Introduction.

Let $u_{1\times \boldsymbol{f}}$, $X_{\boldsymbol{f}\times \boldsymbol{f}}$, and $v_{\boldsymbol{f}\times 1}$ be matrices of indeterminates over a commutative noetherian ring R_0, and let $H(\boldsymbol{f})$ be the ideal $I_1(uX)+I_1(Xv)+I_1(vu-\operatorname{Adj}X)$ of the polynomial ring $R=R_0[\{u_i,v_i,x_{ij}\mid 1\le i,j\le \boldsymbol{f}\}]$. Vasconcelos observed that on numerous occasions, some specialization of $H(\boldsymbol{f})$ is the defining ideal for the symbolic square algebra $A[Pt,P^{(2)}t^2]$ of the prime ideal P in the commutative ring A. He conjectured [19] that $H(\boldsymbol{f})$ is a perfect prime Gorenstein ideal of grade $2\boldsymbol{f}$. In [16], we found the minimal homogeneous resolution of $R/H(\boldsymbol{f})$ by free $R-$modules; thereby establishing Vasconcelos' conjecture. This resolution is obtained by merging four Koszul complexes:

$$(*)\qquad \begin{array}{ccc} \mathbb{F}(1) & \longleftrightarrow & \mathbb{F}(2)\\ \updownarrow & & \updownarrow\\ \mathbb{F}(3) & \longleftrightarrow & \mathbb{F}(4), \end{array}$$

where $\mathbb{F}(1)$ and $\mathbb{F}(4)$ are both Koszul complexes on the entries of $[u\quad v]$, $\mathbb{F}(2)$ is the Koszul complex on the entries of $[uX\quad v]$, and $\mathbb{F}(3)$ is the Koszul complex on the entries of $[u\quad Xv]$. The arrows in (*) represent maps given by the various minors of X.

In the present paper, we consider the next natural question, which is, "What happens when the matrix X is not square?" In this case, the corresponding ideal, K, is equal to $I_1(uX)+I_1(Xv)+I_{\boldsymbol{f}}(X)$, where X is an $\boldsymbol{g}\times\boldsymbol{f}$ matrix, with $\boldsymbol{f}\le\boldsymbol{g}$, v is an $\boldsymbol{f}\times 1$ matrix, and u is a $1\times\boldsymbol{g}$ matrix. In other words, K is the ideal which defines the variety of complexes

$$0\to R\to R^{\boldsymbol{f}}\to R^{\boldsymbol{g}}\to R,$$

where the middle map has rank less than $\boldsymbol{f}$. It quickly becomes clear that the best way to resolve R/K is to produce a family of complexes which resolves "half" of the divisor class group of R/K.

Two distinct starting points give rise to a family of complexes with similar, and very pretty, properties. The first starting point is the theory of residual intersections. Let I be a grade two perfect ideal, or a grade three Gorenstein ideal, or a grade g complete intersection, and let

$$R^n\xrightarrow{P}R^g\xrightarrow{\mathbf{a}}R\to R/I\to 0$$

be exact. Assume that the ring R is the polynomial ring $k[P,X]$, where k is a field, X is a $g\times f$ generic matrix, and P is as generic as possible. Given this data with $\operatorname{grade}I\le f$, let K be the $f-$residual intersection $I_1(\mathbf{a}X):I$, ρ be the map

$$\rho=[\,P\quad X\,]:E=R^n\oplus R^f\to G=R^g,$$

and $\mathbf{m}$ and s be the integers $\mathbf{m}=f+1-\operatorname{grade}I$ and $s=f$. Then, there is a family of complexes $\{\mathcal{C}^{(z)}\}$ which satisfies the following properties.

(a) The complex $\mathcal{C}^{(0)}$ resolves R/K.

(b) The divisor class group of R/K is the infinite cyclic group $\mathbb{Z}[\operatorname{coker}\rho]$.

(c) If $-1 \le z$, then $\mathcal{C}^{(z)}$ resolves a representative of the class $z[\operatorname{coker}\rho]$ from $\mathrm{C}\ell\ R/K$.
(d) The canonical class in the $\mathrm{C}\ell\ R/K$ is equal to $\mathbf{m}[\operatorname{coker}\rho]$.
(e) $\mathcal{C}^{(z)} \cong \left(\mathcal{C}^{(\mathbf{m}-z)}\right)^* [-s]$.
(f) If M is a reflexive $R/K-$module of rank one and $[M] = z[\operatorname{coker}\rho]$ in $\mathrm{C}\ell\ R/K$ for some integer z, then M is a Cohen-Macaulay module if and only if $-1 \le z \le \mathbf{m}+1$.
(g) If $\widetilde{\rho} = \begin{bmatrix} P & \widetilde{X} \end{bmatrix}$, where $\widetilde{X}$ is the submatrix of X which consists of columns 1 to $f-1$, then, for each integer z, there is a short exact sequence of complexes

$$0 \to \mathcal{C}^{(z)}(\widetilde{\rho}) \to \mathcal{C}^{(z)} \to \mathcal{C}^{(z-1)}(\widetilde{\rho})[-1] \to 0.$$

Indeed, if I is a grade two perfect ideal, then $n = g-1$, P is the $g \times g-1$ matrix of indeterminates whose $g-1 \times g-1$ minors generate I, ρ is the $g \times (f+g-1)$ matrix of indeterminates $[P\,X]$, K is generated by the $g \times g$ minors of ρ (see [11, Thm. 4.1] or [12, pg. 4]), and $\mathcal{C}^{(z)}$ is the Eagon-Northcott type complex

$$\cdots \to D_1G^*\otimes {\textstyle\bigwedge}^{z+g+1} E \to D_0G^*\otimes {\textstyle\bigwedge}^{z+g} E \to S_0G\otimes {\textstyle\bigwedge}^{z} E \to S_1G\otimes {\textstyle\bigwedge}^{z-1} E \to \dots,$$

with $S_zG \otimes \bigwedge^0 E$ in position 0; see, for example, [6, Sect. 2C]. If I is a grade three Gorenstein ideal, then $n = g$, $P\colon G^* = R^n \to R^g = G$ is the $g \times g$ alternating matrix of indeterminates whose $g-1$ order pfaffians generate I, $E = G^* \oplus F$, K is generated by the pfaffians of all principal submatrices of $\left(\begin{smallmatrix} P & X \\ -X^{\mathrm{t}} & 0 \end{smallmatrix}\right)$ which contain P, and $\mathcal{C}^{(z)}$ is the complex

$$\cdots \to \left(\overline{S_1G \otimes {\textstyle\bigwedge}^{\mathbf{m}-z-1} E}\right)^* \to \left(\overline{S_0G \otimes {\textstyle\bigwedge}^{\mathbf{m}-z} E}\right)^* \to Q_z$$

$$\to \overline{S_0G \otimes {\textstyle\bigwedge}^{z} E} \to \overline{S_1G \otimes {\textstyle\bigwedge}^{z-1} E} \to \dots,$$

with $\overline{S_zG \otimes \bigwedge^0 E}$ in position 0, where

$$\overline{S_\bullet G \otimes {\textstyle\bigwedge}^\bullet E} = \tfrac{S_\bullet G\otimes \bigwedge^\bullet E}{(S_0G\otimes \bigwedge^g G^*,\eta)},$$

and η is the element of $G \otimes E$ which corresponds to $E^* = G \oplus F^* \xrightarrow{\text{proj}} G$ under the natural identification of $\operatorname{Hom}(E^*, G)$ and $G \otimes E$. See [17]. If I is a grade g complete intersection, then $n = \binom{g}{2}$, $\mathbf{a}$ is a $1 \times g$ matrix of indeterminates, $P\colon \bigwedge^2 R^g \to R^g$ is the Koszul complex map, K is equal to $I_1(\mathbf{a}X) + I_g(X)$, the complex $\mathcal{C}^{(0)}$ is given in [5], and the entire family $\{\mathcal{C}^{(z)}\}$ is given in [13].

There is a second starting point which produces an analogous family of complexes. In this case, there is no ideal I, there is no presentation map P of I, and there is no interpretation in terms of residual intersection. The best examples of this second starting point come from the theory of varieties of complexes. Start with the data

$$0 \to R \xrightarrow{v} F \xrightarrow{X} G \xrightarrow{u} R,$$

where F and G are free R–modules with $\operatorname{rank} F = \boldsymbol{f} \le \boldsymbol{g} = \operatorname{rank} G$, v, X, and u are matrices of indeterminates, and $R = k[v, X, u]$ for some field k. Let K be the R–ideal

$$K = \begin{cases} I_{\boldsymbol{f}}(X) + I_1(uX) & \text{in case 1,} \\ I_1(Xv) + I_{\boldsymbol{f}}(X) + I_1(uX) & \text{in case 2,} \end{cases}$$

and ρ be the R–module homomorphism

$$\rho = \begin{cases} [1 \otimes X^*(u) \quad X^*] \colon (F^* \otimes F) \oplus G^* \to F^* & \text{in case 1,} \\ [\, v \quad 1 \otimes X^*(u) \quad X^* \,] : \bigwedge^2 F^* \oplus (F^* \otimes F) \oplus G^* \to F^* & \text{in case 2.} \end{cases}$$

The integer s plays the role of the projective dimension of R/K as an R–module; hence,

$$s = \begin{cases} \boldsymbol{g} & \text{in case 1, and} \\ \boldsymbol{g} + \boldsymbol{f} - 1 & \text{in case 2.} \end{cases}$$

The integer $\mathbf{m}$ is defined by property (d); hence,

$$\mathbf{m} = \begin{cases} \boldsymbol{g} - \boldsymbol{f} - 1 & \text{in case 1, and} \\ \boldsymbol{g} - \boldsymbol{f} & \text{in case 2.} \end{cases}$$

Then, in each case (1) and (2), there is a family of complexes $\{\mathcal{C}^{(z)}\}$ which satisfy properties (a)—(g), provided (g) is modified to read

$$0 \to \mathcal{C}^{(z)}(\widetilde{\rho}\,) \to \mathcal{C}^{(z)} \otimes_R R/(u_{\boldsymbol{g}}) \to \mathcal{C}^{(z-1)}(\widetilde{\rho}\,)[-1] \to 0. \tag{0.1}$$

Case (1) is treated in [13]; the present paper is devoted to finding the family of complexes $\{\mathcal{C}^{(z)}\}$ in case (2). In fact, given the data of case (2), we produce two families of complexes. The complexes $\{\mathbb{I}^{(z)}\}$ of section 2 are not minimal, but the maps are well understood. The complexes $\{\mathbb{M}^{(z)}\}$ of section 4 are minimal, but the maps are very complicated, and less well understood.

We begin by recording what is known about R/K in case (2). Theorem 0.2 has been established by De Concini and Strickland [10] using Hodge algebra techniques.

Theorem 0.2. *Let R_0 be a commutative noetherian ring, $2 \le \boldsymbol{f} \le \boldsymbol{g}$ be integers, $v_{\boldsymbol{f}\times 1}$, $X_{g\times \boldsymbol{f}}$ and $u_{1\times \boldsymbol{g}}$ be matrices of indeterminates, R be the polynomial ring $R_0[v, X, u]$, and K be the R–ideal $I_1(uX) + I_{\boldsymbol{f}}(X) + I_1(Xv)$.*

(*a*) *The ring R/K is reduced* (*respectively, Cohen-Macaulay, a domain, a normal domain*) *if and only if R_0 satisfies the same property.*

(*b*) *The ideal K is generically perfect of grade $\boldsymbol{f} + \boldsymbol{g} - 1$.*

(*c*) *The ring R/K satisfies Serre's condition (S_i) if and only if R_0 satisfies (S_i).*

The proof and notation of Theorem 0.3 may be found in Bruns [4]. The form of the divisor class group of R/K, but not its generators, may also be found in Yoshino [20].

Theorem 0.3. *Retain the hypotheses of Theorem 0.2, with R_0 a normal domain. Let*

$$\mathfrak{b}_3 = \tfrac{I_1(v)+K}{K},\ \mathfrak{p}_2 = \tfrac{(v_1)+I_{f-1}(\text{columns 2 to } f \text{ of } X)+K}{K},\ \mathfrak{a}_2 = \tfrac{I_{f-1}(\text{rows 1 to } f-1 \text{ of } X)+K}{K},$$
$$\mathfrak{r}_2 = \tfrac{I_{f-1}(X)+K}{K}, \quad \textit{and} \quad \mathfrak{p}_1 = \tfrac{(u_g)+I_{f-1}(\text{rows 1 to } f-1 \text{ of } X)+K}{K}$$

represent various ideals of R/K.

