplus \bigoplus_{\boldsymbol{T}} \mathrm{H}_S(a,c,d) \oplus \bigoplus_{W_0} \bar{A}(a,b,c,d) \oplus \bigoplus_{W_2} A(a,b,c,d).$$

Each of the modules $\bar{B}(d)$, $\bar{A}(a,b,c,d)$, and $\mathrm{H}_S(a,c,d)$ which appears above, is a free module.

Remarks 4.16. (a) Some triples (a_0,c_0,d_0) are elements of more than one of the sets T_i; nonetheless, if (a_0,c_0,d_0) is in $\boldsymbol{T}$, then the module $\mathrm{H}_S(a_0,c_0,d_0)$ is a summand of $\bigoplus_{\boldsymbol{T}} \mathrm{H}_S(a,c,d)$ exactly one time.
(b) In the present context, $\chi(2 \le g)$ may be read "only if $2 \le g$"; see Convention 1.14.

(c) The complex $\mathbb{J}$ has length $eg+1$. Once the integers (a,c,d) have been identified, then $b=a-c+e$.
(d) The listed free modules $\bar{B}(d)$, $\bar{A}(0,0,e,d)$, and $\bar{A}(a,b,c,d)$ have rank equal to $|S(d,d-g)|$, $|S(d,e+d)|$, and $|\mathbb{P}(a,c,d)|$, respectively. (See Example 2.11 and Definition 4.6). The rank of $\mathrm{H}_S(a,c,d)$ is equal $|S(a+d,c+d)|$ for all (a,c,d) in $\cup_{i=1}^{6}T_i$. For each (a,c,d) in $T_7\cup T_8$, the rank of $\mathrm{H}_S(a,c,d)$ is given in Theorem 4.14. If the grading convention of 2.2 is in effect, then the complex $\mathbb{J}$ is homogeneous with twists given in 2.4. The module $\bar{B}(d)$ is a summand of $\mathbb{J}_d$. The module $A(a,b,c,d)$, $\bar{A}(a,b,c,d)$, or $\mathrm{H}_S(a,c,d)$, from the statement of Theorem 4.15, is a summand of $\mathbb{J}_i$, provided $i=a+c+d+1$.
(e) Notice that each module $\mathbb{J}_i$, in the complex $\mathbb{J}$, naturally decomposes into a direct sum of submodules $\bigoplus \mathbb{J}_i(P,Q)$, as (P,Q) varies over all of the strands $S(P,Q)$ of $\mathbb{F}$. See the last sentence in the proof of Observation 4.3, where we realize $\mathbb{J}_i$ as the "$\mathfrak{P}$-graded" summand "$\mathbb{M}_i$" of $\mathbb{F}_i$.

Proof of 4.15. Take U from (4.11). Decompose $\mathbb{F}$ into $\mathbb{K}\oplus\mathbb{L}$, where

$$\mathbb{K}=\bigoplus_{W_2}A(a,b,c,d)\quad\text{and}\quad \mathbb{L}=\bigoplus_{d}B(d)\oplus\bigoplus_{U\setminus W_2}A(a,b,c,d).$$

We use Theorems 4.7 and 4.14 to show that $(\mathbb{L},\boldsymbol{\partial})$ is splittable. Observation 4.3 completes the proof.

The set U decomposes into the disjoint union of $U_1\cup U_2$, with

$$U_1=\{(a,c,d)\in U\mid (a+d,c+d)\in\mathfrak{S}\}\text{ and }U_2=\{(a,c,d)\in U\mid (a+d,c+d)\notin\mathfrak{S}\}.$$

It is clear that $U_1=Z_0\cup W_2$ and $U_2=Z_3\cup Z_4\cup Z_5\cup Z_6$ are disjoint unions, for

$$\begin{aligned}Z_0&=\{(a,c,d)\in U_1\mid\ g-1\le a\text{ and/or }e-1\le c\},\\ Z_6&=\{(a,c,d)\in U_2\mid 1-e\le a-c\le g-1,\ a+d\le eg-e,\text{ and }c+d\le eg-g\},\end{aligned}$$

and Z_3, Z_4, and Z_5 as given in the proof of Theorem 4.10. In the proof of Theorem 4.10, we showed that

$$\bigoplus_{d}B(d)\oplus\bigoplus_{Z_3\cup Z_4\cup Z_5}A(a,b,c,d)$$

is a splittable complex with free homology equal to

$$\bigoplus_{d\le eg-e}\bar{B}(d)\oplus\bigoplus_{d\le eg-e}\bar{A}(0,0,e,d).$$

The argument from the same proof yields that

$$\bigoplus_{Z_0}A(a,b,c,d)=\bigoplus_{W_0}\mathbb{P}(a_0,c_0,d_0);$$

and hence, by Theorem 4.7 (d), $\bigoplus_{Z_0}A(a,b,c,d)$ is a splittable complex with free homology equal to $\bigoplus_{W_0}\bar{A}(a,b,c,d)$.

Let $\mathfrak{R}$ be the set of pairs of integers

$$\mathfrak{R} = \{(P,Q) \mid 1-e \le P-Q \le g-1,\ P \le eg-e,\ Q \le eg-g\}.$$

For each integer i, with $1 \le i \le 8$, let

$$\mathfrak{T}_i = \{(P,Q) \in \mathfrak{R} \mid (P,Q) \text{ satisfies hypothesis } (i) \text{ of Theorem 4.14}\}.$$

Let

$$\mathfrak{T} = \bigcup_{i=1}^{8} \mathfrak{T}_i.$$

Observe that the sets $\mathfrak{R} \setminus \mathfrak{S}$ and $\mathfrak{T}$ are equal. It follows that

$$Z_6 = \{(a,c,d) \in U \mid (a+d, c+d) \in \mathfrak{T}\};$$

and therefore,

$$\bigoplus_{Z_6} A(a,b,c,d) = \bigoplus_{\mathfrak{T}} S(P,Q).$$

For each i, assertion (i) of Theorem 4.14 establishes that the complex $\bigoplus_{\mathfrak{T}_i} S(P,Q)$ is splittable with free homology equal to $\bigoplus_{T_i} \mathrm{H}_S(a,c,d)$. These eight results combine to say that $\bigoplus_{\mathfrak{T}} S(P,Q)$ is a splittable complex with free homology equal to $\bigoplus_T \mathrm{H}_S(a,c,d)$. The triples $(0,0,1)$ and $(g-1, e-1, \boldsymbol{\alpha}-1)$ might be elements of T. However, the corresponding modules $\mathrm{H}_S(0,0,1)$ and $\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha}-1)$ are zero by Theorem 4.14; and as a consequence, $\bigoplus_T \mathrm{H}_S(a,c,d) = \bigoplus_{\boldsymbol{T}} \mathrm{H}_S(a,c,d)$.

The complex $\mathbb{L}$ is a direct sum of splittable subcomplexes and the proof is complete. $\square$

The complex $\mathbb{J}$ may be decomposed as the direct sum of two submodules:

$$\mathbb{T} = \bigoplus_{d \le eg-e} \bar{B}(d) \oplus \bigoplus_{d \le eg-e} \bar{A}(0,0,e,d) \oplus \bigoplus_{\boldsymbol{T}} \mathrm{H}_S(a,c,d) \tag{4.17}$$

and

$$\mathbb{W} = \bigoplus_{W_0} \bar{A}(a,b,c,d) \oplus \bigoplus_{W_2} A(a,b,c,d). \tag{4.18}$$

Theorem 5.4 says that when the term "minimal resolution" makes sense, (that is, the data is local, or graded over a field), then $\mathbb{T}$, which is the tame part of $\mathbb{J}$, is a module direct summand of the minimal resolution of $\mathrm{H}_0(\mathbb{J})$. Section 7 consists of examples of this phenomenon. We have not investigated exactly how much splitting occurs in the wild part, $\mathbb{W}$, of $\mathbb{J}$; but Theorem 6.3 tells us that the answer depends on the characteristic of the base ring.

5. The ideal $\mathcal{J}$ is generically perfect.

The next result is essentially contained in [25], where, among many other things, it is shown that $\mathcal{R}$ is a Cohen-Macaulay ring and a free $\mathbb{Z}$-module. The grade of $\mathcal{J}$ is also known by Tchernev. Our proof of Corollary 5.1 uses Tchernev's description of $\mathcal{R}$, as given in (0.2); but, it is otherwise independent of Tchernev's techniques. Now that we have resolved $\mathcal{R}$, we are able to read the perfection of $\mathcal{R}$ and the grade of $\mathcal{J}$ directly from the resolution.

Corollary 5.1. *If $\mathcal{R} = \mathcal{P}/\mathcal{J}$ is the universal ring of* (0.2)*, then $\mathcal{R}$ is generically perfect as a $\mathcal{P}$-module and $\mathcal{J}$ has grade equal to $eg+1$.*

Proof. In light of Example 4.12, we may assume $2 \le e$ and $2 \le g$. Let $\mathbb{F}$ be the complex of Definition 2.3 which is built using the generic data $(\mathcal{B}, \mathcal{V}, \mathcal{X})$ over the polynomial ring $\mathcal{P}$ and let $\mathbb{J}$ be the length $eg+1$ subcomplex of $\mathbb{F}$ which is introduced in Corollary 4.15. Theorem 3.1 and Observation 2.7 tell us that $\mathbb{F}$ and $\mathbb{J}$ both are resolutions of $\mathcal{R} = \mathcal{P}/\mathcal{J}$ by free $\mathcal{P}$-modules.

Let R_0 be a fixed, but arbitrary, commutative noetherian ring. Let $\mathcal{R}_0$ equal $\mathcal{R} \otimes_{\mathbb{Z}} R_0$, $\mathcal{P}_0 = \mathcal{P} \otimes_{\mathbb{Z}} R_0$, and $\mathcal{J}_0 = \mathcal{J}\mathcal{P}_0$. We apply [6, Proposition 3.2] and prove that $\mathcal{R}$ is generically perfect by showing that $\mathcal{R}_0$ is a perfect $\mathcal{P}_0$-module and $\mathcal{J}_0$ has grade $eg+1$. Theorem 3.1 continues to apply to the generic data $(\mathcal{B}, \mathcal{V}, \mathcal{X})$ over the polynomial ring $\mathcal{P}_0$. Observation 2.7 and Corollary 4.15 also continue to apply after we perform a base change to $\mathcal{P}_0$. It follows that $\mathbb{F} \otimes_{\mathcal{P}} \mathcal{P}_0$ and $\mathbb{J} \otimes_{\mathcal{P}} \mathcal{P}_0$ both are resolutions of $\mathcal{R}_0 = \mathcal{P}_0/\mathcal{J}_0$ by free $\mathcal{P}_0$-modules. Observe that the following string of inequalities holds:

$$eg+1 \le \operatorname{grade} \mathcal{J}_0 \le \operatorname{pd} \mathcal{P}_0/\mathcal{J}_0 \le eg+1. \tag{5.2}$$

Indeed, the middle inequality is true for any ideal of $\mathcal{P}_0$. The right most inequality is clear because $\mathbb{J} \otimes_{\mathcal{P}} \mathcal{P}_0$ is a length $eg+1$ resolution of $\mathcal{P}_0/\mathcal{J}_0$ by free $\mathcal{P}_0$-modules. The left most inequality holds because the acyclic complex $\mathbb{J} \otimes_{\mathcal{P}} \mathcal{P}_0$ looks like

$$0 \to \bar{A}(0,0,e,eg-e) \xrightarrow{\bar{\boldsymbol{d}}_{eg+1}} \dots,$$

with $\bar{A}(0,0,e,eg-e) \cong \mathcal{P}_0$. The exactness criterion of [7] guarantees that if the ideal $I_1(\bar{\boldsymbol{d}}_{eg+1})$ is proper, then $eg+1 \le \operatorname{grade} I_1(\bar{\boldsymbol{d}}_{eg+1})$. Proposition 8.1 shows that $I_1(\bar{\boldsymbol{d}}_{eg+1}) = I_1(\boldsymbol{d}_1) = \mathcal{J}_0$.