(a) If $\boldsymbol{f} < \boldsymbol{g}$, then $\mathfrak{b}_3$, $\mathfrak{a}_2$, and $\mathfrak{p}_2$ all are height one prime ideals of R/K. Furthermore, $\mathrm{C}\ell\, R/K = \mathrm{C}\ell\, R_0 \oplus \mathbb{Z}$, where the summand $\mathbb{Z}$ is generated by the class $[\mathfrak{b}_3]$ and the equations

$$[\mathfrak{b}_3] = [\mathfrak{a}_2] = -[\mathfrak{p}_2]$$

hold in $\mathrm{C}\ell\, R/K$.

(b) If $\boldsymbol{f} = \boldsymbol{g}$, then $\mathfrak{b}_3$, $\mathfrak{p}_2$, $\mathfrak{r}_2$, and $\mathfrak{p}_1$ all are height one prime ideals of R/K. Furthermore, $\mathrm{C}\ell\, R/K = \mathrm{C}\ell\, R_0 \oplus \mathbb{Z} \oplus \mathbb{Z}$ where one summand $\mathbb{Z}$ is generated by the class $[\mathfrak{b}_3]$, the other summand $\mathbb{Z}$ is generated by $[\mathfrak{r}_2]$, and the equations

$$[\mathfrak{b}_3] = -[\mathfrak{p}_2] \quad \textit{and} \quad [\mathfrak{r}_2] = -[\mathfrak{p}_1] - [\mathfrak{p}_2]$$

hold in $\mathrm{C}\ell\, R/K$.

(c) If ω_{R_0} is the canonical module of R_0, then the class of the canonical module of R/K in $\mathrm{C}\ell\, R/K$ is $[\omega_{R_0} R/K] + (\boldsymbol{g} - \boldsymbol{f})[\mathfrak{b}_3]$.

(d) If P is a prime ideal of R, then $(R/K)_P$ is a regular local ring if and only if $(R_0)_{R_0 \cap P}$ is a regular local ring and $I_{\boldsymbol{f}-1}(X) + I_1(u)I_1(v) \not\subseteq P$.

Section 1 is devoted to collecting the relevant facts; especially from the theory of multilinear algebra. In 2, we define $\mathbb{I}^{(z)}$, prove that it is a complex, give examples, and establish the duality between $\mathbb{I}^{(z)}$ and $\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$. In 3, we identify the zeroth homology of the complex $\mathbb{I}^{(z)}$; we establish homomorphisms from $H_0(\mathbb{I}^{(z)})$ to ideals of $H_0(\mathbb{I}^{(0)}) = R/K$ (these homomorphisms are shown to be isomorphisms in section 8); and we record the short exact sequence of complexes (0.1) for the $\mathbb{I}^{(z)}$. In 4, we split off a split exact summand of $\mathbb{I}^{(z)}$ in order to produce the complex $\mathbb{M}^{(z)}$, which is minimal whenever the data is local or homogeneous of positive degree. This section concludes with a list of examples. The modules $\mathcal{M}(p,q,r)$, which comprise the complex $\mathbb{M}^{(z)}$, are defined and shown to be free in 5. Section 6 is a calculation about binomial coefficients which is used to find the rank of $\mathcal{M}(p,q,r)$. In 7, we prove the results which are stated in section 4; thereby completing the proof that $\mathbb{M}^{(z)}$ is homologically equivalent to $\mathbb{I}^{(z)}$. In sections 8 and 9 we prove that the complex $\mathbb{I}^{(z)}$ is acyclic. The proof is by induction on $\boldsymbol{g}$ and uses the short exact sequence (0.1). The inductive step is in 8 and the base case, $\boldsymbol{g} = \boldsymbol{f} - 1$, is in 9.

1. Preliminary results.

In this paper "ring" means commutative noetherian ring with one. The *grade* of a proper ideal I in a ring R is the length of the longest regular sequence on R in I. An $R-$module M is called *perfect* if the grade of the annihilator of M is equal to the projective dimension of M. The ideal I of R is called *perfect* if R/I is a perfect $R-$module. An excellent reference on perfect modules is [6, Sect. 16C]. For any $R-$module F, we write $F^* = \operatorname{Hom}_R(F,R)$. If $f\colon F \to G$ is a map of $R-$modules, then we define $I_r(f)$ to be the image of the map $\bigwedge^r F \otimes (\bigwedge^r G)^* \to R$, which is induced by the map $\bigwedge^r f\colon \bigwedge^r F \to \bigwedge^r G$. (In particular, if F and G are free modules, then $I_r(f)$ is the ideal in R which is generated by the $r\times r$ minors of any matrix representation of f.) Let F be a free $R-$module of finite rank. We make much use of the exterior algebra $\bigwedge^\bullet F$, the symmetric algebra $S_\bullet F$, and the divided power algebra $D_\bullet F$. In particular, $\bigwedge^\bullet F$ and $\bigwedge^\bullet F^*$ are modules over one another, and $S_\bullet F$ and $D_\bullet F^*$ are modules over one another. Indeed, if $\alpha_i \in \bigwedge^i F^*$, $b_j \in \bigwedge^j F$, $A_i \in S_i(F^*)$, and $B_j \in D_j(F)$, then

$$\alpha_i(b_j) \in {\textstyle\bigwedge^{j-i}} F,\ \ b_j(\alpha_i) \in {\textstyle\bigwedge^{i-j}} F^*,\ \ A_i(B_j) \in D_{j-i}(F),\ \text{ and }\ B_j(A_i) \in S_{i-j}(F^*).$$

(We view $\bigwedge^i F$, S_iF, and D_iF to be meaningful for every integer i; in particular, these modules are zero whenever i is negative.) The exterior, symmetric, and divided power algebras A all come equipped with co-multiplication $\Delta\colon A \to A\otimes A$. The following facts are well known; see [7, section 1], [8, Appendix], and [16, section 1].

Proposition 1.1. *Let F be a free module of rank $\boldsymbol{f}$ over a commutative noetherian ring R and let $b_r \in \bigwedge^r F$, $b'_p \in \bigwedge^p F$, and $\alpha_q \in \bigwedge^q F^*$.*

(a) *If $r=1$, then $(b_r(\alpha_q))(b'_p) = b_r \wedge (\alpha_q(b'_p)) + (-1)^{1+q}\alpha_q(b_r\wedge b'_p)$.*
(b) *If $q=\boldsymbol{f}$, then $(b_r(\alpha_q))(b'_p) = (-1)^{(\boldsymbol{f}-r)(\boldsymbol{f}-p)}\left(b'_p(\alpha_q)\right)(b_r)$.*
(c) *If $p=\boldsymbol{f}$, then $[b_r(\alpha_q)](b'_p) = b_r\wedge\alpha_q(b'_p)$.*
(d) *If $X\colon F\to G$ is a homomorphism of free $R-$modules and $\delta_{s+r}\in\bigwedge^{s+r}G^*$, then $(\bigwedge^s X^*)\left[((\bigwedge^r X)(b_r))(\delta_{s+r})\right] = b_r\left[\left(\bigwedge^{s+r}X^*\right)(\delta_{s+r})\right]$.*

Note. The exponent which is given in (b) is correct. An incorrect value has appeared elsewhere in the literature.

The following data is in effect throughout most of the paper.

Data 1.2. Let F and G be free modules of rank $\boldsymbol{f}$ and $\boldsymbol{g}$, respectively, over the commutative noetherian ring R. Let $u\in G^*$, $v\in F$, and $X\colon F\to G$ be an $R-$module homomorphism.

Note 1.3. We will always take $A_p\in S_pF^*$, $B_p\in D_pF$, $\alpha_q\in\bigwedge^q F^*$, $b_r\in\bigwedge^r F$, $c_s\in\bigwedge^s G$, and $\delta_q\in\bigwedge^q G^*$. In particular, a lower case subscript will give the position of a homogeneous element, whenever possible.

Convention 1.4. Orient F and G by fixing basis elements $\omega_F\in\bigwedge^{\boldsymbol{f}}F$, $\omega_{F^*}\in\bigwedge^{\boldsymbol{f}}F^*$, $\omega_G\in\bigwedge^{\boldsymbol{g}}G$, and $\omega_{G^*}\in\bigwedge^{\boldsymbol{g}}G^*$ with $\omega_F(\omega_{F^*})=1$ and $\omega_G(\omega_{G^*})=1$. All of our maps are coordinate free; however, sometimes the easiest way to describe a map is to tell what it does to a basis. Consequently, we fix bases $f^{[1]},\dots,f^{[\boldsymbol{f}]}$ for F and $g^{[1]},\dots,g^{[\boldsymbol{g}]}$ for G. Let $\varphi^{[1]},\dots,\varphi^{[\boldsymbol{f}]}$ and $\gamma^{[1]},\dots,\gamma^{[\boldsymbol{g}]}$ be the corresponding dual bases for F^* and G^*, respectively.

Convention 1.5. (a) Sometimes we think as the data of 1.2 as matrices:

$$u = [\,u_1 \quad \dots \quad u_{\boldsymbol{g}}\,], \quad X = \begin{bmatrix} x_{11} & \dots & x_{1\boldsymbol{f}} \\ \vdots & & \vdots \\ x_{\boldsymbol{g}1} & \dots & x_{\boldsymbol{g}\boldsymbol{f}} \end{bmatrix}, \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_{\boldsymbol{f}} \end{bmatrix}.$$

(b) If $\{u_i\}\cup\{x_{jk}\}\cup\{v_\ell\}$ is a list of indeterminates over a commutative noetherian ring R_0, and R is the polynomial ring $R_0[\{u_i\}\cup\{x_{jk}\}\cup\{v_\ell\}]$, then we say that the data of 1.2 is generic.

Convention 1.6. The bases and orientation elements of Convention 1.4 are related by the following equations:

$$\omega_F = f^{[1]}\wedge\dots\wedge f^{[\boldsymbol{f}]}, \quad \omega_{F^*} = \varphi^{[\boldsymbol{f}]}\wedge\dots\wedge\varphi^{[1]},$$
$$\omega_G = g^{[1]}\wedge\dots\wedge g^{[\boldsymbol{g}]}, \quad \text{and} \quad \omega_{G^*} = \gamma^{[\boldsymbol{g}]}\wedge\dots\wedge\gamma^{[1]}.$$

If I represents the ordered $i-$tuple of integers $a_1 < a_2 < \dots < a_i$, (we write $|I| = i$), then let

$$f_I = f^{[a_1]}\wedge\dots\wedge f^{[a_i]} \quad \text{and} \quad \varphi_I = \varphi^{[a_i]}\wedge\dots\wedge\varphi^{[a_1]}.$$

Notice that the element

$$\sum_{|I|=i}\varphi_I\otimes f_I$$

of $\bigwedge^i F^*\otimes\bigwedge^i F$ is canonical in the sense that it does not depend on the choice of dual bases $f^{[1]},\dots,f^{[\boldsymbol{f}]}$ and $\varphi^{[1]},\dots,\varphi^{[\boldsymbol{f}]}$. (Indeed, this element corresponds to the identity map under the canonical identification of $\operatorname{Hom}(\bigwedge^i F,\bigwedge^i F)$ with $\bigwedge^i F^*\otimes\bigwedge^i F$.) The above sum is taken over all ordered $i-$tuples of $\{1,\dots,\boldsymbol{f}\}$. (The ambient set in which I lies, in this case $\{1,\dots,\boldsymbol{f}\}$, will always be clear from context.)

Convention 1.7. If $b_q\in\bigwedge^q F$, then we use $(b_q\otimes 1)*\_\_$ to represent the homomorphism $\bigwedge^q F^*\otimes M\to M$, which sends $\alpha_q\otimes m$ to $b_q(\alpha_q)\cdot m$, for any $R-$module M.

Example 1.8. Adopt Data 1.2. The easiest way to prove the identity

$$\sum_{|I|=1}\varphi_I\otimes X(f_I) = \sum_{|K|=1}X^*(\gamma_K)\otimes g_K \quad \in F^*\otimes G,$$

is observe that both sides become $X(b_1)$, upon application of $(b_1\otimes 1)*\_\_$, for an arbitrary element b_1 of F. (Notice that $I\subseteq\{1,\dots,\boldsymbol{f}\}$ and $K\subseteq\{1,\dots,\boldsymbol{g}\}$, and, as promised, this is clear from the context.)