Equality holds everywhere in (5.2). It follows that $\mathcal{R}_0$ is perfect and $\mathcal{J}_0$ has grade $eg+1$. The proof is complete. $\square$

Corollary 5.3. *Let R be a commutative noetherian ring, $\mathfrak{b}$ be an element of R, X and V be matrices over R of size $g \times f$ and $f \times e$, respectively. Let J be the ideal of Definition 2.6 and let $\mathbb{F}$ be the complex of Definition 2.3 which is built from the data $(\mathfrak{b}, V, X)$. If J is a proper ideal and $eg+1 \le \operatorname{grade} J$, then J is a perfect Gorenstein ideal of grade equal to $eg+1$ and the complexes $\mathbb{F}$, $\mathbb{G}$, $\mathbb{I}$, and $\mathbb{J}$ all are resolutions of R/J by free R-modules.*

Proof. The complex $\mathbb{F}$, the ideal J, and the quotient ring R/J are obtained from the generic templates $\mathbb{F}(\mathcal{B}, \mathcal{V}, \mathcal{X})$, $\mathcal{J}$, and $\mathcal{R}$ of Corollary 5.1 by way of the base change $\mathcal{P} \to R$. The theorem about the transfer of perfection (see, for example, [6, Theorem 3.5]) tells us that J is a perfect ideal of grade equal to $eg+1$ and the complex $\mathbb{F}$ is a resolution of R/J by free R-modules. Proposition 8.1 tells us that $\mathrm{H}_{-(eg+1)}(\mathbb{F}^*) = \mathrm{H}_{-(eg+1)}(\mathbb{J}^*) = R/J$. We conclude that $\mathrm{Ext}_R^{eg+1}(R/J, R) = R/J$; hence, J is a Gorenstein ideal and the proof is complete. $\square$

The next result does not depend on the characteristic; however section 6 shows that there is not much room for extending this result. We record examples in section 7.

Theorem 5.4. *Let the ideal J and the ring R satisfy the hypotheses of Corollary* 5.3. *Suppose further, that either*
(a) $(R, \mathfrak{m})$ *is a local ring and* $\mathfrak{b}$, $I_1(X)$, *and* $I_1(V)$ *are contained in* $\mathfrak{m}$, *or else,*
(b) $R = \bigoplus_{0 \le i} R_i$ *is a graded ring, with* R_0 *a field, and the data* $(\mathfrak{b}, V, X)$ *satisfies the grading convention* 2.2 *with* $\deg \mathfrak{b}$, d_x, *and* d_v *all positive.*
Let $\mathbb{T}$ *be the direct summand of* $\mathbb{J}$ *which is given in* (4.17). *If* $\mathbb{M}$ *is the minimal resolution of* R/J *by free* R*-modules, then the module* $\mathbb{T}$ *is a direct summand of* $\mathbb{M}$. *If hypothesis* (b) *is in effect then* $\mathbb{T}$ *is a direct summand of* $\mathbb{M}$ *as graded modules.*

Remark. The complement $\mathbb{W}$ of $\mathbb{T}$ in $\mathbb{J}$ is given in (4.18). If $\mathbb{W}_i = 0$, then $\mathbb{J}_i = \mathbb{M}_i$. If $\mathbb{W} = 0$, then $\mathbb{J} = \mathbb{M}$.

Proof. If hypothesis (b) is in effect, then let $\mathfrak{m}$ equal the irrelevant maximal ideal $\bigoplus_{0<i} R_i$ of R. The complex $(\mathbb{J}, \bar{\boldsymbol{d}})$ is defined to be $\mathbb{F}/\mathbb{N}$ for a subcomplex $\mathbb{N}$ of $(\mathbb{F}, \boldsymbol{d})$. Let $'$ represent the functor $\_\_ \otimes_R \boldsymbol{k}$, with $\boldsymbol{k} = R/\mathfrak{m}$. We study $\mathbb{J}' = \mathbb{F}'/\mathbb{N}'$. Remark 2.4 tells us that the differential $\boldsymbol{d}'$ on $\mathbb{F}'$ is equal to the homogeneous map $\boldsymbol{\partial}'$; and therefore, the complex $(\mathbb{F}', \boldsymbol{d}')$ is equal to the direct sum of the complexes $(S(P,Q)', \boldsymbol{\partial}')$. In the proof of Theorem 4.15, we decomposed $\mathbb{F}$, as a module into

$$\bigoplus_{(P,Q)\in\mathfrak{S}} S(P,Q) \oplus \bigoplus_{(P,Q)\notin\mathfrak{S}} S(P,Q). \tag{5.5}$$

The left hand summand of (5.5) is the same as $\bigoplus_{U_1} A(a,b,c,d)$ and this summand contributes the summand $\mathbb{W}$ to $\mathbb{J}$. The right hand summand of (5.5) equals the submodule

$$\bigoplus_{d} B(d) \oplus \bigoplus_{U_2} A(a,b,c,d) \tag{5.6}$$

of $\mathbb{F}$. In 4.15 we decompose the complex (5.6), with differential $\boldsymbol{\partial}$, as the direct sum of two subcomplexes $\mathbb{P} \oplus \mathbb{Q}$, where $\mathbb{P}$ has differential zero and $\mathbb{Q}$ is split exact. We do not put $\mathbb{Q}$ directly into $\mathbb{N}$, there are splitting maps involved. But every map between two different strands $S(P,Q)$ in $\mathbb{F}$ becomes zero in $\mathbb{F}'$, so we do have $\mathbb{Q} \subset \mathbb{N} + \mathfrak{m}\mathbb{F}$. The image of $\mathbb{P}$ in $\mathbb{J}$ is called $\mathbb{T}$ and the compositions

$$\mathbb{J}' \xrightarrow{\bar{\boldsymbol{d}}'} \mathbb{J}' \xrightarrow{\text{proj}} \mathbb{T}' \quad \text{and} \quad \mathbb{T}' \xrightarrow{\text{incl}} \mathbb{J}' \xrightarrow{\bar{\boldsymbol{d}}'} \mathbb{J}'$$

are zero. $\square$

Corollary 5.7. *If the set $\mathfrak{S}$ of Theorem 4.15 is empty, then the complex $\mathbb{J}$ is a generic minimal resolution of the universal ring $\mathcal{R}$ of* (0.2).

Proof. The proof of Theorem 5.4 shows that the differential in $\mathbb{J} \otimes_{\mathcal{P}} \mathbb{Z}$ is identically zero; see Definition 6.1. □

Examples of Corollary 5.7 are given at the beginning of section 7.

6. There does not exist a generic minimal resolution of $\mathcal{R}$ by free $\mathcal{P}$ modules.

Definition 6.1. Let $\mathcal{P} = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring over the ring of integers. View $\mathcal{P}$ as a graded ring in which each indeterminate x_i has positive degree. Let M be a finitely generated, generically perfect, graded $\mathcal{P}$-module and let $\mathbb{X}$ be a homogeneous resolution of M by free $\mathcal{P}$-modules. The resolution $\mathbb{X}$ is a *generic minimal resolution* of M if the differential in the complex $\mathbb{X} \otimes_{\mathcal{P}} \mathbb{Z}$ is zero.

We access the concept of generic minimal resolutions by way of the following result of Paul Roberts [24, Prop. 2 on pg. 105]. (See, also, [13, Prop. II.3.4].) This is the approach that Hashimoto [11,12] took in his proof that determinantal ideals do not, in general, possess a generic minimal resolution.

Proposition 6.2. *Let $\mathcal{P} = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring over the ring of integers. View $\mathcal{P}$ as a graded ring in which each indeterminate x_i has positive degree. If M is a finitely generated, generically perfect, graded $\mathcal{P}$-module, then there exists a generic minimal resolution of M over $\mathcal{P}$ if and only if* $\mathrm{Tor}_i^{\mathcal{P}}(M, \mathbb{Z})$ *is a free abelian group for all integers i.*

In both the Definition and the Proposition, $\mathbb{Z}$ is a $\mathcal{P}$-module by way of the natural quotient map $\mathcal{P} \to \mathcal{P}/(\{x_i\}) = \mathbb{Z}$.

Theorem 6.3. *Let $\mathcal{R} = \mathcal{P}/\mathcal{J}$ be the universal ring of* (0.2). *If $5 \le e$ and $5 \le g$, then there does not exist a generic minimal resolution of $\mathcal{R}$ by free $\mathcal{P}$-modules.*

Proof. We prove that $\mathrm{Tor}_7^{\mathcal{P}}(\mathcal{R}, \mathbb{Z})$ is not a free abelian group. The complex $\mathbb{F}$ of 2.3 is a resolution of $\mathcal{R}$; and therefore, $\mathrm{Tor}_\bullet^{\mathcal{P}}(\mathcal{R}, \mathbb{Z})$ is the homology of the complex $\mathbb{F} \otimes_{\mathcal{P}} \mathbb{Z}$. This is the complex $\mathbb{F}$ with all of the variables set equal to zero. In other words, $\mathbb{F} \otimes_{\mathcal{P}} \mathbb{Z}$ is equal to the direct sum of the complexes $(S(P,Q), \boldsymbol{\partial})$, where each $S(P,Q)$ is constructed with base ring $\mathbb{Z}$. We are particularly interested in the complex $S(5,5)$, which, in the language of [19], is equal to $\mathbb{M}(5,5) \otimes \bigwedge^e F^*$. The homology module $\mathrm{H}_{\mathcal{M}}(1,1,4)$ of the complex $\mathbb{M}(5,5)$ is not a free abelian group. See [19, Cor. 0.8]. □

7. Examples of the complex $\mathbb{J}$.

If the parameters (e,g) satisfy $e \le 2$, or $g \le 2$, or (e,g) is equal to $(3,3)$, or $(3,4)$ or $(4,3)$, then the complex $\mathbb{J}$ of Theorem 4.15 is a generic minimal resolution of the universal ring $\mathcal{R}$ of (0.2) by free $\mathcal{P}$-modules. If $e = 1$ or $g = 1$, then the complex $\mathbb{I} = \mathbb{J}$ is given in Example 4.12. Theorem 7.1 exhibits $\mathbb{J}$ for $e = 2$ or $g = 2$. If $(e,g) = (3,4)$, then the complex $\mathbb{J}$ is given in Example 7.3.

In Examples 7.4 and 7.6, the hypotheses and notation of Theorem 5.4 are in effect with $3 \le e$ and $3 \le g$. In particular, there exists a minimal resolution $\mathbb{M}$ of

R/J by free R-modules and the module $\mathbb{T}$ of (4.17) is a (graded) summand of $\mathbb{M}$. If $i \le 4$ or $eg-3 \le i$, then $\mathbb{W}_i = 0$ and $\mathbb{M}_i = \mathbb{T}_i = \mathbb{J}_i$. The left and right sides of $\mathbb{M}$ are recorded in Examples 7.4 and 7.6.

Theorem 7.1. **(a)** *If $e=2$ and g is arbitrary, then $\mathbb{J}$ is a generic minimal $\mathcal{P}$-free resolution of the universal ring $\mathcal{R}$ of* (0.2) *and $\mathbb{J}$ is given by*

$$\bigoplus_{d\le 2g-2} \bar{B}(d) \oplus \bigoplus_{d\le 2g-2} \bar{A}(0,0,2,d) \oplus \bigoplus_{d\le a\le g-1} \mathrm{H}_S(a,0,d) \oplus \bigoplus_{a\le d\le g-1} \mathrm{H}_S(a,1,d),$$

where

module	*position*	*twist*	*rank*
$\bar{B}(d)$	d	$dd_v + dd_x$	$\binom{2g}{d} - (2g-d+1)\binom{g}{d-g}$
$\bar{A}(0,0,2,d)$	$d+3$	$dd_v + (g+d+2)d_x$	$\binom{2g}{d+2} - (d+3)\binom{g}{d+2}$
$\mathrm{H}_S(a,0,d)$	$a+d+1$	$(a+d)d_v + (g+d)d_x$	$(a-d+1)\binom{g}{d}\binom{g+2}{a+2}$
$\mathrm{H}_S(a,1,d)$	$a+d+2$	$(a+d)d_v + (g+d+1)d_x$	$(d-a+1)\binom{g}{d+1}\binom{g+2}{a+1}$.

(b) *If $g=2$ and e is arbitrary, then $\mathbb{J}$ is a generic minimal $\mathcal{P}$-free resolution of $\mathcal{R}$ and $\mathbb{J}$ is given by*

$$\bigoplus_{d\le e} \bar{B}(d) \oplus \bigoplus_{d\le e} \bar{A}(0,0,e,d) \oplus \bigoplus_{d\le c\le e-1} \mathrm{H}_S(0,c,d) \oplus \bigoplus_{c\le d\le e-1} \mathrm{H}_S(1,c,d),$$

where

module	*position*	*twist*	*rank*
$\bar{B}(d)$	d	$dd_v + dd_x$	$(d+1)\binom{e}{d}$
$\bar{A}(0,0,e,d)$	$d+e+1$	$dd_v + (e+d+2)d_x$	$(e-d+1)\binom{e}{d}$
$\mathrm{H}_S(0,c,d)$	$c+d+1$	$dd_v + (2+c+d)d_x$	$(c-d+1)\binom{e}{d}\binom{e+2}{c+2}$
$\mathrm{H}_S(1,c,d)$	$c+d+2$	$(1+d)d_v + (2+c+d)d_x$	$(d-c+1)\binom{e}{d+1}\binom{e+2}{c+1}$.

Proof. We prove (a). The proof of (b) is similar. The hypothesis $e=2$ ensures that the set $\mathfrak{S}$ is empty since every pair (P,Q) in $\mathfrak{S}$ would have to satisfy $2 \le 2Q-P \le 0$. Apply Corollary 5.7. The resolution $\mathbb{J}$ of $\mathcal{R}$ is given in Theorem 4.15. The resolution $\mathbb{J}$ is equal to

$$\bigoplus_{d\le 2g-2} \bar{B}(d) \oplus \bigoplus_{d\le 2g-2} \bar{A}(0,0,e,d) \oplus \bigoplus_{T} \mathrm{H}_S(a,c,d).$$

It is not difficult to see that T (in the notation of 4.15) is equal to $T_3 \cup T_5 \cup T_7 \cup T_8$ for

$$\begin{aligned}
T_3 &= \{(a,0,d) \mid a \le g-1,\ d \le a+1,\ a+d \le 2g-2\},\\
T_5 &= \{(a,1,d) \mid a \le d+1,\ d \le g-1,\ a \le g-1\},\\
T_7 &= \{(1,2,2)\},\\
T_8 &= \{(1,0,3),(g-2,1,g-4)\}.
\end{aligned}$$

Furthermore, if $(a,c,d) \in T$ and $H_S(a,c,d)$ is not zero, then (a,c,d) is in

$$\{(a,0,d) \mid d \le a \le g-1\} \cup \{(a,1,d) \mid a \le d \le g-1\}.$$

Indeed, if (a,c,d) is in $T_7 \cup T_8$, then Theorem 4.14 shows that the rank of $\mathrm{H}_S(a,c,d)$ is zero. If (a,c,d) is equal to $(a,0,a+1) \in T_3$ or $(a,1,a-1) \in T_5$, then [19, Thm. 4.1] shows that the rank of $\mathrm{H}_S(a,c,d)$ is zero. The modules' positions and twists are given in Remark 2.4; the ranks are given in [19, Thms. 4.1 and 2.1]. $\square$

Example 7.2. If $e = 2$ and $g = 2$, then the generic minimal resolution of $\mathcal{R}$ is the direct sum of the following modules:

module	p.	strand	twist	rank	module	p.	strand	twist	rank
$\bar{B}(0)$	0	$S(0,-2)$	$0d_v + 0d_x$	1	$\bar{A}(0,0,2,2)$	5	$S(2,4)$	$2d_v + 6d_x$	1
$\bar{B}(1)$	1	$S(1,-1)$	$1d_v + 1d_x$	4	$\bar{A}(0,0,2,1)$	4	$S(1,3)$	$1d_v + 5d_x$	4
$\mathrm{H}_S(0,0,0)$	1	$S(0,0)$	$0d_v + 2d_x$	6	$\mathrm{H}_S(1,1,1)$	4	$S(2,2)$	$2d_v + 4d_x$	6
$\mathrm{H}_S(1,0,0)$	2	$S(1,0)$	$1d_v + 2d_x$	8	$\mathrm{H}_S(0,1,1)$	3	$S(1,2)$	$1d_v + 4d_x$	8
$\bar{B}(2)$	2	$S(2,0)$	$2d_v + 2d_x$	3	$\bar{A}(0,0,2,0)$	3	$S(0,2)$	$0d_v + 4d_x$	3
$\mathrm{H}_S(0,1,0)$	2	$S(0,1)$	$0d_v + 3d_x$	8	$\mathrm{H}_S(1,0,1)$	3	$S(2,1)$	$2d_v + 3d_x$	8.

The label "p." stands for "position". We make certain that our meaning is clear by writing $\mathbb{J}$ another way. If $d_v = 1$ and $d_x = \deg \mathfrak{b} = 2$, then the grading convention of 2.2 is satisfied and $\mathbb{J}$ is

$$0 \to R(-14) \to \begin{matrix} R(-10)^6 \\ \oplus \\ R(-11)^4 \end{matrix} \to \begin{matrix} R(-8)^{11} \\ \oplus \\ R(-9)^8 \end{matrix} \to \begin{matrix} R(-5)^8 \\ \oplus \\ R(-6)^{11} \end{matrix} \to \begin{matrix} R(-3)^4 \\ \oplus \\ R(-4)^6 \end{matrix} \to R.$$

We listed each module in the resolution $\mathbb{J}$ next to its dual. In each case, the twist of the module plus the twist of its dual add up to $(eg-e)d_v + (eg+g)d_x$, as expected. Each module $\mathbb{J}_i$, in the complex $\mathbb{J}$, naturally decomposes into a direct sum of submodules $\bigoplus \mathbb{J}_i(P,Q)$, as (P,Q) varies over all of the strands $S(P,Q)$ of $\mathbb{F}$. In Observation 2.9 we imposed the inverse lexicographic order on the set $\{(P,Q)\}$. In the above table, for a fixed position i, we have listed the summands of $\mathbb{J}_i$ in increasing order by strand in the left column and in decreasing order in the right column.

Example 7.3. If $e = 3$ and $g = 4$, then the set $\mathfrak{S}$ of Theorem 4.15 is empty and $\mathbb{J}$ is the minimal resolution of R/J. Furthermore, $\mathbb{J}$ is the direct sum of the following modules:

module	p.	strand	twist	rank	module	p.	strand	twist	rank
$\bar{B}(0)$	0	$S(0,-4)$	$0d_v+0d_x$	1	$\bar{A}(0,0,3,9)$	13	$S(9,12)$	$9d_v+16d_x$	1
$\bar{B}(1)$	1	$S(1,-3)$	$1d_v+1d_x$	12	$\bar{A}(0,0,3,8)$	12	$S(8,11)$	$8d_v+15d_x$	12
$\mathrm{H}_S(0,0,0)$	1	$S(0,0)$	$0d_v+4d_x$	35	$\mathrm{H}_S(3,2,6)$	12	$S(9,8)$	$9d_v+12d_x$	35
$\bar{B}(2)$	2	$S(2,-2)$	$2d_v+2d_x$	66	$\bar{A}(0,0,3,7)$	11	$S(7,10)$	$7d_v+14d_x$	66
$\mathrm{H}_S(1,0,0)$	2	$S(1,0)$	$1d_v+4d_x$	105	$\mathrm{H}_S(2,2,6)$	11	$S(8,8)$	$8d_v+12d_x$	105
$\mathrm{H}_S(0,1,0)$	2	$S(0,1)$	$0d_v+5d_x$	84	$\mathrm{H}_S(3,1,6)$	11	$S(9,7)$	$9d_v+11d_x$	84
$\bar{B}(3)$	3	$S(3,-1)$	$3d_v+3d_x$	220	$\bar{A}(0,0,3,6)$	10	$S(6,9)$	$6d_v+13d_x$	220
$\mathrm{H}_S(2,0,0)$	3	$S(2,0)$	$2d_v+4d_x$	126	$\mathrm{H}_S(1,2,6)$	10	$S(7,8)$	$7d_v+12d_x$	126
$\mathrm{H}_S(1,0,1)$	3	$S(2,1)$	$2d_v+5d_x$	420	$\mathrm{H}_S(2,2,5)$	10	$S(7,7)$	$7d_v+11d_x$	420
$\mathrm{H}_S(0,2,0)$	3	$S(0,2)$	$0d_v+6d_x$	70	$\mathrm{H}_S(3,0,6)$	10	$S(9,6)$	$9d_v+10d_x$	70
$\mathrm{H}_S(0,1,1)$	3	$S(1,2)$	$1d_v+6d_x$	378	$\mathrm{H}_S(3,1,5)$	10	$S(8,6)$	$8d_v+10d_x$	378
$\mathrm{H}_S(3,0,0)$	4	$S(3,0)$	$3d_v+4d_x$	70	$\mathrm{H}_S(0,2,6)$	9	$S(6,8)$	$6d_v+12d_x$	70
$\bar{B}(4)$	4	$S(4,0)$	$4d_v+4d_x$	480	$\bar{A}(0,0,3,5)$	9	$S(5,8)$	$5d_v+12d_x$	480
$\mathrm{H}_S(2,0,1)$	4	$S(3,1)$	$3d_v+5d_x$	672	$\mathrm{H}_S(1,2,5)$	9	$S(6,7)$	$6d_v+11d_x$	672
$\mathrm{H}_S(1,1,1)$	4	$S(2,2)$	$2d_v+6d_x$	630	$\mathrm{H}_S(2,1,5)$	9	$S(7,6)$	$7d_v+10d_x$	630
$\mathrm{H}_S(1,0,2)$	4	$S(3,2)$	$3d_v+6d_x$	350	$\mathrm{H}_S(2,2,4)$	9	$S(6,6)$	$6d_v+10d_x$	350
$\bar{A}(0,0,3,0)$	4	$S(0,3)$	$0d_v+7d_x$	20	$\bar{B}(9)$	9	$S(9,5)$	$9d_v+9d_x$	20
$\mathrm{H}_S(0,2,1)$	4	$S(1,3)$	$1d_v+7d_x$	420	$\mathrm{H}_S(3,0,5)$	9	$S(8,5)$	$8d_v+9d_x$	420
$\mathrm{H}_S(0,1,2)$	4	$S(2,3)$	$2d_v+7d_x$	504	$\mathrm{H}_S(3,1,4)$	9	$S(7,5)$	$7d_v+9d_x$	504
$\mathrm{H}_S(3,0,1)$	5	$S(4,1)$	$4d_v+5d_x$	420	$\mathrm{H}_S(0,2,5)$	8	$S(5,7)$	$5d_v+11d_x$	420
$\bar{B}(5)$	5	$S(5,1)$	$5d_v+5d_x$	696	$\bar{A}(0,0,3,4)$	8	$S(4,7)$	$4d_v+11d_x$	696
$\mathrm{H}_S(2,0,2)$	5	$S(4,2)$	$4d_v+6d_x$	1386	$\mathrm{H}_S(1,2,4)$	8	$S(5,6)$	$5d_v+10d_x$	1386
$\mathrm{H}_S(1,1,2)$	5	$S(3,3)$	$3d_v+7d_x$	1820	$\mathrm{H}_S(2,1,4)$	8	$S(6,5)$	$6d_v+9d_x$	1820
$\bar{A}(0,0,3,1)$	5	$S(1,4)$	$1d_v+8d_x$	135	$\bar{B}(8)$	8	$S(8,4)$	$8d_v+8d_x$	135
$\mathrm{H}_S(0,2,2)$	5	$S(2,4)$	$2d_v+8d_x$	1050	$\mathrm{H}_S(3,0,4)$	8	$S(7,4)$	$7d_v+8d_x$	1050
$\mathrm{H}_S(0,1,3)$	5	$S(3,4)$	$3d_v+8d_x$	210	$\mathrm{H}_S(3,1,3)$	8	$S(6,4)$	$6d_v+8d_x$	210
$\mathrm{H}_S(3,0,2)$	6	$S(5,2)$	$5d_v+6d_x$	1050	$\mathrm{H}_S(0,2,4)$	7	$S(4,6)$	$4d_v+10d_x$	1050
$\bar{B}(6)$	6	$S(6,2)$	$6d_v+6d_x$	662	$\bar{A}(0,0,3,3)$	7	$S(3,6)$	$3d_v+10d_x$	662
$\mathrm{H}_S(2,1,2)$	6	$S(4,3)$	$4d_v+7d_x$	840	$\mathrm{H}_S(1,1,4)$	7	$S(5,5)$	$5d_v+9d_x$	840
$\mathrm{H}_S(2,0,3)$	6	$S(5,3)$	$5d_v+7d_x$	1260	$\mathrm{H}_S(1,2,3)$	7	$S(4,5)$	$4d_v+9d_x$	1260
$\mathrm{H}_S(1,2,2)$	6	$S(3,4)$	$3d_v+8d_x$	420	$\mathrm{H}_S(2,0,4)$	7	$S(6,4)$	$6d_v+8d_x$	420
$\mathrm{H}_S(1,1,3)$	6	$S(4,4)$	$4d_v+8d_x$	2100	$\mathrm{H}_S(2,1,3)$	7	$S(5,4)$	$5d_v+8d_x$	2100
$\bar{A}(0,0,3,2)$	6	$S(2,5)$	$2d_v+9d_x$	396	$\bar{B}(7)$	7	$S(7,3)$	$7d_v+7d_x$	396
$\mathrm{H}_S(0,2,3)$	6	$S(3,5)$	$3d_v+9d_x$	1400	$\mathrm{H}_S(3,0,3)$	7	$S(6,3)$	$6d_v+7d_x$	1400.