Lemma 1.9. *Adopt Data 1.2. If k is a fixed integer, $\alpha_p\in\bigwedge^p F^*$, $b_q\in\bigwedge^q F$, and $b_r\in\bigwedge^r F$, then*

(a) $\Delta(\alpha_p) = \sum_i\sum_{|I|=i}\varphi_I\otimes f_I(\alpha_p)$,

(b) $\sum\limits_{|K|=k} \varphi_K \otimes f_K(\alpha_p) = (-1)^{k(p-k)} \sum\limits_{|K|=p-k} f_K(\alpha_p) \otimes \varphi_K,$

(c) $\sum\limits_{|K|=k} \varphi_K \wedge f_K(\alpha_p) = \binom{p}{k}\alpha_p,$

(d) $\alpha_p(b_q \wedge b_r) = \sum\limits_i \sum\limits_{|I|=i} (-1)^{q(p-i)}\varphi_I(b_q) \wedge [f_I(\alpha_p)](b_r),$

(e) $\sum\limits_{|K|=k} \varphi_K \otimes f_K \wedge b_q \wedge b_r = \sum\limits_{|K|=k+q} b_q(\varphi_K) \otimes f_K \wedge b_r,$

(f) $\sum\limits_{|K|=k} [b_q(\varphi_K)](f_K \wedge b_r) = \binom{\boldsymbol{f}-q-r}{k-q} b_q \wedge b_r,$ *and*

(g) $\sum\limits_{|K|=k} \varphi_K \otimes f_K = \sum\limits_{|K|=\boldsymbol{f}-k} (-1)^{k(1+\boldsymbol{f})} f_K(\omega_{F^*}) \otimes \varphi_K(\omega_F).$

Proof. To prove (a), fix i and project onto $\bigwedge^i F^* \otimes \bigwedge^{p-i} F^*$. Both sides become $b_i(\alpha_p)$, upon application of $(b_i \otimes 1) * \_\_$. Apply $(b_k \otimes 1) * \_\_$ to both sides of (b). The left side becomes $\sum\limits_{|K|=k} b_k(\varphi_K) \cdot f_K(\alpha_p) = b_k(\alpha_p)$. The right side becomes

$$(-1)^{k(p-k)} \sum_{|K|=p-k} (b_k \wedge f_K)(\alpha_p) \cdot \varphi_K = \sum_{|K|=p-k} f_K[b_k(\alpha_p)] \cdot \varphi_K = b_k(\alpha_p).$$

Part (c) follows from (a), together with the well-known fact that the composition

$$\bigwedge^p F^* \xrightarrow{\Delta} \bigwedge^k F^* \otimes \bigwedge^{p-k} F^* \xrightarrow{\mu} \bigwedge^p F^*$$

is equal to multiplication by $\binom{p}{k}$. Part (d) is an immediate consequence the measuring identity, [8, Proposition A.2], together with (a). Apply $(b_k \otimes 1) * \_\_$ to both sides of (e). The left side becomes

$$\sum_{|K|=k} b_k(\varphi_K) \cdot f_K \wedge b_q \wedge b_r = b_k \wedge b_q \wedge b_r.$$

The right side becomes

$$\sum_{|K|=k+q} b_k \left(b_q(\varphi_K)\right) \cdot f_K \wedge b_r = \sum_{|K|=k+q} \left(b_k \wedge b_q\right) (\varphi_K) \cdot f_K \wedge b_r = b_k \wedge b_q \wedge b_r.$$

Part (e) shows that the left side of (f) is equal to $\sum\limits_{|K|=k-q} \varphi_K(f_K \wedge b_q \wedge b_r)$. One may finish the proof of (f) by establishing the assertion when $b_q \wedge b_r$ is a basis vector from $\bigwedge^{q+r} F$. Apply $(b_k \otimes 1) * \_\_$ to each side of (g); then use part (b) of Proposition 1.1. $\square$

Remark 1.10. With the exception of section 6, binomial coefficients play only a minor role in this paper. Nonetheless, it should be mentioned, at the beginning, that $\binom{m}{i}$ is defined for all integers m and i. This binomial coefficient is zero whenever $i < 0$ or $0 \le m < i$. See [14,15] for more details.

Each complex $\mathbb{I}^{(z)}$ of section 2 is obtained by splicing together two smaller complexes. The next result is the multilinear algebra which is used in the proof of Theorem 2.11 at this splice. It is not apparent, at first glance, but identities (a) and (b) are actually dual to one another. The proof we have given of (b) emphasizes this realtionship. On the other hand, one can give a proof of (b) which mimics the proof of (a).

Lemma 1.11. *Adopt Data 1.2. Let $\alpha_p \in \bigwedge^p F^*$, $b_q \in \bigwedge^q F$, $\alpha_r \in \bigwedge^r F^*$, and M and w be integers.*

(a) *If $\boldsymbol{f}+1+w \le p+r$, then* $\sum\limits_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{ip}\binom{M-i}{w} f_I(\alpha_p)\wedge[\varphi_I(b_q)](\alpha_r) = 0.$

(b) *If $w+p+1 \le q$, then* $\sum\limits_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{i(p+1)}\binom{M-i}{w} f_I\left[\alpha_p\wedge[\varphi_I(b_q)](\alpha_r)\right] = 0.$

Proof. We first prove (a). For each pair of integers (M,w), let

$$h_{M,w}\colon \bigwedge^\bullet F^*\otimes\bigwedge^\bullet F\otimes\bigwedge^\bullet F^*\to\otimes\bigwedge^\bullet F^*$$

be the homomorphism which is given by

$$h_{M,w}(\alpha_p\otimes b_q\otimes\alpha_r) = \sum_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{ip}\binom{M-i}{w} f_I(\alpha_p)\wedge[\varphi_I(b_q)](\alpha_r).$$

It is clear that $h_{M,w}$ is the zero homomorphism whenever $w<0$. Lemma 1.9.d shows that

$$h_{M,0}(\alpha_p\otimes b_q\otimes\alpha_r) = (-1)^{pq} b_q(\alpha_p\wedge\alpha_r);$$

thus, the conclusion holds whenever $w=0$. The proof proceeds by induction on w. Observe that

$$\begin{aligned} &b_1\left[h_{M,w}(\alpha_p\otimes b_q\otimes\alpha_r)\right]\\ &= (-1)^p h_{M,w}(\alpha_p\otimes b_1\wedge b_q\otimes\alpha_r) + h_{M-1,w-1}(b_1(\alpha_p)\otimes b_q\otimes\alpha_r) \end{aligned} \tag{1.12}$$

for all $b_1\in\bigwedge^1 F$. Indeed, the left side of (1.12) is equal to $A+B$, where

$$A = \sum_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{ip}\binom{M-i}{w} b_1\left[f_I(\alpha_p)\right]\wedge[\varphi_I(b_q)](\alpha_r) \quad\text{and}$$

$$B = \sum_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{ip+p-i}\binom{M-i}{w} f_I(\alpha_p)\wedge b_1\left([\varphi_I(b_q)](\alpha_r)\right).$$

Use Proposition 1.1.a to write

$$b_1\left([\varphi_I(b_q)](\alpha_r)\right) = [b_1\wedge\varphi_I(b_q)](\alpha_r) = \left([b_1(\varphi_I)](b_q)\right)(\alpha_r) + (-1)^i\left(\varphi_I(b_1\wedge b_q)\right)(\alpha_r).$$

Apply Lemma 1.9.e to see that $B=B_1+B_2$ for

$$B_1 = \sum_{\substack{i\in\mathbb{Z}\\ |I|=i-1}} (-1)^{ip+p-i}\binom{M-i}{w} f_I[b_1(\alpha_p)]\wedge[\varphi_I(b_q)](\alpha_r) \quad\text{and}$$

$$B_2 = \sum_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{ip+p}\binom{M-i}{w} f_I(\alpha_p)\wedge[\varphi_I(b_1\wedge b_q)](\alpha_r) = (-1)^p h_{M,w}(\alpha_p\otimes b_1\wedge b_q\otimes\alpha_r).$$

A short calculation yields that $A+B_1 = h_{M-1,w-1}(b_1(\alpha_p)\otimes b_q\otimes \alpha_r)$. Now that (1.12) is established, we continue with the induction. If $\boldsymbol{f}+1+w\le p+r$ and the induction hypothesis is known to hold at $w-1$, then (1.12) shows that

$$b_1\left[h_{M,w}(\alpha_p\otimes b_q\otimes\alpha_r)\right] = (-1)^p h_{M,w}(\alpha_p\otimes b_1\wedge b_q\otimes\alpha_r);$$

hence,

$$h_{M,w}(\alpha_p\otimes b_q\otimes\alpha_r) = (-1)^{pq} b_q\left[h_{M,w}(\alpha_p\otimes 1\otimes\alpha_r)\right] = (-1)^{pq}\tbinom{M}{w} b_q(\alpha_p\wedge\alpha_r) = 0.$$

Now we prove (b). Assume that $w+p+1\le q$. We prove

$$\sum_{\substack{i\in\mathbb{Z}\\ |I|=i}} (-1)^{i(p+1)}\tbinom{M-i}{w}\left(f_I\left[\alpha_p\wedge[\varphi_I(b_q)](\alpha_r)\right]\right)(\omega_F) = 0. \tag{1.13}$$

Let $\Delta(\alpha_p) = \sum\limits_{s,\,[j]} \alpha_s^{[j]}\otimes\alpha'^{[j]}_{p-s}$, with $\alpha_s^{[j]}\in\bigwedge^s F^*$ and $\alpha'^{[j]}_{p-s}\in\bigwedge^{p-s}F^*$. Fix I, with $|I|=i$. Proposition 1.1.c, together with the measuring identity, gives

$$\begin{aligned}\left(f_I\left[\alpha_p\wedge[\varphi_I(b_q)](\alpha_r)\right]\right)(\omega_F) &= f_I\wedge\alpha_p\left(\varphi_I(b_q)\wedge\alpha_r(\omega_F)\right)\\ &= \sum_{s,\,[j]}(-1)^{(q-i)(p-s)} f_I\wedge\alpha_s^{[j]}[\varphi_I(b_q)]\wedge\alpha'^{[j]}_{p-s}[\alpha_r(\omega_F)]\\ &= \sum_{s,\,[j]}(-1)^{(q-i)(p-s)+iq+i}\varphi_I[\alpha_s^{[j]}(b_q)]\wedge[f_I(\alpha'^{[j]}_{p-s}\wedge\alpha_r)](\omega_F).\end{aligned}$$

It follows that the left side of (1.13) is

$$\sum_{s,\,[j]}(-1)^{q(p-s)}\sum_{\substack{i\in\mathbb{Z}\\ |I|=i}}(-1)^{i(q-s)}\tbinom{M-i}{w}\varphi_I[\alpha_s^{[j]}(b_q)]\wedge[f_I(\alpha'^{[j]}_{p-s}\wedge\alpha_r)](\omega_F),$$

which is zero by part (a). $\square$

The phrase "Koszul complex" has two meanings in this paper. If $X\colon F\to R$ is a map of free $R-$modules, then the Koszul complex associated to X is

$$\dots\xrightarrow{\partial}\textstyle\bigwedge^q F\xrightarrow{\partial}\bigwedge^{q-1}F\xrightarrow{\partial}\dots, \tag{1.14}$$

where

$$\partial\,(b_q) = \sum_{|I|=1} X(f_I)\cdot\varphi_I(b_q),$$

for all $b_q\in\bigwedge^q F$. Of course, if $b_q = b^{[1]}\wedge\dots\wedge b^{[q]}$, with $b^{[i]}\in F$, then

$$\partial\,(b_q) = \sum_{i=1}^{q}(-1)^{i+1}X(b^{[i]})\cdot b^{[1]}\wedge\dots\wedge b^{[i-1]}\wedge b^{[i+1]}\wedge\dots\wedge b^{[q]};$$

see Lemma 1.9.a, if necessary. If $X\colon F \to G$ is a map of free $R-$modules, then the Koszul complex associated to X is

$$\dots \xrightarrow{\partial} S_\bullet G \otimes_R \bigwedge^q F \xrightarrow{\partial} S_\bullet G \otimes_R \bigwedge^{q-1} F \xrightarrow{\partial} \dots, \tag{1.15}$$

where

$$\partial\,(s \otimes b_q) = \sum_{|I|=1} s \cdot X(f_I) \otimes \varphi_I(b_q),$$

for all $s \in S_\bullet G$ and all $b_q \in \bigwedge^q F$. If the G of (1.15) is equal to R, then the two complexes are much different. We will always make our meaning clear.