The triples $(1,0,3)$ and $(g-2,e-1,\boldsymbol{\alpha}-3)$ are in $\boldsymbol{T}$; however, Theorem 4.14 tells us that the corresponding modules $\mathrm{H}_S(1,0,3)$ and $\mathrm{H}_S(g-2,e-1,\boldsymbol{\alpha}-3)$ are zero because the binomial coefficient $\binom{e}{4}$ is zero.

In the graded case, the module $R(-3d_v-8d_x)^{210}$ is a summand of both $\mathbb{J}_5$ and $\mathbb{J}_6$, and the module $R(-6d_v-8d_x)^{210}$ is a summand of both $\mathbb{J}_8$ and $\mathbb{J}_7$. These summands can not be predicted if one only knows the Hilbert function of R/J. In this particular example, every other summand of the $\mathbb{J}$ can be correctly predicted from knowledge of the Hilbert function of R/J, together with the assumption that the minimal resolution of R/J is as simple as possible.

Example 7.4. Assume that the hypotheses and notation of Theorem 5.4 are in effect with $3 \le e$ and $3 \le g$. The beginning and the end of the minimal resolution of R/J are given by the corresponding parts of the complex $\mathbb{J}$ of Theorem 4.15. The beginning of the minimal resolution of R/J is the direct sum of the modules

module	p.	strand	twist	rank
$\bar{B}(0)$	0	$S(0,-g)$	$0d_v + 0d_x$	1
$\bar{B}(1)$	1	$S(1,1-g)$	$1d_v + 1d_x$	eg
$\mathrm{H}_S(0,0,0)$	1	$S(0,0)$	$0d_v + gd_x$	$\binom{f}{e}$
$\bar{B}(2)$	2	$S(2,2-g)$	$2d_v + 2d_x$	$\binom{eg}{2}$
$\mathrm{H}_S(1,0,0)$	2	$S(1,0)$	$1d_v + gd_x$	$e\binom{f}{e+1}$
$\mathrm{H}_S(0,1,0)$	2	$S(0,1)$	$0d_v + (g+1)d_x$	$g\binom{f}{e-1}$
$\bar{B}(3)$	3	$S(3,3-g)$	$3d_v + 3d_x$	see 7.5
$\mathrm{H}_S(2,0,0)$	3	$S(2,0)$	$2d_v + gd_x$	$\binom{e+1}{2}\binom{f}{e+2}$
$\mathrm{H}_S(1,0,1)$	3	$S(2,1)$	$2d_v + (g+1)d_x$	$\lvert S(2,1)\rvert$
$\mathrm{H}_S(0,2,0)$	3	$S(0,2)$	$0d_v + (g+2)d_x$	$\binom{f}{e-2}\binom{g+1}{2}$
$\mathrm{H}_S(0,1,1)$	3	$S(1,2)$	$1d_v + (g+2)d_x$	$\lvert S(1,2)\rvert$,

and the end of the minimal resolution of R/J is the direct sum of the modules

module	p.	strand	twist	rank
$\bar{A}(0,0,e,eg-e)$	$eg+1$	$S(eg-e,eg)$	$(eg-e)d_v$ $+ (eg+g)d_x$	1
$\bar{A}(0,0,e,eg-e-1)$	eg	$S(eg-e-1,eg-1)$	$(eg-e-1)d_v$ $+ (eg+g-1)d_x$	eg
$\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha})$	eg	$S(eg-e,eg-g)$	$(eg-e)d_v$ $+ egd_x$	$\binom{f}{e}$
$\bar{A}(0,0,e,eg-e-2)$	$eg-1$	$S(eg-e-2,eg-2)$	$(eg-e-2)d_v$ $+ (eg+g-2)d_x$	$\binom{eg}{2}$
$\mathrm{H}_S(g-2,e-1,\boldsymbol{\alpha})$	$eg-1$	$S(eg-e-1,eg-g)$	$(eg-e-1)d_v$ $+ egd_x$	$e\binom{f}{e+1}$

$\mathrm{H}_S(g-1,e-2,\boldsymbol{\alpha})$	$eg-1$	$S(eg-e,eg-g-1)$	$(eg-e)d_v$ $+(eg-1)d_x$	$g\binom{f}{e-1}$
$\bar{A}(0,0,e,eg-e-3)$	$eg-2$	$S(eg-e-3,eg-3)$	$(eg-e-3)d_v$ $+(eg+g-3)d_x$	rank $\bar{B}(3)$
$\mathrm{H}_S(g-3,e-1,\boldsymbol{\alpha})$	$eg-2$	$S(eg-e-2,eg-g)$	$(eg-e-2)d_v$ $+egd_x$	$\binom{e+1}{2}\binom{f}{e+2}$
$\mathrm{H}_S(g-2,e-1,\boldsymbol{\alpha}-1)$	$eg-2$	$S(eg-e-2,eg-g-1)$	$(eg-e-2)d_v$ $+(eg-1)d_x$	$\lvert S(2,1)\rvert$
$\mathrm{H}_S(g-1,e-3,\boldsymbol{\alpha})$	$eg-2$	$S(eg-e,eg-g-2)$	$(eg-e)d_v$ $+(eg-2)d_x$	$\binom{f}{e-2}\binom{g+1}{2}$
$\mathrm{H}_S(g-1,e-2,\boldsymbol{\alpha}-1)$	$eg-2$	$S(eg-e-1,eg-g-2)$	$(eg-e-1)d_v$ $+(eg-2)d_x$	$\lvert S(1,2)\rvert$.

The rank of $\bar{B}(d)$ is

$$\binom{eg}{d}-\chi(g\le d)\sum_{i=0}^{d-g}(-1)^i\operatorname{rank}A(g+i,f,i,d-g-i). \tag{7.5}$$

The ranks $|S(2,1)|$ and $|S(1,2)|$ are easy to compute; see Example 2.11.

Example 7.6. Assume that the hypotheses and notation of Theorem 5.4 are in effect with $3\le e$ and $3\le g$. The free module in position four in the minimal resolution of R/J is the direct sum of the following modules:

module	strand	twist	rank	comment
$\bar{B}(4)$	$S(4,4-g)$	$4d_v+4d_x$	see 7.5	
$\chi(4\le g)\,\mathrm{H}_S(3,0,0)$	$S(3,0)$	$3d_v+gd_x$	$\binom{e+2}{3}\binom{f}{e+3}$	1.
$\mathrm{H}_S(2,0,1)$	$S(3,1)$	$3d_v+(g+1)d_x$	$\lvert S(3,1)\rvert$	
$\mathrm{H}_S(1,1,1)$	$S(2,2)$	$2d_v+(g+2)d_x$	$\lvert S(2,2)\rvert$	
$\mathrm{H}_S(1,0,2)$	$S(3,2)$	$3d_v+(g+2)d_x$	$\lvert S(3,2)\rvert$	
$A(0,e-3,3,0)$	$S(0,3)$	$0d_v+(g+3)d_x$	$\binom{f}{e-3}\binom{g+2}{3}$	2.
$\mathrm{H}_S(0,2,1)$	$S(1,3)$	$1d_v+(g+3)d_x$	$\lvert S(1,3)\rvert$	
$\mathrm{H}_S(0,1,2)$	$S(2,3)$	$2d_v+(g+3)d_x$	$\lvert S(2,3)\rvert$,	

and the module in position $eg-3$ in the minimal resolution of R/J is the direct sum of the modules

module	strand	twist	rank
$\bar{A}(0,0,e,eg-e-4)$	$S(eg-e-4,eg-4)$	$(eg-e-4)d_v$ $+(eg+g-4)d_x$	rank $\bar{B}(4)$
$\mathrm{H}_S(g-4,e-1,\boldsymbol{\alpha})$	$S(eg-e-3,eg-g)$	$(eg-e-3)d_v+egd_x$	see 7.7
$\mathrm{H}_S(g-3,e-1,\boldsymbol{\alpha}-1)$	$S(eg-e-3,eg-g-1)$	$(eg-e-3)d_v+(eg-1)d_x$	$\lvert S(3,1)\rvert$
$\mathrm{H}_S(g-2,e-2,\boldsymbol{\alpha}-1)$	$S(eg-e-2,eg-g-2)$	$(eg-e-2)d_v+(eg-2)d_x$	$\lvert S(2,2)\rvert$

$\mathrm{H}_S(g-2, e-1, \boldsymbol{\alpha}-2)$	$S(eg-e-3, eg-g-2)$	$(eg-e-3)d_v+(eg-2)d_x$	$\|S(3,2)\|$
$\chi(e=3)\bar{B}(eg-3)$	$S(eg-e, eg-g-3)$	$(eg-e)d_v+(eg-3)d_x$	$\binom{g+2}{3}$
$\chi(4\le e)\,\mathrm{H}_S(g-1, e-4, \boldsymbol{\alpha})$	$S(eg-e, eg-g-3)$	$(eg-e)d_v+(eg-3)d_x$	$\binom{f}{e-3}\binom{g+2}{3}$
$\mathrm{H}_S(g-1, e-3, \boldsymbol{\alpha}-1)$	$S(eg-e-1, eg-g-3)$	$(eg-e-1)d_v+(eg-3)d_x$	$\|S(1,3)\|$
$\mathrm{H}_S(g-1, e-2, \boldsymbol{\alpha}-2)$	$S(eg-e-2, eg-g-3)$	$(eg-e-2)d_v+(eg-3)d_x$	$\|S(2,3)\|$.

The rank of $\mathrm{H}_S(g-4, e-1, \boldsymbol{\alpha})$ is

$$\chi(4\le g)\binom{e+2}{3}\binom{f}{e+3}. \tag{7.7}$$

Comments. 1. The module $\mathrm{H}_S(3,0,0)$ does not contribute to the minimal resolution when $3=g$. In this case, the strands $S(3,3-g)$ and $S(3,0)$ coincide and the rank of $\mathrm{H}_S(3,0,0)$ has been subtracted from the rank of $B(3)$ in (7.5) in order to produce the rank of $\bar{B}(3)$.
2. In Theorem 4.15, the module "$A(0,e-3,2,0)$" is called $\bar{A}(0,0,3,0)$, when $e=3$; and it is called $\mathrm{H}_S(0,3,0)$ when $4\le e$.

8. The right side of the complex $\mathbb{J}$.

The main result in this section is Proposition 8.1 which we have already used in the proof of Corollaries 5.1 and 5.3. The complexes $\mathbb{I}$ and $\mathbb{J}$ are understood for $e=1$ and $g=1$ in Example 4.12; consequently, throughout the present section we insist that $2\le e$ and $2\le g$.

Proposition 8.1. *Assume $2\le e$ and $2\le g$. If $(\mathbb{J},\bar{\boldsymbol{d}})$ is the complex of Theorem 4.15, then the truncated complex*

$$\cdots \xrightarrow{\bar{\boldsymbol{d}}} \mathbb{J}_{eg+2} \xrightarrow{\bar{\boldsymbol{d}}} \mathbb{J}_{eg+1} \xrightarrow{\bar{\boldsymbol{d}}} \mathbb{J}_{eg}$$

is isomorphic to

$$\cdots \to 0 \to R \xrightarrow{d} \begin{matrix} B(eg-1) \\ \oplus \\ \bigwedge^g F^*, \end{matrix}$$

with R in position $eg+1$, and

$$d(1) = \begin{bmatrix} (X \mathbin{\check{\circ}} V)(\omega_{E\otimes G^*}) \\ \lambda(\omega_{F^*}) \end{bmatrix}.$$

The proof of Proposition 8.1 appears after (8.8). In theory we "know" the differential $\bar{\boldsymbol{d}}$ on the complex $\mathbb{J}=\mathbb{F}/\mathbb{N}$ which is induced by the differential $\boldsymbol{d}$ of $\mathbb{F}$. However, in practice, we don't really want to record the splitting of the various strands $S(P,Q)$ and, we certainly don't want to record the map $\theta_{i-1}\colon B_{i-1}\to A_i$ which is the inverse of the map (4.4). The next lemma is the technical calculation about the complex map ψ of 4.3 which allows us to make some computations involving the differential in the complex $\mathbb{J}$. Recall that each module $\mathbb{J}_i$, in the complex $\mathbb{J}$, naturally decomposes into a direct sum of submodules $\bigoplus \mathbb{J}_i(P,Q)$, as (P,Q) varies over all of the strands $S(P,Q)$ of $\mathbb{F}$. Convention 1.12 describes the two partial orders which appear in the next result.