Remark 1.16. It is well known that if the map X, of (1.15), is an isomorphism, then the graded strand

$$\dots \xrightarrow{\partial} S_p G \otimes_R \bigwedge^q F \xrightarrow{\partial} S_{p+1} G \otimes_R \bigwedge^{q-1} F \xrightarrow{\partial} \dots,$$

of the Koszul complex associated to X, is split exact for all integers p and q, provided $p+q \neq 0$.

A *quasi-isomorphism* of complexes is a homomorphism of complexes which induces an isomorphism on homology. Two complexes $\mathbb{A}$ and $\mathbb{B}$ are *homologically equivalent* if there is a sequence quasi-isomorphisms between them:

$$\mathbb{A} = \mathbb{A}^{(0)} \to \mathbb{A}^{(1)} \leftarrow \mathbb{A}^{(2)} \to \dots \leftarrow \mathbb{A}^{(n-1)} \to \mathbb{A}^{(n)} = \mathbb{B}.$$

We close this section by recording two conventions which simplify the description of the differential in the complexes of section 2.

Notation 1.17. For each integer n, let $\theta_n = (-1)^{\frac{n(n+1)}{2}}$.

Observation 1.18. *If p, q and r are integers then*

(a) $\theta_p\theta_{p+1} = (-1)^{p+1}$,
(b) $\theta_p\theta_r\theta_{p+r} = (-1)^{pr}$, *and*
(c) $\theta_p\theta_q\theta_{p+r}\theta_{q+r} = (-1)^{(p+q)r}$.

Proof. It suffices to prove (b) and this is trivial.

Convention 1.19. For each statement "S", let

$$\chi(\mathrm{S}) = \begin{cases} 1, & \text{if S is true, and} \\ 0, & \text{if S is false.} \end{cases}$$

In particular, $\chi(i=j)$ has the same value as the Kronecker delta δ_{ij}.

2. The complex $\mathbb{I}^{(z)}$.

Given the data of 1.2, we create a family of complexes $\{\mathbb{I}^{(z)} \mid z \in \mathbb{Z}\}$. The free $R-$modules which are the building blocks for the $\mathbb{I}^{(z)}$ are introduced in Definition 2.1. The official modules, maps, and grading of the $\mathbb{I}^{(z)}$ are given in Definition 2.3. The proof that $\mathbb{I}^{(z)}$ is a complex occurs in Theorem 2.11. The duality between $\mathbb{I}^{(z)}$ and $\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$ is established in Proposition 2.12. An informal description of the complexes $\mathbb{I}^{(z)}$ is given in Remark 2.2. The calculations which verify all of the assertions in this remark are equivalent to the proof of Theorem 2.11 and Proposition 2.12. Some example of $\mathbb{I}^{(z)}$ for small $\boldsymbol{f}$ and $\boldsymbol{g}$ are given in Examples 2.5 — 2.10.

Definition 2.1. Adopt Data 1.2. Let

$$\begin{aligned}
\mathbb{L}(p,q,r,s,t) &= S_pF^* \otimes \textstyle\bigwedge^q F^* \otimes \bigwedge^r F \otimes \bigwedge^s G \otimes R\nu^{(t)},\\
\mathbb{U}(p,q,r) &= \textstyle\bigwedge^p F \otimes \bigwedge^q G^* \otimes R\mu^{(r)},\\
\mathbb{T}(p,q,r) &= \textstyle\bigwedge^p F^* \otimes \bigwedge^q G \otimes R\lambda^{(r)}, \quad \text{and}\\
\mathbb{W}(p,q,r,s,t) &= D_pF \otimes \textstyle\bigwedge^q F \otimes \bigwedge^r F^* \otimes \bigwedge^s G^* \otimes R\xi^{(t)},
\end{aligned}$$

where each of the modules $R\nu^{(t)}$, $R\mu^{(r)}$, $R\lambda^{(r)}$, and $R\xi^{(t)}$ is a free $R-$module of rank one.

Remark 2.2. Retain Data 1.2. Let S be the symmetric algebra $S_\bullet^R F^*$. Define a DG$-$algebra $\mathbb{L}$ over S and a DG$-$algebra $\mathbb{U}$ over R as follows. Let

$$\textstyle\bigwedge_S^\bullet (S \otimes_R (F^* \oplus F \oplus G))$$

be the Koszul complex, in the sense of (1.14), which is associated to the $S-$module map

$$S \otimes_R (F^* \oplus F \oplus G) \to S: \qquad \begin{bmatrix} \alpha_1 \\ b_1 \\ c_1 \end{bmatrix} \mapsto v(\alpha_1) + \alpha_1 + [X^*(u)](b_1) + u(c_1),$$

for $\alpha_1 \in F^*$, $b_1 \in F$, and $c_1 \in G$. (Notice that $v(\alpha_1)$, $[X^*(u)](b_1)$, and $u(c_1)$ are all in S_0F^*; however, α_1 is in S_1F^*.) The DG$-$algebra

$$\mathbb{L} = \left[\textstyle\bigwedge_S^\bullet (S \otimes_R (F^* \oplus F \oplus G))\right] <\nu>$$

is obtained from this Koszul complex by adjoining a divided power variable ν which kills the cycle

$$\sum_{|I|=1} \varphi_I \otimes \begin{bmatrix} 1 \\ f_I \\ 1 \end{bmatrix} - \sum_{|I|=1} X^*(\gamma_I) \otimes \begin{bmatrix} 1 \\ 1 \\ g_I \end{bmatrix}.$$

Let $\bigwedge_R^\bullet(F \oplus G^*)$ be the Koszul complex, in the sense of (1.14), associated to the $R-$module map

$$F \oplus G^* \to R: \qquad \begin{bmatrix} b_1 \\ \delta_1 \end{bmatrix} \mapsto [X^*(u)](b_1) + [X(v)](\delta_1)$$

for $b_1 \in F$ and $\delta_1 \in G^*$. The DG$-$algebra

$$\mathbb{U} = \left(\textstyle\bigwedge_R^\bullet(F \oplus G^*)\right) <\mu>$$

is obtained from this Koszul complex by adjoining a divided power variable μ which kills the cycle $\left[\begin{smallmatrix} v \\ -u \end{smallmatrix}\right]$. In the language of Definition 2.1, we have

$$\mathbb{L} = \bigoplus_{0 \le t} \mathbb{L}(p,q,r,s,t) \quad \text{and} \quad \mathbb{U} = \bigoplus_{0 \le r} \mathbb{U}(p,q,r)$$

as $R-$modules.

Fix a non-negative integer z. Observe that

$$\mathbb{L}' = \bigoplus_{s+t\le \boldsymbol{g}-1} \mathbb{L}(p,q,r,s,t) + \bigoplus_{z\le p+t} \mathbb{L}(p,q,r,s,t) + \bigoplus_{p+q+t\le z-1} \mathbb{L}(p,q,r,s,t)$$

is a subcomplex of $\mathbb{L}$ and

$$\mathbb{U}' = \bigoplus_{q+r\le z-1} \mathbb{U}(p,q,r)$$

is a subcomplex of $\mathbb{U}$. Let $\overline{\mathbb{L}}$ represent the complex $\mathbb{L}/\mathbb{L}'$ and $\overline{\mathbb{U}}$ represent the complex $\mathbb{U}/\mathbb{U}'$. The left most summand of $\overline{\mathbb{U}}$ is $\mathbb{U}(0,z,0)$. For each element δ_z of $\bigwedge^z G^*$, observe that

$$Y(\delta_z) = \sum_{\substack{t\le z-1\\ |J|=z-t}} 1\otimes (\textstyle\bigwedge^{z-t} X^*)(\gamma_J)\otimes 1\otimes g_J\wedge \delta_z(\omega_G)\otimes \nu^{(t)} \in \sum_{t=0}^{z-1} \mathbb{L}(0,z-t,0,\boldsymbol{g}-t,t)$$

is a cycle in $\overline{\mathbb{L}}$. Define $\tau\colon \overline{\mathbb{U}} \to \overline{\mathbb{L}}$ by

$$\tau(b_p\otimes\delta_q\otimes\mu^{(r)}) = \sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}} (-1)^{ps+t+q+s} 1\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.$$

Observe that τ is a map of complexes which extends the map

$$Y\colon \mathbb{U}(0,z,0)\to \sum_{t=0}^{z-1}\mathbb{L}(0,z-t,0,\boldsymbol{g}-t,t).$$

Let $\mathbb{C}^{(z)}$ be the subcomplex

$$\bigoplus_{\{(p,q,r)\mid p+q+r\le \boldsymbol{f}-1+z \text{ and } p+r\le \boldsymbol{f}-1\}} \mathbb{U}(p,q,r) \oplus \bigoplus_{\{(p,q,r,s,t)\mid r+s+t\le \boldsymbol{f}+\boldsymbol{g}-1\}} \mathbb{L}(p,q,r,s,t)$$

of the mapping cone of τ. The left most summand of $\mathbb{C}^{(z)}$ is $\mathbb{L}(z-1,1,0,\boldsymbol{g},0)$. We give this module position 0 in $\mathbb{C}^{(z)}$ The complex $\mathbb{I}^{(z)}$ is obtained by splicing $\left(\mathbb{C}^{(\boldsymbol{g}-\boldsymbol{f}-z)}\right)^*[-(\boldsymbol{g}+\boldsymbol{f}-1)]$ and $\mathbb{C}^{(z)}$.

The conventions of 1.17, 1.19, and 1.3 are used in the next definition.

Definition 2.3. Adopt Data 1.2. Fix an integer z. Define the free $R-$module $\mathbb{I}^{(z)}$ by

$$\mathbb{I}^{(z)} = \mathbb{W}^{(z)}\oplus\mathbb{T}^{(z)}\oplus\mathbb{U}^{(z)}\oplus\mathbb{L}^{(z)}, \text{ where}$$

$$\mathbb{W}^{(z)} = \bigoplus_{T_{\mathbb{W}}^{(z)}} \mathbb{W}(p,q,r,s,t),\quad \mathbb{T}^{(z)} = \bigoplus_{T_{\mathbb{T}}^{(z)}} \mathbb{T}(p,q,r),$$

$$\mathbb{U}^{(z)} = \bigoplus_{T_{\mathbb{U}}^{(z)}} \mathbb{U}(p,q,r), \text{ and }\quad \mathbb{L}^{(z)} = \bigoplus_{T_{\mathbb{L}}^{(z)}} \mathbb{L}(p,q,r,s,t),$$

for

$$T_{\mathbb{W}}^{(z)}=\left\{(p,q,r,s,t)\;\middle|\; \begin{array}{l} 2\boldsymbol{f}+z+1\le q+r+s+t,\quad p+q+r+t\le \boldsymbol{f},\\ r+s+t\le 2\boldsymbol{f}+z,\quad \text{and}\quad 1\le p+q+t\end{array}\right\},$$

$$T_{\mathbb{T}}^{(z)}=\{(p,q,r)\mid p+q+r\le \boldsymbol{f}-1,\quad \boldsymbol{f}-\boldsymbol{g}+z\le r,\quad 0\le q+r,\quad \text{and}\quad p+r\le 2\boldsymbol{f}-\boldsymbol{g}+z-1\},$$

$$T_{\mathbb{U}}^{(z)}=\{(p,q,r)\mid 0\le r,\quad p+q+r\le \boldsymbol{f}-1+z,\quad p+r\le \boldsymbol{f}-1,\quad \text{and}\quad z\le q+r\},\ \text{and}$$

$$T_{\mathbb{L}}^{(z)}=\{(p,q,r,s,t)\mid p+t\le z-1,\quad \boldsymbol{g}\le s+t,\quad z\le p+q+t,\quad \text{and}\quad r+s+t\le \boldsymbol{f}+\boldsymbol{g}-1\}.$$

The module $\mathbb{I}^{(z)}$ is graded by the following rules:

(a) the position of $\mathbb{L}(p,q,r,s,t)$ is $q+r+s+2t-1-\boldsymbol{g}$,
(b) the position of $\mathbb{U}(p,q,r)$ is $p+q+2r$,
(c) the position of $\mathbb{T}(p,q,r)$ is $p+q+2r+\boldsymbol{g}-\boldsymbol{f}+1$, and
(d) the position of $\mathbb{W}(p,q,r,s,t)$ is $2p+q+r+s+2t-\boldsymbol{f}$.

Define an $R-$module homomorphism $d\colon \mathbb{I}^{(z)}\to\mathbb{I}^{(z)}$ as follows. If

$$x=A_p\otimes\alpha_q\otimes b_r\otimes c_s\otimes\nu^{(t)}\in\mathbb{L}(p,q,r,s,t)\subseteq\mathbb{L}^{(z)},$$

then

$$d(x)=\begin{cases}\chi(z+1\le p+q+t)A_p\otimes v(\alpha_q)\otimes b_r\otimes c_s\otimes\nu^{(t)}\\ +\chi(p+t\le z-2)\sum\limits_{|I|=1}\varphi_I\cdot A_p\otimes f_I(\alpha_q)\otimes b_r\otimes c_s\otimes\nu^{(t)}\\ +(-1)^qA_p\otimes\alpha_q\otimes[X^*(u)](b_r)\otimes c_s\otimes\nu^{(t)}\\ +\chi(\boldsymbol{g}+1\le s+t)(-1)^{q+r}A_p\otimes\alpha_q\otimes b_r\otimes u(c_s)\otimes\nu^{(t)}\\ +(-1)^q\chi(\boldsymbol{g}+1\le s+t)\sum\limits_{|J|=1}\varphi_J\cdot A_p\otimes\alpha_q\otimes f_J\wedge b_r\otimes c_s\otimes\nu^{(t-1)}\\ +(-1)^{q+r+1}\sum\limits_{|K|=1}X^*(\gamma_K)\cdot A_p\otimes\alpha_q\otimes b_r\otimes g_K\wedge c_s\otimes\nu^{(t-1)}.\end{cases}$$

If $x=b_p\otimes\delta_q\otimes\mu^{(r)}\in\mathbb{U}(p,q,r)\subseteq\mathbb{U}^{(z)}$, then

$$d(x)=\begin{cases}[X^*(u)](b_p)\otimes\delta_q\otimes\mu^{(r)}\\ +\chi(z+1\le q+r)(-1)^pb_p\otimes[X(v)](\delta_q)\otimes\mu^{(r)}\\ +\chi(1\le r)\chi(z+1\le q+r)v\wedge b_p\otimes\delta_q\otimes\mu^{(r-1)}\\ +\chi(1\le r)(-1)^{p+1}b_p\otimes u\wedge\delta_q\otimes\mu^{(r-1)}\\ +\sum\limits_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+t+p+s}1\otimes(\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.\end{cases}$$

If $x=\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}(p,q,r)\subseteq\mathbb{T}^{(z)}$, then

$$d(x)=\begin{cases}(-1)^{p-1}v(\alpha_p)\otimes c_q\otimes\lambda^{(r)}\\ +\chi(1\le q+r)(-1)^{p+q}\alpha_p\otimes u(c_q)\otimes\lambda^{(r)}\\ +\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le r)\chi(1\le q+r)\alpha_p\wedge[X^*(u)]\otimes c_q\otimes\lambda^{(r-1)}\\ +\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le r)(-1)^p\alpha_p\otimes c_q\wedge[X(v)]\otimes\lambda^{(r-1)}\\ +\sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p,q,r,t)f_I\otimes\Big[(\bigwedge^{\boldsymbol{f}-p+t-q-r}X)\Big((\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\Big](\omega_{G^*})\otimes\mu^{(t)},\end{cases}$$

where $\sigma_z(p,q,r,t)=(-1)^{r+z+q\boldsymbol{f}}\theta_q\theta_p\binom{\boldsymbol{f}-1-p-q-r+t}{r+\boldsymbol{g}-\boldsymbol{f}-z}$. If

$$x=B_p\otimes b_q\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t)}\in\mathbb{W}(p,q,r,s,t)\subseteq\mathbb{W}^{(z)},$$

then $d(x)$ is equal to

$$\begin{cases} B_p\otimes b_q\wedge v\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t-1)}\\ +\sum\limits_{|I|=1}\varphi_I(B_p)\otimes b_q\wedge f_I\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t)}\\ +\chi(2\le p+q+t)(-1)^q B_p\otimes b_q\otimes\alpha_r\wedge[X^*(u)]\otimes\delta_s\otimes\xi^{(t-1)}\\ +\chi(2\le p+q+t)(-1)^{q+r}B_p\otimes b_q\otimes\alpha_r\otimes\delta_s\wedge u\otimes\xi^{(t-1)}\\ +(-1)^{q+r+1}\sum\limits_{|I|=1}\varphi_I(B_p)\otimes b_q\otimes f_I(\alpha_r)\otimes\delta_s\otimes\xi^{(t+1)}\\ +(-1)^{q+r+s}\sum\limits_{|I|=1}\varphi_I(B_p)\otimes b_q\otimes\alpha_r\otimes[X(f_I)](\delta_s)\otimes\xi^{(t+1)}\\ +\delta_{p0}B_0\sum\limits_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-1\\|I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+\varepsilon+q+s\boldsymbol{g}}[\varphi_I(b_q)](\alpha_r)\otimes\delta_s(\omega_G)\wedge(\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-1-\varepsilon)}. \end{cases}$$

Notes. (a) When we want to emphasize the data which was used to construct $(\mathbb{I}^{(z)},d)$, we write $\mathbb{I}^{(z)}[u,X,v]$.
(b) The definition of d uses many module and algebra operations. For example, the module action of $\bigwedge^\bullet F$ on $\bigwedge^\bullet F^*$ is used in $v(\alpha_q)$, the multiplication of the symmetric algebra $S_\bullet F^*$ is used in $\varphi_J\cdot A_p$, the exterior multiplication of $\bigwedge^\bullet G$ is used in $g_K\wedge c_s$, and the module action of $S_\bullet F^*$ on $D_\bullet F$ is used in $\varphi_I(B_p)$.

Remark 2.4. Retain Data 1.2. Suppose that R is a graded ring and that each map of

$$0\to R(-3)\xrightarrow{v}R(-2)^{\boldsymbol{f}}\xrightarrow{X}R(-1)^{\boldsymbol{g}}\xrightarrow{u}R$$

is homogeneous of degree zero. A quick check verifies that, if $0\le z$, then $\mathbb{I}^{(z)}$ is a homogeneous complex with degree zero maps, provided

(a) the shift of $\mathbb{L}(p,q,r,s,t)$ is $p+q+2r+s+3t-\boldsymbol{g}-z$,
(b) the shift of $\mathbb{U}(p,q,r)$ is $2p+2q+3r-z$,
(c) the shift of $\mathbb{T}(p,q,r)$ is $p+q+3r+2\boldsymbol{g}-\boldsymbol{f}-z$, and
(d) the shift of $\mathbb{W}(p,q,r,s,t)$ is $2p+2q+r+2s+3t-\boldsymbol{f}-3-z$.

For example, by (b) we mean that

$$\mathbb{U}(p,q,r)=R[-(2p+2q+3r-z)]^{\binom{\boldsymbol{f}}{p}\binom{\boldsymbol{g}}{q}}.$$

(If $z=-1$, then the appropriate grading on $\mathbb{I}^{(z)}$ is obtained by subtracting 1 from each shift in (a)–(d). This convention allows $R[0]$ to be summand of $\mathbb{I}_0^{(z)}$, whenever $-1\le z$. See Corollary 4.11.b or Example 4.12.)

Example 2.5. If $\boldsymbol{g}=0$ and $\boldsymbol{f}=1$, then $\mathbb{I}^{(z)}$ is acyclic for $-1\le z$; indeed, $\mathbb{I}^{(z)}$ is equal to

$$\begin{array}{ll} 0\to\mathbb{T}(0,0,0)=R, & \text{if } -1=z,\\ 0\to\mathbb{U}(0,0,0)=R, & \text{if } 0=z,\\ 0\to\mathbb{L}(z-1,1,0,0,0)=R, & \text{if } 1\le z, \end{array}$$

where each module R is in position zero.

Example 2.6. If $\boldsymbol{g} = \boldsymbol{f} = 1$, then the complexes $\mathbb{I}^{(z)}$ of Definition 2.3 are

$$\begin{aligned}
-1 = z: &\quad 0 \to \mathbb{W}(0,1,0,1,0) \xrightarrow{d_1} \mathbb{T}(0,1,-1) \to 0,\\
0 = z: &\quad 0 \to \mathbb{T}(0,0,0) \xrightarrow{d_1} \mathbb{U}(0,0,0) \to 0,\\
1 = z: &\quad 0 \to \mathbb{U}(0,1,0) \xrightarrow{d_1} \mathbb{L}(0,1,0,1,0) \to 0, \text{ and}\\
2 \le z: &\quad 0 \to \mathbb{L}(z-2,1,0,0,1) \xrightarrow{d_1} \mathbb{L}(z-1,1,0,1,0) \to 0.
\end{aligned}$$

Furthermore, each complex is $0 \to R \xrightarrow{X} R \to 0$.

Example 2.7. If $\boldsymbol{g} = 1$ and $\boldsymbol{f} = 2$, then $\mathbb{I}^{(0)}$ is

$$0 \to \begin{matrix} \mathbb{U}(0,0,1) \\ \oplus \\ \mathbb{T}(0,0,1) \end{matrix} \xrightarrow{d_2} \begin{matrix} \mathbb{U}(0,1,0) \\ \oplus \\ \mathbb{U}(1,0,0) \end{matrix} \xrightarrow{d_1} \mathbb{U}(0,0,0),$$

where, in the notation of Convention 1.5.a,

$$d_2 = \begin{bmatrix} -u_1 & 0 \\ v_1 & -x_2 \\ v_2 & x_1 \end{bmatrix} \quad \text{and} \quad d_1 = \begin{bmatrix} x_1v_1 + x_2v_2 & u_1x_1 & u_1x_2 \end{bmatrix}.$$

If the ideal generated by the entries of d_1 has grade 2, then $\mathbb{I}^{(0)}$ and $\mathbb{I}^{(-1)}$ (which is the shifted dual of $\mathbb{I}^{(0)}$) are both acyclic.

Example 2.8. If $\boldsymbol{g} = 1$ and $\boldsymbol{f} = 2$, then $\mathbb{I}^{(1)}$ is

$$0 \to \mathbb{U}(0,1,1) \xrightarrow{d_3} \begin{matrix} \mathbb{L}(0,2,1,1,0) \\ \oplus \\ \mathbb{U}(0,0,1) \\ \oplus \\ \mathbb{U}(1,1,0) \end{matrix} \xrightarrow{d_2} \begin{matrix} \mathbb{L}(0,2,0,1,0) \\ \oplus \\ \mathbb{L}(0,1,1,1,0) \\ \oplus \\ \mathbb{U}(0,1,0) \end{matrix} \xrightarrow{d_1} \mathbb{L}(0,1,0,1,0),$$

with

$$d_3 = \begin{bmatrix} -x_2 \\ x_1 \\ x_1v_1 + x_2v_2 \\ v_1 \\ v_2 \end{bmatrix}, \quad d_2 = \begin{bmatrix} u_1x_1 & u_1x_2 & 0 & 0 & 0 \\ v_2 & 0 & 1 & -x_1 & 0 \\ 0 & v_2 & 0 & 0 & -x_1 \\ -v_1 & 0 & 0 & -x_2 & 0 \\ 0 & -v_1 & 1 & 0 & -x_2 \\ 0 & 0 & -u_1 & u_1x_1 & u_1x_2 \end{bmatrix}, \text{ and}$$

$$d_1 = \begin{bmatrix} v_2 & -u_1x_1 & -u_1x_2 & 0 & 0 & -x_1 \\ -v_1 & 0 & 0 & -u_1x_1 & -u_1x_2 & -x_2 \end{bmatrix}.$$

The above complex is homologically equivalent to

$$0 \to R \xrightarrow{\delta_3} R^4 \xrightarrow{\delta_2} R^5 \xrightarrow{\delta_1} R^2, \text{ with}$$

$$\delta_3 = \begin{bmatrix} -x_2 \\ x_1 \\ v_1 \\ v_2 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} u_1x_1 & u_1x_2 & 0 & 0 \\ 0 & v_2 & 0 & -x_1 \\ -v_1 & 0 & -x_2 & 0 \\ -v_2 & -v_1 & x_1 & -x_2 \\ u_1v_2 & 0 & 0 & u_1x_2 \end{bmatrix}, \text{ and}$$

$$\delta_1 = \begin{bmatrix} v_2 & -u_1x_2 & 0 & 0 & -x_1 \\ -v_1 & 0 & -u_1x_1 & -u_1x_2 & -x_2 \end{bmatrix}.$$

This complex is easily seen to be acyclic when the data is generic in the sense of Convention 1.5.b.