Lemma 8.2. *If $\mathbb{J}$ is the complex of Theorem 4.15 and $\psi\colon \mathbb{F}\to\mathbb{J}$ is the map of complexes which is given in Observation 4.3, then ψ is a non-increasing map with respect to the natural partial order on $\mathbb{Z}\times\mathbb{Z}$.*

Proof. Apply the technique of the proof of Observation 4.3. Decompose each module $S_i(P,Q)$ into

$$A_i(P,Q)\oplus B_i(P,Q)\oplus \mathbb{P}_i(P,Q)\oplus \mathbb{K}_i(P,Q),$$

where the composition

$$A_{i+1}(P,Q)\xrightarrow{\text{incl}} S_{i+1}(P,Q)\xrightarrow{\boldsymbol{\partial}} S_i(P,Q)\xrightarrow{\text{proj}} B_i(P,Q)$$

is an isomorphism, $B_i(P,Q)\oplus\mathbb{P}_i(P,Q)$ is contained in the kernel of $\boldsymbol{\partial}$, and

$$\mathbb{J}_i(P,Q)=\mathbb{P}_i(P,Q)\oplus\mathbb{K}_i(P,Q).$$

The submodules A_i and B_i of $\mathbb{F}_i$ are defined to be $\bigoplus A_i(P,Q)$ and $\bigoplus B_i(P,Q)$, respectively, where each sum is taken over all strands $S(P,Q)$. According to Observation 2.9, the differential $\boldsymbol{d}_i\colon \mathbb{F}_i\to\mathbb{F}_{i-1}$ is non-increasing with respect to the inverse lexicographic order on $\mathbb{Z}\times\mathbb{Z}$. However, the behavior of the restriction of $\boldsymbol{d}_i$ to A_i is even better. Indeed,

$$\boldsymbol{d}_i(A_i(P,Q))\subseteq\bigoplus S_{i-1}(P',Q'), \tag{8.3}$$

where the sum is taken over all pairs (P',Q'), with $(P',Q')\le_{\text{np}}(P,Q)$. Look at the proof of Observation 2.9. Almost all of the components of $\boldsymbol{d}_i(S_i(P,Q))$ are contained in the right side of (8.3). The only exception occurs only when $S_i(P,Q)$ is equal to $A(a,b,c,d)$ with $a=0$. However, in this case, $A_i(P,Q)$ is zero because $A_i(P,Q)$ is isomorphic to

$$B_{i-1}(P,Q)\subset S_{i-1}(P,Q)=0.$$

It follows that the map

$$A_i\xrightarrow{\text{incl}}\mathbb{F}_i\xrightarrow{\boldsymbol{d}_i}\mathbb{F}_{i-1}\xrightarrow{\text{proj}} B_{i-1}, \tag{4.4}$$

as well as its inverse $\theta_{i-1}\colon B_{i-1}\to A_i$, are non-increasing with respect to the natural partial order on $\mathbb{Z}\times\mathbb{Z}$. Recall that the map ψ is defined by

$$\psi_i=\pi_i^{\mathbb{P}\oplus\mathbb{K}}\circ(1-\boldsymbol{d}_{i+1}\circ\theta_i\circ\pi_i^B).$$

Let $x=\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{p}+\boldsymbol{k}$ be an element of $S_i(P,Q)$ with $\boldsymbol{a}\in A$, $\boldsymbol{b}\in B$, $\boldsymbol{p}\in\mathbb{P}$, and $\boldsymbol{k}\in\mathbb{K}$. The right most factor of ψ_i leaves $\boldsymbol{a}+\boldsymbol{p}+\boldsymbol{k}$ unchanged and replaces $\boldsymbol{b}$ with a sum of terms from strands $S(P',Q')$ with $(P',Q')<_{\text{np}}(P,Q)$. It follows that $\psi_i(x)$ is equal to $\boldsymbol{p}+\boldsymbol{k}$ plus a sum of terms from strands $S(P',Q')$ with $(P',Q')<_{\text{np}}(P,Q)$. $\square$

The official description of the strand $S(P,Q)$ is given in Definition 2.8 and Example 2.11. Let $Z_i(P,Q)$ be the module of cycles in the strand $S(P,Q)$ at position i, and let $\mathrm{H}_i(S(P,Q))$ be the homology of $S(P,Q)$ at position i; that is,

$$\mathrm{H}_i(S(P,Q))=\frac{Z_i(P,Q)}{\boldsymbol{\partial}(S_{i+1}(P,Q))}.$$

In the context of the above result, if $\mathbb{K}_i(P,Q)=0$, then $Z_i(P,Q)$ is equal to $B_i(P,Q)\oplus\mathbb{P}_i(P,Q)$, $\mathrm{H}_i(S(P,Q))=\mathbb{P}_i(P,Q)$, and the restriction of the homogeneous part of ψ_i to $Z_i(P,Q)$ is the natural quotient map $Z_i(P,Q)\to\mathrm{H}_i(S(P,Q))$. The next result follows immediately.

Corollary 8.4. *Let $\mathbb{N}$ be the subcomplex of $\mathbb{F}$ from Theorem 4.15 and $\psi\colon \mathbb{F} \to \mathbb{J}$ be the map of complexes which is given in Observation 4.3. Fix a strand $S(P,Q)$ of $\mathbb{F}$ and a position i.*

(a) *If $\mathbb{J}_i(P',Q') = 0$ for all (P',Q') with $(P',Q') \le_{\mathrm{np}} (P,Q)$, then $\psi(S_i(P,Q))$ is equal to zero.*

(b) *If $\mathbb{J}_i(P,Q) = \mathrm{H}_i(S(P,Q))$ and $\mathbb{J}_i(P',Q') = 0$ for all (P',Q') with $(P',Q') <_{\mathrm{np}} (P,Q)$, then the restriction of map ψ to $Z_i(P,Q)$ is the natural quotient map $Z_i(P,Q) \to \mathrm{H}_i(S(P,Q))$.*

The calculations throughout the rest of this section use bases and are very similar to the calculations of [20]. We adopt the notation of that paper by fixing bases

(8.5) $\quad x_1,\dots,x_g$ for G; $y_1,\dots,y_g$ for G^*; $u_1,\dots,u_e$ for E; and $v_1,\dots,v_e$ for E^*,

with $\{x_i\}$ dual to $\{y_j\}$, and $\{u_i\}$ dual to $\{v_j\}$. Keep in mind that the indexing convention of 1.5 which held in sections 1 through 7 no longer applies.

Lemma 8.6.

(a) *If $0 \le s \le g-1$, then the homomorphism*

$$\gamma\colon A(0,0,e,eg-e-s) \to B(eg-s),$$

which is given by $\gamma(1 \otimes 1 \otimes Y \otimes Z) = (\omega_E \bowtie Y) \wedge Z$, induces an isomorphism from $\bar{A}(0,0,e,eg-e-s)$ to $B(eg-s)$.

(b) *If $M_{g-1}\colon A(g-1,g,e-1,\boldsymbol{\alpha}) \to \bigwedge^g F^*$ is the homomorphism which is given by $M_{g-1}(U \otimes \alpha_g \otimes y_1^{(b_1)} \cdots y_g^{(b_g)} \otimes Z)$ is equal to*

$$\begin{cases} [(\omega_E \bowtie y_g^{(e)}) \wedge Z \wedge (U \bowtie (y_1 \wedge \ldots \wedge y_{g-1}))](\omega_{E^*\otimes G}) \cdot \alpha_g, & \textit{if } b_g = e-1, \textit{ and} \\ 0, & \textit{otherwise,} \end{cases}$$

then M_{g-1} induces an isomorphism

$$\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha}) \to \bigwedge^g F^*.$$

Proof. The calculations are taken from [20]; see, especially, section 4. The module $A(0,0,e,eg-e-s)$ is equal to $\mathcal{M}(0,e,eg-e-s)$ in the split exact complex $\mathfrak{C}^{e+g,s}$, and $A(g-1,g,e-1,\boldsymbol{\alpha})$ is equal to

$$\mathcal{M}(g-1,e-1,\boldsymbol{\alpha}) \otimes \bigwedge^g F^*$$

in the split exact complex $\mathfrak{C}^{e,0} \otimes \bigwedge^g F^*$. Each of the complexes $\mathfrak{C}^{r,s}$ may be viewed as the mapping cone of two complexes; see [20, Rmk. 1.10]. In the situation of the present proof, the top complex is $S(P,Q)$, the bottom complex is concentrated in one position, and the indicated maps γ and M_{g-1} are the maps of complexes from the top complex to the bottom complex. The mapping cone is split exact; hence, the map from the top complex to the bottom complex induces an isomorphism on homology. $\square$

We name the components of the differential $\boldsymbol{d}$ of the complex $\mathbb{F}$ of 2.3.

Notation 8.7. Consider the maps

$$\begin{aligned}
\boldsymbol{A}&\colon A(a,b,c,d)\to A(a-1,b-1,c,d)\\
\boldsymbol{B}&\colon A(a,b,c,d)\to A(a,b+1,c-1,d)\\
\boldsymbol{C}&\colon A(a,b,c,d)\to A(a,b,c,d-1)\\
\boldsymbol{D}&\colon A(a,b,c,d)\to A(a-1,b,c-1,d+1)\\
\boldsymbol{E}&\colon A(a,b,c,d)\to B(a+d)\\
\boldsymbol{F}&\colon A(a,b,c,d)\to B(c+d)
\end{aligned}$$

which are given by

$$\begin{aligned}
\boldsymbol{A}(x)&=\varepsilon_1^{(a-1)}\otimes[V(\varepsilon_1)](\alpha_b)\otimes\gamma_1^{(c)}\otimes z_d\\
\boldsymbol{B}(x)&=\varepsilon_1^{(a)}\otimes X^*(\gamma_1)\wedge\alpha_b\otimes\gamma_1^{(c-1)}\otimes z_d\\
\boldsymbol{C}(x)&=\varepsilon_1^{(a)}\otimes\alpha_b\otimes\gamma_1^{(c)}\otimes(X\check{\circ}V)(z_d)\\
\boldsymbol{D}(x)&=\varepsilon_1^{(a-1)}\otimes\alpha_b\otimes\gamma_1^{(c-1)}\otimes(\varepsilon_1\otimes\gamma_1)\wedge z_d\\
\boldsymbol{E}(x)&=\chi(c=0)\varepsilon_1^{(a)}\bowtie\left[(\textstyle\bigwedge^{f-b}X)(\alpha_b[\omega_F])\right](\omega_{G^*})\wedge z_d\\
\boldsymbol{F}(x)&=\chi(a=0)[(\textstyle\bigwedge^{b}V^*)(\alpha_b)](\omega_E)\bowtie\gamma_1^{(c)}\wedge z_d
\end{aligned}$$

for $x=\varepsilon_1^{(a)}\otimes\alpha_b\otimes\gamma_1^{(c)}\otimes z_d\in A(a,b,c,d)$. If $P=a+d$ and $Q=c+d$, then

$$\boldsymbol{d}(x)=\begin{cases}
+\boldsymbol{A}(x)\in A(a-1,b-1,c,d)\subset S(P-1,Q)\\
-\boldsymbol{B}(x)\in A(a,b+1,c-1,d)\subset S(P,Q-1)\\
+(-1)^{a+c}\boldsymbol{C}(x)\in A(a,b,c,d-1)\subset S(P-1,Q-1)\\
+(-1)^{a+c}\boldsymbol{D}(x)\in A(a-1,b,c-1,d+1)\subset S(P,Q)\\
+(-1)^{a+d}\boldsymbol{E}(x)\in B(a+d)\subset S(P,Q+a-g)\\
+(-1)^{d}\mathfrak{b}\cdot\boldsymbol{F}(x)\in B(c+d)\subset S(P+c,Q-g).
\end{cases}\tag{8.8}$$

Proof of Proposition 8.1. We know, from Example 7.4, that

$$\mathbb{J}_i\cong\begin{cases}
0, & \text{if } eg+2\le i,\\
\bar{A}(0,0,e,eg-e), & \text{if } eg+1=i, \text{ and}\\
\bar{A}(0,0,e,eg-e-1)\oplus\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha}), & \text{if } eg=i.
\end{cases}$$

Lemma 8.6 exhibits isomorphisms

$$\bar{A}(0,0,e,eg-e)\to R,\qquad \bar{A}(0,0,e,eg-e-1)\to B(eg-1),\quad\text{and}\quad \mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha})\to\textstyle\bigwedge^g F^*.$$

In Definition 8.12 we identify elements

$$\begin{aligned}
&\Xi_r\in A(r,r,e,eg-e-r), && \text{for } 0\le r\le g-1, \text{ and}\\
&\Theta_r\in A(g,f-r,r,eg-g-r), && \text{for } 0\le r\le e-1,
\end{aligned}$$

and constants A_0 and B_0. Let Y be the element

$$Y=\begin{cases}
+\displaystyle\sum_{r=0}^{g-1}(-1)^{\binom{r+1}{2}+re}A_0\Xi_r\in A(r,r,e,eg-e-r)\\
+\displaystyle\sum_{r=0}^{e-1}(-1)^{rg+\binom{r}{2}}B_0\Theta_r\in A(g,f-r,r,eg-g-r)
\end{cases}\tag{8.9}$$

of $\mathbb{F}_{eg+1}$. Notice that Y consists of $A_0\Xi_0$ plus an element of $\bigoplus S_{eg+1}(P,Q)$ as (P,Q) varies over tuples with $Q < eg$. Every module $S_i(eg-e,eg)$, with $i \le eg$, is zero; so, $A_0\Xi_0$ is automatically a cycle in $Z_{eg+1}(eg-e,eg)$. Corollary 8.4 tells us that $\psi(Y)$ is equal to the class of $A_0\Xi_0$ in $\bar{A}(0,0,e,eg-e)$. We learn in (8.15) that

$$A_0\Xi_0 = 1 \otimes 1 \otimes y_g^{(e)} \otimes [\omega_{E^*} \bowtie x_g^{(e)}](\omega_{E\otimes G^*});$$

thus, the isomorphism $\bar{A}(0,0,e,eg-e) \to B(eg) \to R$ of Lemma 8.6 sends $A_0\Xi_0$ to

$$(\omega_E \bowtie y_g^{(e)}) \wedge [\omega_{E^*} \bowtie x_g^{(e)}](\omega_{E\otimes G^*}) = \omega_{E\otimes G^*} \mapsto 1.$$

We conclude that $\psi(Y)$ is the element 1 in $\mathbb{J}_{eg+1}$. The map $\psi\colon \mathbb{F} \to \mathbb{J}$ is a map of complexes, so $d(1) = \psi(\boldsymbol{d}(Y))$.