Example 2.9. If $\boldsymbol{g} = \boldsymbol{f} = 2$, then the complex $\mathbb{I}^{(0)}$ is

$$0 \to \mathbb{T}(0,0,1) \xrightarrow{d_3} \begin{matrix} \mathbb{T}(1,0,0) \\ \oplus \\ \mathbb{T}(0,1,0) \\ \oplus \\ \mathbb{U}(0,0,1) \end{matrix} \xrightarrow{d_2} \begin{matrix} \mathbb{U}(1,0,0) \\ \oplus \\ \mathbb{U}(0,1,0) \\ \oplus \\ \mathbb{T}(0,0,0) \end{matrix} \xrightarrow{d_1} \mathbb{U}(0,0,0) \to 0.$$

If the bases $\lambda^{(1)}$ for $\mathbb{T}(0,0,1)$; $\varphi^{[1]}, \varphi^{[2]}$ for $\mathbb{T}(1,0,0)$; $g^{[1]}, g^{[2]}$ for $\mathbb{T}(0,1,0)$; $\mu^{(1)}$ for $\mathbb{U}(0,0,1)$; $f^{[1]}, f^{[2]}$ for $\mathbb{U}(1,0,0)$; $\gamma^{[1]}, \gamma^{[2]}$ for $\mathbb{U}(0,1,0)$; $-\lambda^{(0)}$ for $\mathbb{T}(0,0,0)$; and $\mu^{(0)}$ for $\mathbb{U}(0,0,0)$, are chosen, then, in the notation of Convention 1.5.a,

$$d_2 = \begin{bmatrix} 0 & 0 & x_{22} & -x_{12} & v_1 \\ 0 & 0 & -x_{21} & x_{11} & v_2 \\ -x_{22} & x_{21} & 0 & 0 & -u_1 \\ x_{12} & -x_{11} & 0 & 0 & -u_2 \\ -v_1 & -v_2 & u_1 & u_2 & 0 \end{bmatrix},$$

and the entries of d_1 and d_3 are the maximal order pfaffians of d_2.

Example 2.10. If $\boldsymbol{g} = 4$ and $\boldsymbol{f} = 2$, then the complex $\mathbb{I}^{(2)}$ is

$$0 \to \mathbb{T}(0,0,1) \xrightarrow{d_5} \begin{matrix} \mathbb{U}(0,2,1) \\ \oplus \\ \mathbb{T}(0,1,0) \\ \oplus \\ \mathbb{T}(1,0,0) \end{matrix} \xrightarrow{d_4} \begin{matrix} \mathbb{L}(0,2,0,4,1) \\ \oplus \\ \mathbb{L}(0,2,1,3,1) \\ \oplus \\ \mathbb{U}(0,3,0) \\ \oplus \\ \mathbb{U}(0,1,1) \\ \oplus \\ \mathbb{U}(1,2,0) \\ \oplus \\ \mathbb{T}(0,0,0) \end{matrix} \xrightarrow{d_3} \begin{matrix} \mathbb{L}(0,2,1,4,0) \\ \oplus \\ \mathbb{L}(0,1,1,3,1) \\ \oplus \\ \mathbb{L}(0,2,0,3,1) \\ \oplus \\ \mathbb{L}(0,1,0,4,1) \\ \oplus \\ \mathbb{L}(1,2,1,4,0) \\ \oplus \\ \mathbb{U}(0,2,0) \end{matrix} \xrightarrow{d_2} \begin{matrix} \mathbb{L}(0,2,0,4,0) \\ \oplus \\ \mathbb{L}(0,1,0,3,1) \\ \oplus \\ \mathbb{L}(1,2,0,4,0) \\ \oplus \\ \mathbb{L}(1,1,1,4,0) \end{matrix} \xrightarrow{d_1} \mathbb{L}(1,1,0,4,0).$$

Theorem 2.11. *The modules and maps of Definition 2.3 form a complex.*

Proof. A straightforward calculation shows that if x is an element of $\mathbb{I}^{(z)}$ of position i, then $d(x)$ is an element of $\mathbb{I}^{(z)}$ of position $i-1$. We record the interesting parts of the calculation that $d \circ d(x) = 0$. If x is in $\mathbb{L}^{(z)}$, then the calculation is completely routine. Next, we let $x = b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}(p,q,r) \subseteq \mathbb{U}^{(z)}$. Decompose $d \circ d(x)$

as $A+B+C+D$, where every term of A is in $\mathbb{U}^{(z)}$, every term of $B+C+D$ is in $\mathbb{L}^{(z)}$, every term of B involves u, every term of C involves v, and the terms of D involve neither u nor v. There is no difficulty seeing that $A=0$. We have $B=B_1+B_2+B_3+B_4$, where

$$B_1=\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+t+p+1}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge[X^*(u)](b_p)\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)},$$

$$B_2=\chi(1\le r)\sum_{\substack{\{(t,s)\mid t\le z-1,\ q+1\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+t+1+s}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge(u\wedge\delta_q)(\omega_G)\otimes\nu^{(t)},$$

$$B_3=\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+p+s+r+q}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes[X^*(u)](f_I\wedge b_p)\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)},$$

and

$$B_4=\sum_{\substack{\{(t,s)\mid t\le z-1,\ q+1\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+t}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes u(g_J\wedge\delta_q(\omega_G))\otimes\nu^{(t)}.$$

The module action of $\bigwedge^\bullet F^*$ on $\bigwedge^\bullet F$ yields that B_1+B_3 is equal to

$$\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+p+s+r+q}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes[X^*(u)](f_I)\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)},$$

which, according to Lemma 1.9.e, is the same as

$$\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t-1}}(-1)^{ps+p+t+1}\otimes(\textstyle\bigwedge^{s+1} X^*)(\gamma_J\wedge u)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.$$

The factor $\chi(1\le r)$ may be removed from B_2, without affecting its value, because if $r\le 0$, then $|I|<0$ and the rest of B_2 is already zero. We see, from Lemmas 1.9.e and 1.1, that B_2+B_4 is equal to

$$\begin{aligned}&\sum_{\substack{\{(t,s)\mid t\le z-1,\ q+1\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}}(-1)^{ps+t}\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes u(g_J)\wedge\delta_q(\omega_G)\otimes\nu^{(t)}\\
=&\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-1-t}}(-1)^{ps+p+t}\otimes(\textstyle\bigwedge^{s+1} X^*)(\gamma_J\wedge u)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.\end{aligned}$$

It is now clear that $B=0$. The calculation that $C=0$ is similar to the calculation for B; hence, we omit it. We see that $D=D_1+D_2+D_3$, where

$$D_1=\sum_{\substack{\{(t,s)\mid t\le z-2,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t\\ |K|=1}}(-1)^{ps+t+p+s}\varphi_K\otimes f_K((\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I)\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)},$$

$$D_2=\sum_{\substack{\{(t,s)\mid t\le z-2,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t-1\\ |K|=1}}(-1)^{ps+p+s+r+q}\varphi_K\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_K\wedge f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)},$$

and

$$D_3=\sum_{\substack{\{(t,s)\mid t\le z-2,\ q-1\le s+t\}\\ |J|=s\\ |I|=r+q-s-t-1\\ |K|=1}}(-1)^{ps+t}X^*(\gamma_K)\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_K\wedge g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.$$

Apply Lemmas 1.9.e and 1.1.d to see that D_1+D_2 is equal to

$$\sum_{\substack{\{(t,s)\mid t\le z-2,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t\\ |K|=1}} (-1)^{ps+t+p+s}\varphi_K\otimes f_K\left((\textstyle\bigwedge^s X^*)(\gamma_J)\right)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}$$

$$=\sum_{\substack{\{(t,s)\mid t\le z-2,\ q-1\le s+t\}\\ |J|=s+1\\ |I|=r+q-s-t-1\\ |K|=1}} (-1)^{ps+t+1+s}\varphi_K\otimes(\textstyle\bigwedge^s X^*)\left([X(f_K)](\gamma_J)\right)\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.$$

On the other hand, Lemma 1.9.e yields that D_3 is equal to

$$\sum_{\substack{\{(t,s)\mid t\le z-2,\ q-1\le s+t\}\\ |J|=s+1\\ |I|=r+q-s-t-1\\ |K|=1}} (-1)^{ps+t+s}X^*(\gamma_K)\otimes(\textstyle\bigwedge^s X^*)(g_K(\gamma_J))\wedge\varphi_I\otimes f_I\wedge b_p\otimes g_J\wedge\delta_q(\omega_G)\otimes\nu^{(t)}.$$

Example 1.8 shows that $D=0$.

Let $x=\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}(p,q,r)\subseteq\mathbb{T}^{(z)}$. Decompose $d\circ d(x)$ as $A+B+C+D$, where every term of A is in $\mathbb{T}^{(z)}$, every term of $B+C$ is in $\mathbb{U}^{(z)}$, every term of B involves v, every term of C involves u, and every term of D is in $\mathbb{L}^{(z)}$. There is no difficulty seeing that $A=0$. We have $B=B_1+B_2+B_3+B_4$, where

$$B_1=\begin{cases}(-1)^{p-1}\sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p-1,q,r,t)f_I\otimes\left[(\bigwedge^{\boldsymbol{f}+1-p-q-r+t}X)\left((\varphi_I\wedge v(\alpha_p))[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\\ \qquad\otimes\mu^{(t)},\end{cases}$$

$$B_2=\begin{cases}\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le r)(-1)^p\sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p,q+1,r-1,t)f_I\\ \qquad\otimes\left[(\bigwedge^{\boldsymbol{f}-p-q-r+t}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\wedge[X(v)]\right](\omega_{G^*})\otimes\mu^{(t)},\end{cases}$$

$$B_3=\begin{cases}\chi(\boldsymbol{f}+z-\boldsymbol{g}+1\le p+r)\sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p,q,r,t)(-1)^{q+r-t}f_I\\ \qquad\otimes[X(v)]\left(\left[(\bigwedge^{\boldsymbol{f}-p-q-r+t}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\right)\otimes\mu^{(t)},\quad\text{and}\end{cases}$$

$$B_4=\begin{cases}\chi(\boldsymbol{f}+z-\boldsymbol{g}+1\le p+r)\sum\limits_{\substack{0\le t\\ |I|=q+r-t-1}}\sigma_z(p,q,r,t+1)v\wedge f_I\\ \qquad\otimes\left[(\bigwedge^{\boldsymbol{f}+1-p-q-r+t}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(t)}.\end{cases}$$

Notice that the factor $\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le p+r)$ may be adjoined to B_1 without changing its value, because $\boldsymbol{f}-\boldsymbol{g}+z\le r$ (since $(p,q,r)\in T_{\mathbb{T}}^{(z)}$) and $1\le p$ (or else B_1, which contains the factor $v(\alpha_p)$, is already zero). Also notice that

$$\sigma_z(p-1,q,r,t)=(-1)^p\sigma_z(p,q,r,t+1).$$

It follows that B_1+B_4 is equal to

$$\begin{cases}\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le p+r)\sum\limits_{\substack{0\le t\\ |I|=q+r-t}}(-1)^{q+r-t-1}\sigma_z(p,q,r,t+1)f_I\\ \qquad\otimes\left[(\bigwedge^{\boldsymbol{f}+1-p-q-r+t}X)\left(v\wedge(\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(t)}.\end{cases}$$

In B_2, the condition $\boldsymbol{f}-\boldsymbol{g}+z+1\le r$ holds automatically (if this hypothesis is not met, then the binomial coefficient is zero, see Remark 1.10); consequently, in term B_2,

$$\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le r)=1=\chi(\boldsymbol{f}-\boldsymbol{g}+z+1\le r+p).$$

Observe that

$$(-1)^{q+r-t-1}\sigma_z(p,q,r,t+1)+(-1)^{\boldsymbol{f}+t-r}\sigma_z(p,q+1,r-1,t)+(-1)^{q+r-t}\sigma_z(p,q,r,t)=0;$$

and therefore, $B=0$. The calculation of C is similar; we have omitted it. We have D is equal to

$$\begin{cases} \sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p,q,r,t)\sum\limits_{\substack{\{(\tau,s)|\tau\le z-1,\ \boldsymbol{g}-\boldsymbol{f}+p+r-t\le s+\tau\}\\ |J|=s\\ |K|=\boldsymbol{g}-\boldsymbol{f}+p+r-s-\tau}}(-1)^{(q+r-t)s+\tau+q+r-t+s}\\ \quad\otimes(\bigwedge^s X^*)(\gamma_J)\wedge\varphi_K\otimes f_K\wedge f_I\\ \quad\otimes g_J\wedge\left(\left[(\bigwedge^{\boldsymbol{f}-p-q-r+t}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\right)(\omega_G)\otimes\nu^{(\tau)}.\end{cases}$$

Apply Lemma 1.9.e and Proposition 1.1.d to see that D is equal to

$$\begin{cases} \sum\limits_{\substack{0\le t\\ |I|=q+r-t}}\sigma_z(p,q,r,t)\sum\limits_{\substack{\{(\tau,s)|\tau\le z-1,\ \boldsymbol{g}-\boldsymbol{f}+p+r-t\le s+\tau\}\\ |J|=s+\boldsymbol{f}-p+t-q-r\\ |K|=\boldsymbol{g}-\boldsymbol{f}+p-s-\tau+q-t+2r}}(-1)^{\tau+q+r-t+(\boldsymbol{g}-\boldsymbol{f}+p-\tau+q-t)s}\\ \quad\otimes f_I(\varphi_K)\wedge\left(\varphi_I\left[\alpha_p[\omega_F]\right]\right)\left[(\bigwedge^{\boldsymbol{f}-p+t-q-r+s}X^*)(\gamma_J)\right]\otimes f_K\otimes g_J\wedge c_q\otimes\nu^{(\tau)}.\end{cases}$$

Replace s by $\ell-\boldsymbol{f}-t+p+q+r$ and t by $q+r-i$ in order to obtain that D is equal to

$$\begin{cases} \sum\limits_{\substack{i\le q+r\\ |I|=i}}(-1)^{r+z+q\boldsymbol{f}}\theta_p\theta_q\binom{\boldsymbol{f}-1-p-i}{r+\boldsymbol{g}-\boldsymbol{f}-z}\sum\limits_{\substack{\{(\tau,\ell)|\tau\le z-1,\ \boldsymbol{g}-q\le\ell+\tau\}\\ |J|=\ell\\ |K|=\boldsymbol{g}-\ell-\tau+r}}(-1)^{\tau+i+(\boldsymbol{g}-\boldsymbol{f}+p-\tau+i+r)(\ell-\boldsymbol{f}+i+p)}\\ \quad\otimes f_I(\varphi_K)\wedge\left(\varphi_I\left[\alpha_p[\omega_F]\right]\right)\left[(\bigwedge^{\ell}X^*)(\gamma_J)\right]\otimes f_K\otimes g_J\wedge c_q\otimes\nu^{(\tau)}.\end{cases}$$

Notice that $i\le q+r$ is not a real restriction, because we every non-zero term already has

$$i=|I|\le|K|=\boldsymbol{g}-\ell-\tau+r\le q+r;$$

thus, D is equal to

$$\begin{cases} (-1)^{r+z+q\boldsymbol{f}}\theta_p\theta_q\sum\limits_{\substack{\{(\tau,\ell)|\tau\le z-1,\ \boldsymbol{g}-q\le\ell+\tau\}\\ |J|=\ell\\ |K|=\boldsymbol{g}-\ell-\tau+r}}(-1)^{\tau+(\boldsymbol{g}-\boldsymbol{f}+p-\tau+r)(\ell-\boldsymbol{f}+p)}\otimes\\ \quad\sum\limits_{\substack{i\\ |I|=i}}(-1)^{i(\ell+\boldsymbol{g}-\tau+r)}\binom{\boldsymbol{f}-1-p-i}{r+\boldsymbol{g}-\boldsymbol{f}-z}f_I(\varphi_K)\wedge\left(\varphi_I\left[\alpha_p[\omega_F]\right]\right)\left[(\bigwedge^{\ell}X^*)(\gamma_J)\right]\otimes f_K\\ \quad\otimes g_J\wedge c_q\otimes\nu^{(\tau)}.\end{cases}$$

In the language of the proof of Lemma 1.11, we have that D is equal to

$$\begin{cases} (-1)^{r+z+q\boldsymbol{f}}\theta_p\theta_q\sum\limits_{\substack{\{(\tau,\ell)|\tau\le z-1,\ \boldsymbol{g}-q\le\ell+\tau\}\\ |J|=\ell\\ |K|=\boldsymbol{g}-\ell-\tau+r}}(-1)^{\tau+(\boldsymbol{g}-\boldsymbol{f}+p-\tau+r)(\ell-\boldsymbol{f}+p)}\otimes\\ \quad h_{\boldsymbol{f}-1-p,r+\boldsymbol{g}-\boldsymbol{f}-z}\left(\varphi_K\otimes\alpha_p[\omega_F]\otimes(\bigwedge^{\ell}X^*)(\gamma_J)\right)\otimes f_K\otimes g_J\wedge c_q\otimes\nu^{(\tau)}.\end{cases}$$

The hypothesis "$\boldsymbol{f}+1+w\le p+r$" from Lemma 1.11 is satisfied because

$$\boldsymbol{f}+1+(r+\boldsymbol{g}-\boldsymbol{f}-z)\le |K|+|J|,$$

since $\tau\le z-1$. We conclude that $D=0$.

Let $x=B_p\otimes b_q\otimes \alpha_r\otimes \delta_s\otimes \xi^{(t)}\in \mathbb{W}(p,q,r,s,t)\subseteq \mathbb{W}^{(z)}$. Decompose $d\circ d(x)$ as $A+B+C+D+E$, where every term of A is in $\mathbb{W}^{(z)}$, every term of $B+C+D$ is in $\mathbb{T}^{(z)}$, every term of B involves v, every term of C involves u, the terms of D involve neither u nor v, and every term of E is in $\mathbb{U}^{(z)}$. There is no difficulty seeing that $A=0$. We have $B=B_1+B_2+B_3$, where

$$B_1=\begin{cases}\delta_{p0}B_0\displaystyle\sum_{\substack{\varepsilon\le q+s+t-1-\boldsymbol{f}-z\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+r+\varepsilon+s\boldsymbol{g}}[\varphi_I(v\wedge b_q)](\alpha_r)\\ \otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-1-\varepsilon)},\end{cases}$$

$$B_2=\begin{cases}\delta_{p0}B_0\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-1\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+s\boldsymbol{g}+r-1}[v\wedge\varphi_I(b_q)](\alpha_r)\\ \otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-1-\varepsilon)},\quad\text{and}\end{cases}$$

$$B_3=\begin{cases}\delta_{p0}B_0\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-1\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+s\boldsymbol{g}+r}\left[\Big(v(\varphi_I)\Big)(b_q)\right](\alpha_r)\\ \otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-\varepsilon-1)}.\end{cases}$$

Apply Proposition 1.1.a to see that $B=0$. We have $C=C_1+C_2+C_3+C_4$, where

$$C_1=\begin{cases}\delta_{p0}B_0\chi(2\le q+t)\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-2\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t-1+qr+\varepsilon+q+s\boldsymbol{g}+r}\\ [\varphi_I(b_q)]([X^*(u)]\wedge\alpha_r)\otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-2-\varepsilon)},\end{cases}$$

$$C_2=\begin{cases}\delta_{p0}B_0\chi(2\le q+t)\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-1\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+t-1+qr+\varepsilon+r+s\boldsymbol{g}}\\ [\varphi_I(b_q)](\alpha_r)\otimes(\delta_s\wedge u)(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-1-\varepsilon)},\end{cases}$$

$$C_3=\begin{cases}\delta_{p0}B_0\chi(2\le q+t)\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-1\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+t+qr+s\boldsymbol{g}+r-s+\varepsilon}\\ [\varphi_I(b_q)](\alpha_r)\otimes u(\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I))\otimes\lambda^{(q+s+t-\boldsymbol{g}-1-\varepsilon)},\quad\text{and}\end{cases}$$

$$C_4=\begin{cases}\delta_{p0}B_0\chi(2\le q+t)\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-2\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+s\boldsymbol{g}+r}\\ [X^*(u)]\wedge[\varphi_I(b_q)](\alpha_r)\otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-\varepsilon-2)}.\end{cases}$$

Apply Lemma 1.9.e to see that C_2+C_3 is equal to

$$\begin{cases}\delta_{p0}B_0\chi(2\le q+t)\displaystyle\sum_{\substack{\varepsilon\le q+s+t-\boldsymbol{f}-z-2\\ |I|=\varepsilon}}(-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+s\boldsymbol{g}+r+1}\\ \left[[X^*(u)]\Big(\varphi_I(b_q)\Big)\right](\alpha_r)\otimes\delta_s(\omega_G)\wedge(\textstyle\bigwedge^\varepsilon X)(f_I)\otimes\lambda^{(q+s+t-\boldsymbol{g}-2-\varepsilon)}.\end{cases}$$

Proposition 1.1.a yields that $C = 0$. We have D is equal to

$$\begin{cases} \delta_{p1} \sum\limits_{\substack{\varepsilon \le q+s+t-\boldsymbol{f}-z \\ |J|=\varepsilon}} (-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+r+\varepsilon+1+s\boldsymbol{g}} [\varphi_J(B_p \wedge b_q)](\alpha_r) \otimes \delta_s(\omega_G) \wedge (\bigwedge^{\varepsilon} X)(f_J) \\ \qquad \otimes \lambda^{(q+s+t-\boldsymbol{g}-\varepsilon)} \\ + \delta_{p1} \sum\limits_{\substack{\varepsilon \le q+s+t-\boldsymbol{f}-z \\ |J|=\varepsilon}} (-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+s\boldsymbol{g}+r} [B_p \wedge \varphi_J(b_q)](\alpha_r) \otimes \delta_s(\omega_G) \wedge (\bigwedge^{\varepsilon} X)(f_J) \\ \qquad \otimes \lambda^{(q+s+t-\boldsymbol{g}-\varepsilon)} \\ + \delta_{p1} \sum\limits_{\substack{\varepsilon \le q+s+t-\boldsymbol{f}-z \\ |J|=\varepsilon}} (-1)^{\boldsymbol{f}+t+1+qr+s\boldsymbol{g}+r+\boldsymbol{g}} \left[\Big(B_p(\varphi_J)\Big)(b_q)\right](\alpha_r) \otimes \delta_s(\omega_G) \wedge (\bigwedge^{\varepsilon} X)(f_J) \\ \qquad \otimes \lambda^{(q+s+t-\boldsymbol{g}-\varepsilon)}, \end{cases}$$

which is zero by Proposition 1.1.a. We have E is equal to

$$\begin{cases} \delta_{p0} B_0 \sum\limits_{\substack{\varepsilon \le q+s+t-\boldsymbol{f}-z-1 \\ |I|=\varepsilon}} (-1)^{\boldsymbol{f}+\boldsymbol{g}+t+qr+\varepsilon+q+s\boldsymbol{g}} \sum\limits_{\substack{0 \le \tau \\ |J|=q+t-1-\tau}} \\ \quad \sigma_z(r-q+\varepsilon, \boldsymbol{g}-s+\varepsilon, q+s+t-\boldsymbol{g}-1-\varepsilon, \tau) f_J \otimes \\ \quad \Big[(\bigwedge^{\boldsymbol{f}-r-\varepsilon-t+1+\tau} X)\Big((\varphi_J \wedge [\varphi_I(b_q)](\alpha_r))[\omega_F]\Big) \wedge \delta_s(\omega_G) \wedge (\bigwedge^{\varepsilon} X)(f_I)\Big](\omega_{G^*}) \otimes \mu^{(\tau)} \end{cases}$$

$$= \begin{cases} \delta_{p0} B_0 \sum\limits_{\substack{\varepsilon \le q+s+t-\boldsymbol{f}-z-1 \\ |I|=\varepsilon}} (-1)^{\boldsymbol{f}+qr+s\boldsymbol{g}+\varepsilon(t+\tau+q)+s-1+z+(\boldsymbol{g}-s)\boldsymbol{f}} \sum\limits_{\substack{0 \le \tau \\ |J|=q+t-1-\tau}} \\ \quad \theta_{r-q}\theta_{\boldsymbol{g}-s} \binom{\boldsymbol{f}+\tau-t-r-\varepsilon}{q+s+t-\varepsilon-\boldsymbol{f}-z-1} f_J \otimes \\ \quad \left[(\bigwedge^{\boldsymbol{f}-r-t+1+\tau} X)\left(\left(f_I\left[\varphi_J \wedge [\varphi_I(b_q)](\alpha_r)\right]\right)[\omega_F]\right)\right](\delta_s) \otimes \mu^{(\tau)}. \end{cases}$$

No harm is done if we remove the condition $\varepsilon \le q+s+t-\boldsymbol{f}-z-1$; because, if this condition fails, then the binomial coefficient is zero. The fact that (p,q,r,s,t) is in $T_{\mathbb{W}}^{(z)}$ ensures that parameters in every non-zero term of E satisfies

$$\varepsilon + t + r \le q + t + r = p + q + r + t \le \boldsymbol{f};$$

consequently, the top line of the binomial coefficient is non-negative and we may apply $\binom{a}{b} = \binom{a}{a-b}$. It follows that E is equal to

$$\begin{cases} \delta_{p0} B_0 (-1)^{\boldsymbol{f}+qr+s\boldsymbol{g}+s-1+z+(\boldsymbol{g}-s)\boldsymbol{f}} \sum\limits_{\substack{0 \le \tau \\ |J|=q+t-1-\tau}} \theta_{r-q}\theta_{\boldsymbol{g}-s} f_J \otimes \\ \left[(\bigwedge^{\boldsymbol{f}-r-t+1+\tau} X)\left(\sum\limits_{\substack{\varepsilon \in \mathbb{Z} \\ |I|=\varepsilon}} (-1)^{\varepsilon(|J|+1)} \binom{\boldsymbol{f}+\tau-t-r-\varepsilon}{2\boldsymbol{f}+\tau-2t-r-q-s+1+z} \Big(f_I\left[\varphi_J \wedge [\varphi_I(b_q)](\alpha_r)\right]\Big)[\omega_F]\right)\right](\delta_s) \\ \otimes \mu^{(\tau)}. \end{cases}$$

The fact that (p,q,r,s,t) is in $T_{\mathbb{W}}^{(z)}$ ensures that $w + |J| + 1 \le q$, for

$$w = 2\boldsymbol{f} + \tau - 2t - r - q - s + 1 + z.$$

Apply Lemma 1.11.b in order to conclude that $E = 0$. $\square$

Proposition 2.12. *The complexes* $\mathbb{I}^{(z)}$ *and* $\left(\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}\right)^*[-(\boldsymbol{g}+\boldsymbol{f}-1)]$, *of Definition 2.3, are isomorphic for all integers* z.