In Lemma 8.18, we calculate that $\boldsymbol{d}(Y)$ is equal to

$$\begin{cases} A_0(-1)^e[\boldsymbol{C}(\Xi_0) - \boldsymbol{A}(\Xi_1)] \in A(0,0,e,eg-e-1) \subset S(eg-e-1,eg-1) \\ +(-1)^{eg+1}[(-1)^{\binom{g}{2}-e}A_0\boldsymbol{B}(\Xi_{g-1}) - (-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})] \\ \qquad \in A(g-1,g,e-1,\boldsymbol{\alpha}) \subset S(eg-e,eg-g) \end{cases}$$

plus an element of $\bigoplus_{(P,Q)} S(P,Q)$ with $P \le eg-e-1$ and $Q \le eg-2$; or $P \le eg-1$ and $Q \le eg-g-1$. Every element of $A(0,0,e,eg-e-1)$ is a cycle in $S(eg-e-1,eg-1)$. Lemma 8.16 shows that $\boldsymbol{B}(\Xi_{g-1})$ and $\boldsymbol{A}(\Theta_{e-1})$ both are cycles in $S(eg-e,eg-g)$. Once again, Corollary 8.4 applies. We see that $\psi(\boldsymbol{d}(Y))$ is the class of

$$A_0(-1)^e[\boldsymbol{C}(\Xi_0) - \boldsymbol{A}(\Xi_1)]$$

in $\bar{A}(0,0,e,eg-e-1)$ plus the class of

$$(-1)^{eg+1}[(-1)^{\binom{g}{2}-e}A_0\boldsymbol{B}(\Xi_{g-1}) - (-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})]$$

in $\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha})$. Lemma 8.19 shows that the isomorphism

$$\bar{A}(0,0,e,eg-e-1) \to B(eg-1)$$

of Lemma 8.6 sends the class of $A_0(-1)^e(\boldsymbol{C}(\Xi_0) - A_0\boldsymbol{A}(\Xi_1))$ to $(X \check{\circ} V)(\omega_{E\otimes G^*})$. Lemma 8.21 shows that the isomorphism

$$\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha}) \to \bigwedge^g F^*$$

of Lemma 8.6 sends the class of

$$(-1)^{\binom{g}{2}+e}A_0\boldsymbol{B}(\Xi_{g-1}) - (-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})$$

to $\omega_G(y_1 \wedge \ldots \wedge y_g) \cdot \lambda(\omega_{F^*})$. $\square$

Definition 8.10. For each non-negative integer r, let

$$\begin{aligned}\mathcal{I}_r &= \{(i) = (i_1, \dots, i_r) \mid 1 \le i_1 \le i_2 \le \dots \le i_r \le g\} \text{ and}\\ \mathcal{H}_r &= \{(h) = (h_1, \dots, h_r) \mid 1 \le h_1 \le h_2 \le \dots \le h_r \le e\},\end{aligned}$$

where each i_j and h_j is an integer.

(a) Fix $(i) = (i_1, \dots, i_r) \in \mathcal{I}_r$ and $(h) = (h_1, \dots, h_r) \in \mathcal{H}_r$. For notational convenience, we give meaning to the symbols i_0, i_{r+1}, h_0, and h_{r+1}. We allow the symbols i_0 and h_0 to mean 1, i_{r+1} to mean g, and h_{r+1} to mean e. Notice that neither i_0 nor i_{r+1} is an element of the r-tuple (i) and neither h_0 nor h_{r+1} is an element of the r-tuple (h).

(b) If $(j) = (j_1, \dots, j_r)$ is an r-tuple of integers, and s is an integer, then we define the number $\#_s(j)$ to be equal to the number of subscripts p, with $1 \le p \le r$, and $j_p = s$.

(c) If $(j) = (j_1, \dots, j_r)$ is an r-tuple of integers, then we define $|(j)|$ to be $\sum_{p=1}^{r} j_p$.

Lemma 8.11. *If $(i) = (i_1, \dots, i_K, \dots, i_r)$ and $(i)' = (i_1, \dots, \widehat{i_K}, \dots, i_r)$, then*

$$x_{i_K}\left(y_1^{(\#_1(i))} \cdots y_g^{(\#_g(i))}\right) = y_1^{(\#_1(i)')} \cdots y_g^{(\#_g(i)')}.$$

Proof. The exponent of $y_{i_K}^{(\#_{i_K}(i))}$ is positive, so

$$x_{i_K}\left(y_1^{(\#_1(i))} \cdots y_{i_K}^{(\#_{i_K}(i))} \cdots y_g^{(\#_g(i))}\right) = y_1^{(\#_1(i))} \cdots y_{i_K}^{(\#_{i_K}(i)-1)} \cdots y_g^{(\#_g(i))}.$$

On the other hand,

$$\#_p(i)' = \begin{cases} \#_p(i) & \text{if } p \ne i_K \\ \#_p(i) - 1 & \text{if } p = i_K. \end{cases} \qquad \square$$

Definition 8.12. (a) For $0 \le r \le e-1$, define

$$\Theta_r \in A(g, f-r, r, eg-g-r) \quad \text{and} \quad t_r \in A(g-1, f-r-1, r, eg-g-r)$$

by Θ_r is equal to

$$\left\{\begin{aligned}&\sum_{(i)\in\mathcal{I}_r} (-1)^{r+|(i)|} u_{e-r}^{(i_1)} \prod_{p=1}^{r} u_{e-r+p}^{(i_{p+1}-i_p)} \otimes \left[\bigwedge_{q=1}^{r} V(u_{e-r+q})\right](\omega_{F^*}) \otimes \prod_{s=1}^{g} y_s^{(\#_s(i))}\\ &\qquad \otimes \left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}} (v_{e-r+t} \otimes x_w)\right)\right](\omega_{E\otimes G^*})\end{aligned}\right.$$

and t_r is equal to

$$\left\{\begin{aligned}&\sum_{(i)\in\mathcal{I}_r} (-1)^{r+|(i)|} \prod_{p=0}^{r} u_{e-r+p}^{(i_{p+1}-i_p)} \otimes \left[\bigwedge_{q=0}^{r} V(u_{e-r+q})\right](\omega_{F^*}) \otimes \prod_{s=1}^{g} y_s^{(\#_s(i))}\\ &\qquad \otimes \left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}} (v_{e-r+t} \otimes x_w)\right)\right](\omega_{E\otimes G^*}).\end{aligned}\right.$$

(b) For $0 \le r \le g-1$, define

$$\Xi_r \in A(r,r,e,eg-e-r) \quad \text{and} \quad \xi_r \in A(r,r+1,e-1,eg-e-r)$$

by Ξ_r is equal to

$$\begin{cases} \displaystyle\sum_{(h)\in\mathcal{H}_r} (-1)^{r+|(h)|} \prod_{s=1}^{e} u_s^{(\#_s(h))} \otimes \bigwedge_{q=1}^{r} X^*(y_{g-r+q}) \otimes y_{g-r}^{(h_1)} \prod_{p=1}^{r} y_{g-r+p}^{(h_{p+1}-h_p)} \\ \qquad \otimes \left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=h_t}^{h_{t+1}} (v_w \otimes x_{g-r+t})\right)\right](\omega_{E\otimes G^*}) \end{cases}$$

and ξ_r is equal to

$$\begin{cases} \displaystyle\sum_{(h)\in\mathcal{H}_r} (-1)^{r+|(h)|} \prod_{s=1}^{e} u_s^{(\#_s(h))} \otimes \bigwedge_{q=0}^{r} X^*(y_{g-r+q}) \otimes \prod_{p=0}^{r} y_{g-r+p}^{(h_{p+1}-h_p)} \\ \qquad \otimes \left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=h_t}^{h_{t+1}} (v_w \otimes x_{g-r+t})\right)\right](\omega_{E\otimes G^*}). \end{cases}$$

(c) Define the constants A_0, A_1, and B_0 in R to be $A_0 = (u_e \wedge \ldots \wedge u_1)(\omega_{E^*})$, $A_1 = (y_g \wedge \ldots \wedge y_1)(\omega_G)$, and $B_0 = (-1)^{e+1}A_1\mathfrak{b}$.

Remark 8.13. The elements $(v_1 \wedge \ldots \wedge v_e)(\omega_E) \cdot \omega_{E^*}$ and $v_1 \wedge \ldots \wedge v_e$ of $\bigwedge^e E^*$ are equal, as may be seen by applying ω_E to each element. Apply $u_e \wedge \ldots \wedge u_1$ to these two elements to see that

$$(v_1 \wedge \ldots \wedge v_e)(\omega_E) \cdot A_0 = 1.$$

A similar argument gives

$$(x_1 \wedge \ldots \wedge x_g)(\omega_{G^*}) \cdot A_1 = 1.$$

Lemma 8.14. *The following statements hold:*
(a) $A_1\boldsymbol{E}(\Theta_0) = \omega_{E\otimes G^*}$,
(b) $A_0\boldsymbol{F}(\Xi_0) = \omega_{E\otimes G^*}$,
(c) $M_{g-1}(t_{e-1}) = (-1)^{\binom{e}{2}+\binom{g-1}{2}} \cdot [(\bigwedge^e V)(\omega_E)](\omega_{F^*})$, *and*
(d) $A_0 M_{g-1}(\xi_{g-1}) = (-1)^{eg+e} A_1 \cdot (\bigwedge^g X^*)(\omega_{G^*})$.

Proof. Notice that Θ_0 is equal to

$$\begin{aligned} &u_e^{(g)} \otimes \omega_{F^*} \otimes 1 \otimes \left(\bigwedge_{w=1}^{g} (v_e \otimes x_w)\right)(\omega_{E\otimes G^*}) \\ &= u_e^{(g)} \otimes \omega_{F^*} \otimes 1 \otimes (v_e^{(g)} \bowtie (x_1 \wedge \ldots \wedge x_g))(\omega_{E\otimes G^*}) \\ &= (x_1 \wedge \ldots \wedge x_g)(\omega_{G^*}) \cdot u_e^{(g)} \otimes \omega_{F^*} \otimes 1 \otimes (v_e^{(g)} \bowtie \omega_G)(\omega_{E\otimes G^*}); \end{aligned}$$

and therefore, Remark 8.13 yields that

$$A_1\Theta_0 = u_e^{(g)} \otimes \omega_{F^*} \otimes 1 \otimes (v_e^{(g)} \bowtie \omega_G)(\omega_{E\otimes G^*}).$$

Apply the map $\boldsymbol{E}$ of 8.7 to obtain (a). A similar straightforward calculation shows that

$$A_0\Xi_0 = 1\otimes 1\otimes y_g^{(e)}\otimes[\omega_{E^*}\bowtie x_g^{(e)}](\omega_{E\otimes G^*}); \tag{8.15}$$

thereby establishing (b).

The homomorphism $M_{g-1}\colon A(g-1,g,e-1,\boldsymbol{\alpha})\to\bigwedge^g F^*$ is defined in Lemma 8.6. The only term of t_{e-1} which is not in the kernel of M_{g-1} is the term which corresponds to the element $(i)\in\mathcal{I}_{e-1}$ with $i_p=g$ for $1\le p\le e-1$. This term is

$$\begin{cases} (-1)^{\boldsymbol{\alpha}}u_1^{(g-1)}\otimes[(\bigwedge^e V)(u_1\wedge\ldots\wedge u_e)](\omega_{F^*})\otimes y_g^{(e-1)}\\ \qquad\otimes\left[\left((v_1^{(g)}\bowtie(x_1\wedge\ldots\wedge x_g)\right)\wedge\left((v_2\wedge\ldots\wedge v_e)\bowtie x_g^{(e-1)}\right)\right](\omega_{E\otimes G^*}).\end{cases}$$

Thus, $M_{g-1}(t_{e-1})$ is equal to

$$(-1)^{\boldsymbol{\alpha}}[W_1\wedge W_2](\omega_{E^*\otimes G})\cdot[(\textstyle\bigwedge^e V)(u_1\wedge\ldots\wedge u_e)](\omega_{F^*}),$$

where W_1 is equal to

$$(\omega_E\bowtie y_g^{(e)})\wedge\left[\left(v_1^{(g)}\bowtie(x_1\wedge\ldots\wedge x_g)\right)\wedge\left((v_2\wedge\ldots\wedge v_e)\bowtie x_g^{(e-1)}\right)\right](\omega_{E\otimes G^*})$$

and $W_2=(u_1^{(g-1)}\bowtie Y_{g-1}^-)$, for $Y_{g-1}^-=y_1\wedge\ldots\wedge y_{g-1}$. Apply Proposition 1.2 to see that $M_{g-1}(t_{e-1})$ is equal to

$$\omega_E(v_1\wedge v_2\wedge\ldots\wedge v_e)\cdot Y_{g-1}^-(x_1\wedge\ldots\wedge x_{g-1})\cdot[(\textstyle\bigwedge^e V)(u_1\wedge\ldots\wedge u_e)](\omega_{F^*}).$$

It is easy to see that $Y_{g-1}^-(x_1\wedge\ldots\wedge x_{g-1})=(-1)^{\binom{g-1}{2}}$ and that

$$u_1\wedge\ldots\wedge u_e=(-1)^{\binom{e}{2}}u_e\wedge\ldots\wedge u_1=(-1)^{\binom{e}{2}}A_0\cdot\omega_E.$$

Remark 8.13 completes the proof of (c).

The only term of ξ_{g-1} which is not in the kernel of M_{g-1} is

$$\begin{cases} u_1^{(g-1)}\otimes(\bigwedge^g X^*)(y_1\wedge\ldots\wedge y_g)\otimes y_g^{(e-1)}\otimes\\ \qquad\left[\left(v_1^{(g-1)}\bowtie(x_1\wedge\ldots\wedge x_{g-1})\right)\wedge\left((v_1\wedge\ldots\wedge v_e)\bowtie x_g^{(e)}\right)\right](\omega_{E\otimes G^*}).\end{cases}$$

A calculation similar to the proof of (c) completes the proof of (d). □

Lemma 8.16.

(a) *If* $0\le r\le e-1$, *then* $\boldsymbol{A}(\Theta_r)=t_r$.
(b) *If* $0\le r\le g-1$, *then* $\boldsymbol{B}(\Xi_r)=\xi_r$.
(c) *If* $1\le r\le e-1$, *then* $\boldsymbol{D}(\Theta_r)=t_{r-1}$.
(d) *If* $1\le r\le g-1$, *then* $\boldsymbol{D}(\Xi_r)=\xi_{r-1}$.
(e) $(\boldsymbol{D}\circ\boldsymbol{B})(\Xi_{g-1})=(\boldsymbol{D}\circ\boldsymbol{A})(\Theta_{e-1})=0$.

Proof. Assertions (a) and (b) are obvious. Assertion (e) follows from (a) through (d) because the maps $\boldsymbol{D}$ and $\boldsymbol{B}$ commute and $\boldsymbol{B}^2=0$:

$$(\boldsymbol{D}\circ\boldsymbol{B})(\Xi_{g-1})=(\boldsymbol{B}\circ\boldsymbol{D})(\Xi_{g-1})=\boldsymbol{B}(\xi_{g-2})=\boldsymbol{B}^2(\Xi_{g-2})=0.$$

The same argument works for $(\boldsymbol{D}\circ\boldsymbol{A})(\Theta_{e-1})$. The proofs of (c) and (d) are fairly complicated but totally analogous to one another. We exhibit the proof of (c) and surpress the details of the proof of (d). The expression $\boldsymbol{D}(\Theta_r)$ is equal to

$$\begin{cases} \displaystyle\sum_{\substack{1\le k\le e\\ 1\le \ell\le g}}\sum_{(i)\in\mathcal{I}_r}(-1)^{r+|(i)|}v_k\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)\otimes\left[\bigwedge_{q=1}^{r}V(u_{e-r+q})\right](\omega_{F^*})\\ \qquad\otimes x_\ell\left(\prod_{s=1}^{g}y_s^{(\#_s(i))}\right)\otimes(u_k\otimes y_\ell)\wedge\left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\right](\omega_{E\otimes G^*}).\end{cases}$$

Notice that $v_k\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)$ is zero unless $k=e-r+K$ for some K with $0\le K\le r$. Notice, also, that $(u_{e-r+K}\otimes y_\ell)(v_{e-r+t}\otimes x_w)$ is equal to zero unless $K=t$ and $\ell=w$. Furthermore, the graded product rule tells us that

$$(u_{e-r+K}\otimes y_\ell)\left[\bigwedge_{t=0}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\right]$$

is equal to $(-1)^{i_K+K-1}$ times

$$\begin{cases}\displaystyle\bigwedge_{t=0}^{K-1}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\wedge(u_{e-r+K}\otimes y_\ell)\left(\bigwedge_{w=i_K}^{i_{K+1}}(v_{e-r+K}\otimes x_w)\right)\\ \qquad\wedge\displaystyle\bigwedge_{t=K+1}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right).\end{cases}$$

The sign was calculated using the fact that we have moved across each subscript w from 1 to i_K at least once and we have moved across the subscripts w equal to $i_1,\dots,i_{K-1}$ exactly twice. Notice that if $1\le K$, then $v_{e-r+K}\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)$ is equal to zero unless $i_K<i_{K+1}$. We split $\boldsymbol{D}(\Theta_r)$ into two pieces $A+C$. In A we have $K=0$ and $i_1=1$. In C, we have K arbitrary and $i_K<i_{K+1}$. Thus, A is equal to

$$\begin{cases}\displaystyle\sum_{1\le\ell\le g}\sum_{\substack{(i)\in\mathcal{I}_r\\ i_1=1}}(-1)^{r+|(i)|}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\otimes\left[\bigwedge_{q=1}^{r}V(u_{e-r+q})\right](\omega_{F^*})\otimes x_\ell\left(\prod_{s=1}^{g}y_s^{(\#_s(i))}\right)\\ \qquad\otimes\left[(u_{e-r}\otimes y_\ell)\left(\bigwedge_{w=1}^{i_1}(v_{e-r}\otimes x_w)\right)\wedge\bigwedge_{t=1}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\right](\omega_{E\otimes G^*})\end{cases}$$

and C is equal to

$$\begin{cases}\displaystyle\sum_{\substack{0\le K\le r\\ 1\le\ell\le g}}\sum_{\substack{(i)\in\mathcal{I}_r\\ i_K<i_{K+1}}}(-1)^{r+|(i)|+i_K+K-1}v_{e-r+K}\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)\\ \qquad\otimes\left[\bigwedge_{q=1}^{r}V(u_{e-r+q})\right](\omega_{F^*})\otimes x_\ell\left(\prod_{s=1}^{g}y_s^{(\#_s(i))}\right)\\ \qquad\otimes\left[\begin{array}{l}\displaystyle\bigwedge_{t=0}^{K-1}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\wedge(u_{e-r+K}\otimes y_\ell)\left(\bigwedge_{w=i_K}^{i_{K+1}}(v_{e-r+K}\otimes x_w)\right)\\ \qquad\wedge\displaystyle\bigwedge_{t=K+1}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\end{array}\right](\omega_{E\otimes G^*}).\end{cases}$$

In A, the expression $(u_{e-r}\otimes y_\ell)\left(\bigwedge\limits_{w=1}^{i_1}(v_{e-r}\otimes x_w)\right)$ is zero unless $\ell=1$. Notice that there is a bijection between

$$\{(i)=(i_1,\dots,i_r)\in\mathcal{I}_r\mid i_1=1\}\quad\text{and}\quad\mathcal{I}_{r-1}.$$

The r-tuple $(i_1,\dots,i_r)$ from the set on the left corresponds to the $(r-1)$-tuple $(i)'=(i_2,\dots,i_r)$ in $\mathcal{I}_{r-1}$. Lemma 8.11 tells us that $x_1\left(\prod\limits_{s=1}^{g}y_s^{(\#_s(i))}\right)=\prod\limits_{s=1}^{g}y_s^{(\#_s(i)')}$. Notice that $|(i)|=|(i)'|+1$. Also, if $1\le p\le r+1$, then $i_p=i'_{p-1}$; so,

$$\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}=\prod_{p=0}^{r-1}u_{e-(r-1)+p}^{(i'_{p+1}-i'_p)},\ \text{and}$$
$$\left(\bigwedge_{t=1}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\right)=\left(\bigwedge_{t=0}^{r-1}\left(\bigwedge_{w=i'_t}^{i'_{t+1}}(v_{e-(r-1)+t}\otimes x_w)\right)\right).$$

We conclude that $A=t_{r-1}$.

The expression $(u_{e-r+K}\otimes y_\ell)\left(\bigwedge\limits_{w=i_K}^{i_{K+1}}(v_{e-r+K}\otimes x_w)\right)$, in C, is equal to zero unless $i_K\le\ell\le i_{K+1}$. On the other hand, $x_\ell\left(\prod\limits_{s=1}^{g}y_s^{(\#_s(i))}\right)$ is zero whenever $i_K<\ell<i_{K+1}$. So C decomposes into C_1+C_2. In C_1, we have $\ell=i_K$. In C_2, we have $\ell=i_{K+1}$. In C_2, the contribution when $K=r$ is necessarily zero because $i_K<i_{K+1}$, tells us that $i_r<g$; hence, $x_\ell\left(\prod\limits_{s=1}^{g}y_s^{(\#_s(i))}\right)$ is zero at $\ell=i_{K+1}=g$. The contribution to C_1, when $K=0$ is also zero. Use Lemma 8.11 to see that C_1 is equal to

$$\left\{\begin{array}{l}\displaystyle\sum_{1\le K\le r}\ \sum_{\substack{(i)\in\mathcal{I}_r\\ i_K<i_{K+1}}}(-1)^{r+|(i)|+i_K+K-1}v_{e-r+K}\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)\\ \qquad\displaystyle\otimes\left[\bigwedge_{q=1}^{r}V(u_{e-r+q})\right](\omega_{F^*})\otimes\prod_{s=1}^{g}y_s^{(\#_s(i_1,\dots,\widehat{i_K},\dots,i_r))}\\ \qquad\displaystyle\otimes\left[\begin{array}{l}\displaystyle\bigwedge_{t=0}^{K-1}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\wedge\left(\bigwedge_{w=i_K+1}^{i_{K+1}}(v_{e-r+K}\otimes x_w)\right)\\ \qquad\displaystyle\wedge\bigwedge_{t=K+1}^{r}\left(\bigwedge_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\end{array}\right](\omega_{E\otimes G^*}).\end{array}\right.$$

The derivation $(u_{e-r+K}\otimes y_{i_{K+1}})$ still has to move across $i_{K+1}-i_K$ one forms before it finds the one form which it does not annihilate. We see that C_2 is equal to

$$\begin{cases} \sum\limits_{0\le K\le r-1} \sum\limits_{\substack{(i)\in\mathcal{I}_r\\ i_K<i_{K+1}}} (-1)^{r+|(i)|+i_{K+1}+K-1} v_{e-r+K}\left(u_{e-r}^{(i_1)}\prod\limits_{p=1}^{r} u_{e-r+p}^{(i_{p+1}-i_p)}\right)\\ \quad\otimes\left[\bigwedge\limits_{q=1}^{r} V(u_{e-r+q})\right](\omega_{F^*})\otimes\prod\limits_{s=1}^{g} y_s^{(\#_s(i_1,\dots,\widehat{i_{K+1}},\dots,i_r))}\\ \quad\otimes\left[\begin{array}{l}\bigwedge\limits_{t=0}^{K-1}\left(\bigwedge\limits_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\wedge\left(\bigwedge\limits_{w=i_K}^{i_{K+1}-1}(v_{e-r+K}\otimes x_w)\right)\\ \qquad\wedge\bigwedge\limits_{t=K+1}^{r}\left(\bigwedge\limits_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\end{array}\right](\omega_{E\otimes G^*}).\end{cases}$$

We shift the indices in C_1 twice. First, replace K by $K+1$ to get C_1 is equal to

$$\begin{cases} \sum\limits_{0\le K\le r-1} \sum\limits_{\substack{(i)\in\mathcal{I}_r\\ i_{K+1}<i_{K+2}}} (-1)^{r+|(i)|+i_{K+1}+K} v_{e-r+K+1}\left(u_{e-r}^{(i_1)}\prod\limits_{p=1}^{r} u_{e-r+p}^{(i_{p+1}-i_p)}\right)\\ \quad\otimes\left[\bigwedge\limits_{q=1}^{r} V(u_{e-r+q})\right](\omega_{F^*})\otimes\prod\limits_{s=1}^{g} y_s^{(\#_s(i_1,\dots,\widehat{i_{K+1}},\dots,i_r))}\\ \quad\otimes\left[\begin{array}{l}\bigwedge\limits_{t=0}^{K}\left(\bigwedge\limits_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\wedge\left(\bigwedge\limits_{w=i_{K+1}+1}^{i_{K+2}}(v_{e-r+K+1}\otimes x_w)\right)\\ \qquad\wedge\bigwedge\limits_{t=K+2}^{r}\left(\bigwedge\limits_{w=i_t}^{i_{t+1}}(v_{e-r+t}\otimes x_w)\right)\end{array}\right](\omega_{E\otimes G^*}).\end{cases}$$

Fix K with $0\le K\le r-1$. Observe that there is a one-to-one correspondence between

$$\{(i)=(i_1,\dots,i_r)\in\mathcal{I}_r \mid i_{K+1}<i_{K+2}\}\quad\text{and}\quad\{(j)=(j_1,\dots,j_r)\in\mathcal{I}_r \mid j_K<j_{K+1}\}.$$

If (i) is in the left hand set, then the corresponding (j) in the right hand set has

$$j_\ell=\begin{cases} i_\ell & \text{if } \ell\ne K+1\\ i_{K+1}+1 & \text{if } \ell=K+1\end{cases}\tag{8.17}$$

It is clear that

$$1\le i_1\le\cdots\le i_K\le i_{K+1}<i_{K+2}\le i_{K+3}\le\cdots\le i_r\le g$$

if and only if

$$1\le i_1\le\cdots\le i_K<i_{K+1}+1\le i_{K+2}\le i_{K+3}\le\cdots\le i_r\le g.$$

Use (8.17) to transform the sum C_1. Observe that $|(i)|+1=|(j)|$,

$$v_{e-r+K+1}\left(u_{e-r}^{(i_1)}\prod_{p=1}^{r}u_{e-r+p}^{(i_{p+1}-i_p)}\right)=v_{e-r+K}\left(u_{e-r}^{(j_1)}\prod_{p=1}^{r}u_{e-r+p}^{(j_{p+1}-j_p)}\right),\text{ and}$$

$$\left(\bigwedge_{w=i_K}^{i_{K+1}}(v_{e-r+K}\otimes x_w)\right)\wedge\left(\bigwedge_{w=i_{K+1}+1}^{i_{K+2}}(v_{e-r+K+1}\otimes x_w)\right)$$

is equal to

$$\left(\bigwedge_{w=j_K}^{j_{K+1}-1}(v_{e-r+K}\otimes x_w)\right)\wedge\left(\bigwedge_{w=j_{K+1}}^{j_{K+2}}(v_{e-r+K+1}\otimes x_w)\right).$$

It is now clear that $C_1+C_2=0$ and $\boldsymbol{D}(\Theta_r)=t_{r-1}$. $\square$

Lemma 8.18. *The element $\boldsymbol{d}(Y)$ of $\mathbb{F}_{eg}$ is equal to*

$$\begin{cases} A_0(-1)^e[\boldsymbol{C}(\Xi_0)-\boldsymbol{A}(\Xi_1)] \in A(0,0,e,eg-e-1)\\ +(-1)^{eg+1}[(-1)^{\binom{g}{2}+e}A_0\boldsymbol{B}(\Xi_{g-1})-(-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})]\\ \qquad \in A(g-1,g,e-1,\boldsymbol{\alpha}),\end{cases}$$

plus an element of $\bigoplus_{(P,Q)} S(P,Q)$ with $P\le eg-e-1$ and $Q\le eg-2$; or $P\le eg-1$ and $Q\le eg-g-1$.

Proof. The element Y is defined in (8.9). Apply (8.8) to see that $\boldsymbol{d}(Y)$ is equal to

$$\begin{cases} A_0(-1)^e[\boldsymbol{C}(\Xi_0)-\boldsymbol{A}(\Xi_1)] \in A(0,0,e,eg-e-1)\\ +[(-1)^{eg+g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})-(-1)^{\binom{g}{2}+eg+e}A_0\boldsymbol{B}(\Xi_{g-1})] \in A(g-1,g,e-1,\boldsymbol{\alpha})\\ +A_0(-1)^{eg-e}\mathfrak{b}\boldsymbol{F}(\Xi_0)+B_0(-1)^{eg}\boldsymbol{E}(\Theta_0)\in B(eg)\\ \left.\begin{array}{l} -\sum\limits_{r=0}^{g-2}(-1)^{\binom{r+1}{2}+re}A_0\boldsymbol{B}(\Xi_r)\in A(r,r+1,e-1,eg-e-r)\\ +\sum\limits_{r=1}^{g-1}(-1)^{\binom{r+1}{2}+re+r+e}A_0\boldsymbol{D}(\Xi_r)\in A(r-1,r,e-1,eg-e-r+1)\end{array}\right\}*\\ \left.\begin{array}{l} +\sum\limits_{r=0}^{e-2}(-1)^{rg+\binom{r}{2}}B_0\boldsymbol{A}(\Theta_r)\in A(g-1,f-r-1,r,eg-g-r)\\ +\sum\limits_{r=1}^{e-1}(-1)^{rg+g+r+\binom{r}{2}}B_0\boldsymbol{D}(\Theta_r)\in A(g-1,f-r,r-1,eg-g-r+1)\end{array}\right\}*\\ +\sum\limits_{r=2}^{g-1}(-1)^{\binom{r+1}{2}+re}A_0\boldsymbol{A}(\Xi_r)\in S(eg-e-1,eg-r)\\ +\sum\limits_{r=1}^{g-1}(-1)^{\binom{r+1}{2}+re+r+e}A_0\boldsymbol{C}(\Xi_r)\in S(eg-e-1,eg-r-1)\\ -\sum\limits_{r=1}^{e-1}(-1)^{rg+\binom{r}{2}}B_0\boldsymbol{B}(\Theta_r)\in S(eg-r,eg-g-1)\\ +\sum\limits_{r=0}^{e-1}(-1)^{rg+g+r+\binom{r}{2}}B_0\boldsymbol{C}(\Theta_r)\in S(eg-r-1,eg-g-1)\end{cases}$$

Lemma 8.14 shows that the $B(eg)$-component of $\boldsymbol{d}(Y)$ is zero. Each of the sums labeled $*$ is zero by Lemma 8.16. □

Lemma 8.19. *The isomorphism $\bar{A}(0,0,e,eg-e-1)\to B(eg-1)$ of Lemma 8.6 sends the class of $A_0(-1)^e(\boldsymbol{C}(\Xi_0)-A_0\boldsymbol{A}(\Xi_1))$ to $(X\,\check{\circ}\,V)(\omega_{E\otimes G^*})$.*

Proof. The element $A_0\Xi_0$ is calculated in (8.15) and the map

$$\gamma\colon A(0,0,e,eg-e-1)\to B(eg-1)$$

carries $A_0\boldsymbol{C}(\Xi_0)$ to

$$(\omega_E\bowtie y_g^{(e)})\wedge(X\,\check{\circ}\,V)\left(\left[\omega_{E^*}\bowtie x_g^{(e)}\right](\omega_{E\otimes G^*})\right). \tag{8.20}$$

We know, from Definition 8.12, that Ξ_1 is the element

$$\begin{cases}\sum\limits_{h=1}^{e}(-1)^{1+h}u_h\otimes X^*(y_g)\otimes y_{g-1}^{(h)}y_g^{(e-h)}\\ \qquad\otimes\left[\left(\bigwedge\limits_{w=1}^{h}(v_w\otimes x_{g-1})\right)\wedge\left(\bigwedge\limits_{w=h}^{e}(v_w\otimes x_g)\right)\right](\omega_{E\otimes G^*})\end{cases}$$

of $A(1,1,e,eg-e-1)$. Remark 8.13 shows that $A_0\omega_E = u_e \wedge \ldots \wedge u_1$. It follows that $(-1)^{1+e}A_0(\gamma\circ \boldsymbol{A})(\Xi_1)$ is equal to

$$\begin{cases} \sum_{h=1}^{e}(-1)^{h+e}(X\check{\circ}V)(u_h\otimes y_g)\cdot\Big((u_e\wedge\ldots\wedge u_1)\bowtie y_{g-1}^{(h)}y_g^{(e-h)}\Big) \\ \qquad \wedge\left[\left(\bigwedge_{w=1}^{h}(v_w\otimes x_{g-1})\right)\wedge\left(\bigwedge_{w=h}^{e}(v_w\otimes x_g)\right)\right](\omega_{E\otimes G^*}).\end{cases}$$

Most of the terms of $(u_e\wedge\ldots\wedge u_1)\bowtie y_{g-1}^{(h)}y_g^{(e-h)}$ send

$$\left(\bigwedge_{w=1}^{h}(v_w\otimes x_{g-1})\right)\wedge\left(\bigwedge_{w=h}^{e}(v_w\otimes x_g)\right)$$

to zero. The only living term is

$$(u_e\otimes y_g)\wedge\ldots\wedge(u_{h+1}\otimes y_g)\wedge(u_h\otimes y_{g-1})\wedge\ldots\wedge(u_1\otimes y_{g-1}).$$

It follows that

$$\Big((u_e\wedge\ldots\wedge u_1)\bowtie y_{g-1}^{(h)}y_g^{(e-h)}\Big)\left[\left(\bigwedge_{w=1}^{h}(v_w\otimes x_{g-1})\right)\wedge\left(\bigwedge_{w=h}^{e}(v_w\otimes x_g)\right)\right]$$

is equal to $(-1)^{e-h}v_h\otimes x_g$, and $(-1)^{1+e}A_0(\gamma\circ\boldsymbol{A})(\Xi_1)$ is equal to

$$\begin{aligned} &\sum_{h=1}^{e}(X\check{\circ}V)(u_h\otimes y_g)\cdot(v_h\otimes x_g)(\omega_{E\otimes G^*})\\ &=\sum_{h=1}^{e}(X\check{\circ}V)(u_h\otimes y_g)\cdot(v_h\otimes x_g)\left((\omega_E\bowtie y_g^{(e)})\wedge\left[\omega_{E^*}\bowtie x_g^{(e)}\right](\omega_{E\otimes G^*})\right)\\ &=\sum_{h=1}^{e}(X\check{\circ}V)(u_h\otimes y_g)\cdot(v_h\otimes x_g)(\omega_E\bowtie y_g^{(e)})\wedge\left[\omega_{E^*}\bowtie x_g^{(e)}\right](\omega_{E\otimes G^*})\\ &=(X\check{\circ}V)(\omega_E\bowtie y_g^{(e)})\wedge\left[\omega_{E^*}\bowtie x_g^{(e)}\right](\omega_{E\otimes G^*}).\end{aligned}$$

Combine this with (8.20) to complete the proof. $\square$

Lemma 8.21. *The isomorphism* $\mathrm{H}_S(g-1,e-1,\boldsymbol{\alpha})\to\bigwedge^g F^*$ *of Lemma 8.6 sends the class of*

$$(-1)^{\binom{g}{2}+e}A_0\boldsymbol{B}(\Xi_{g-1})-(-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})$$

to $\omega_G(y_1\wedge\ldots\wedge y_g)\cdot\lambda(\omega_{F^*})$.

Proof. The element λ of $\bigwedge^e F$ is defined in 2.6. Apply Lemmas 8.16 and 8.14, together with the fact that

$$(-1)^{\binom{e-1}{2}+\binom{e}{2}+e-1}=1,$$

to see that $M_{g-1}\colon A(g-1,g,e-1,\boldsymbol{\alpha})\to\bigwedge^g F^*$ carries

$$(-1)^{\binom{g}{2}+e}A_0\boldsymbol{B}(\Xi_{g-1})-(-1)^{g+\binom{e-1}{2}}B_0\boldsymbol{A}(\Theta_{e-1})$$

to $(-1)^{\binom{g}{2}}A_1\cdot\lambda(\omega_{F^*})$. $\square$

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