Proof. Consider the map $\psi\colon \mathbb{I}^{(z)} \to \left(\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}\right)^*[-(\boldsymbol{g}+\boldsymbol{f}-1)]$, which is given by

$$\psi(x) = \begin{cases} <x\,,\,\_\_>, & \text{if } x \in \mathbb{L}^{(z)} \oplus \mathbb{U}^{(z)}, \\ -\theta_{\boldsymbol{f}+\boldsymbol{g}}<x\,,\,\_\_>, & \text{if } x \in \mathbb{T}^{(z)} \oplus \mathbb{W}^{(z)}, \end{cases}$$

where the perfect pairings

$$\begin{aligned} &\mathbb{W}(p,q,r,s,t) \otimes \mathbb{L}(p',q',r',s',t') \xrightarrow{<\_\_,\_\_>} R, \quad \text{and} \\ &\mathbb{L}(p',q',r',s',t') \otimes \mathbb{W}(p,q,r,s,t) \xrightarrow{<\_\_,\_\_>} R, \end{aligned}$$

are defined by:

$$<x,y> = <y,x> = \delta_{p\,p'}\delta_{q\,q'}\delta_{r\,r'}\delta_{s\,s'} \cdot \chi\Big(p+q+r+s+t+t'=\boldsymbol{f}+\boldsymbol{g}\Big) \cdot B_p(A_{p'}) \cdot b_q(\alpha_{q'}) \cdot \alpha_r(b_{r'}) \cdot \delta_s(c_{s'}),$$

for

$$x = B_p \otimes b_q \otimes \alpha_r \otimes \delta_s \otimes \xi^{(t)} \in \mathbb{W}(p,q,r,s,t) \quad \text{and} \quad y = A_{p'} \otimes \alpha_{q'} \otimes b_{r'} \otimes c_{s'} \otimes \nu^{(t')} \in \mathbb{L}(p',q',r',s',t'),$$

and the perfect pairings

$$\begin{aligned} &\mathbb{T}(p,q,r) \otimes \mathbb{U}(p',q',r') \xrightarrow{<\_\_,\_\_>} R, \quad \text{and} \\ &\mathbb{U}(p',q',r') \otimes \mathbb{T}(p,q,r) \xrightarrow{<\_\_,\_\_>} R, \end{aligned}$$

are defined by

$$<x,y> = <y,x> = \delta_{p\,p'}\delta_{q\,q'} \cdot \chi\Big(p+q+r+r' = \boldsymbol{f}-1\Big) \cdot \alpha_p(b_{p'}) \cdot c_q(\delta_{q'}), \quad \text{for}$$

$$x = \alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbb{T}(p,q,r) \quad \text{and} \quad y = b_{p'} \otimes \delta_{q'} \otimes \mu^{(r')} \in \mathbb{U}(p',q',r').$$

A short calculation shows that

$$\begin{aligned} (p,q,r,s,t) \in T_{\mathbb{L}}^{(z)} &\iff (p,q,r,s,\boldsymbol{f}+\boldsymbol{g}-p-q-r-s-t) \in T_{\mathbb{W}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}, \quad \text{and} \\ (p,q,r) \in T_{\mathbb{U}}^{(z)} &\iff (p,q,\boldsymbol{f}-1-p-q-r) \in T_{\mathbb{T}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}. \end{aligned}$$

Moreover, it is easy to see that ψ carries the module in $\mathbb{I}^{(z)}$ of position i to the dual of the module in $\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$ of position $\boldsymbol{g}+\boldsymbol{f}-1-i$. For example, $\mathbb{U}(p,q,r)$ has position $p+q+2r$ in $\mathbb{I}^{(z)}$, and $\mathbb{T}(p,q,\boldsymbol{f}-1-p-q-r)$ has position

$$p+q+2(\boldsymbol{f}-1-p-q-r)+\boldsymbol{g}-\boldsymbol{f}+1 = \boldsymbol{g}+\boldsymbol{f}-1-(p+q+2r)$$

in $\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$. At this point we have established that ψ is an isomorphism of graded modules. It remains to show that if x is an element of $\mathbb{L}^{(z)}$, $\mathbb{U}^{(z)}$, $\mathbb{T}^{(z)}$, or $\mathbb{W}^{(z)}$ in position i, and y is an element of $\mathbb{L}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$, $\mathbb{U}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$, $\mathbb{T}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$, or $\mathbb{W}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$ in position $\boldsymbol{g}+\boldsymbol{f}-i$, then

$$<x,dy> = \begin{cases} -\theta_{\boldsymbol{f}+\boldsymbol{g}}<x,dy>, & \text{if } x\in\mathbb{T}^{(z)} \text{ and } y\in\mathbb{T}^{(\boldsymbol{g}-\boldsymbol{f}-z)},\\ <x,dy>, & \text{otherwise.}\end{cases} \tag{2.13}$$

There are four cases to consider:
(1) $x\in\mathbb{L}^{(z)}$ and $y\in\mathbb{W}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$, (2) $x\in\mathbb{U}^{(z)}$ and $y\in\mathbb{W}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$,
(3) $x\in\mathbb{U}^{(z)}$ and $y\in\mathbb{T}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$, (4) $x\in\mathbb{T}^{(z)}$ and $y\in\mathbb{T}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$.
Cases (1) and (3) are easy; we omit them.

Case (2). Take $x=b_p\otimes\delta_q\otimes\mu^{(r)}\in\mathbb{U}(p,q,r)\subseteq\mathbb{U}^{(z)}$ and

$$y=B_{p'}\otimes b_{q'}\otimes\alpha_{r'}\otimes\delta_{s'}\otimes\xi^{(t')}\in\mathbb{W}(p',q',r',s',t')\subseteq\mathbb{W}^{(\boldsymbol{g}-\boldsymbol{f}-z)}.$$

We compute that $<x,dy>$ is equal to

$$\begin{cases}\delta_{p'0}B_0\sum\limits_{\substack{\varepsilon\le q'+s'+t'-\boldsymbol{g}+z-1\\|I|=\varepsilon}}\delta_{p\,r'+\varepsilon-q'}\delta_{q\,\boldsymbol{g}-s'+\varepsilon}\chi(p+q+r+q'+s'+t'-\boldsymbol{g}-\varepsilon=\boldsymbol{f})\\ \qquad(-1)^{\boldsymbol{f}+\boldsymbol{g}+t'+q'r'+\varepsilon+q'+s'\boldsymbol{g}}b_p\left([\varphi_I(b_{q'})](\alpha_{r'})\right)\cdot\delta_q\left(\delta_{s'}(\omega_G)\wedge(\bigwedge^{\varepsilon}X)(f_I)\right)\end{cases}$$

$$=\begin{cases}\delta_{p'0}B_0\chi(-p+q-q'+r'+s'=\boldsymbol{g})\chi(p+q'+r+t'=\boldsymbol{f})\chi(p-r'\le s'+t'-\boldsymbol{g}+z-1)\\ \qquad(-1)^{\boldsymbol{f}+\boldsymbol{g}+t'+q'r'+p-r'+s'\boldsymbol{g}}\\ \qquad\sum\limits_{|I|=p+q'-r'}b_p\left([\varphi_I(b_{q'})](\alpha_{r'})\right)\cdot[\delta_{s'}(\omega_G)]\left(\left[(\bigwedge^{p+q'-r'}X)(f_I)\right](\delta_q)\right).\end{cases}$$

In $<x,dy>$, we have $\chi(p-r'\le s'+t'-\boldsymbol{g}+z-1)=\chi(p+q+r\le\boldsymbol{f}+z-1)=1$. The last equality holds because $(p,q,r)\in\mathbb{T}^{(z)}_{\mathbb{U}}$. Apply Proposition 1.1 to see that $<x,dy>$ is equal to

$$\begin{cases}\delta_{p'0}B_0\chi(-p+q-q'+r'+s'=\boldsymbol{g})\chi(p+q'+r+t'=\boldsymbol{f})(-1)^{r+p+q+q'r'}\\ \qquad\sum\limits_{|I|=p+q'-r'}b_p\left([\varphi_I(b_{q'})](\alpha_{r'})\right)\cdot\left[(\bigwedge^{p+q'-r'}X)(f_I)\wedge\delta_q(\omega_G)\right](\delta_{s'}).\end{cases}$$

On the other hand, we compute that $<dx,y>$ is equal to

$$\begin{cases}\sum\limits_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\|J|=s\\|I|=r+q-s-t}}(-1)^{ps+t+p+s}\delta_{0\,p'}\delta_{r+q-t\,q'}\delta_{p+r+q-s-t\,r'}\delta_{\boldsymbol{g}-q+s\,s'}\chi(p+q+2r-t+t'=\boldsymbol{f})\\ \qquad 1(B_{p'})\cdot\left[(\bigwedge^{s}X^*)(\gamma_J)\wedge\varphi_I\right](b_{q'})\cdot\left[f_I\wedge b_p\right](\alpha_{r'})\cdot\left[g_J\wedge\delta_q(\omega_G)\right](\delta_{s'})\end{cases}$$

$$=\begin{cases}\delta_{p'\,0}B_0\chi(-p+q-q'+r'+s'=\boldsymbol{g})\chi(p+r+q'+t'=\boldsymbol{f})\chi(r+q-q'\le z-1)\\ \qquad\chi(r'\le p+r)(-1)^{pq'+r+q'+p+s'-\boldsymbol{g}}\\ \qquad\sum\limits_{\substack{|J|=q+s'-\boldsymbol{g}\\|I|=\boldsymbol{g}-q+q'-s'}}\left[(\bigwedge^{q+s'-\boldsymbol{g}}X^*)(\gamma_J)\right][\varphi_I(b_{q'})]\cdot b_p\left[f_I(\alpha_{r'})\right]\cdot\left[g_J\wedge\delta_q(\omega_G)\right](\delta_{s'}).\end{cases}$$

In $<dx,y>$, we have $\chi(r'\le p+r)=\chi(q'+r'+t'\le\boldsymbol{f})=1$, and

$$\chi(r+q-q'\le z-1)=\chi(\boldsymbol{f}+\boldsymbol{g}-z+1\le q'+r'+s'+t')=1.$$

In each case, the last equality holds because $(p',q',r',s',t')\in T_{\mathbb{W}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$. Apply Lemma 1.9.b to see that $<dx,y>$ is equal to

$$\begin{cases}\delta_{p'\,0}B_0\chi(-p+q-q'+r'+s'=\boldsymbol{g})\chi(p+r+q'+t'=\boldsymbol{f})(-1)^{r+q'+p+q'(\boldsymbol{g}-q-s'+p)-q}\\ \qquad\sum\limits_{\substack{|J|=q+s'-\boldsymbol{g}\\|I|=q+s'-\boldsymbol{g}}}\left[(\bigwedge^{q+s'-\boldsymbol{g}}X^*)(\gamma_J)\right][f_I]\cdot b_p\left[[\varphi_I(b_{q'})](\alpha_{r'})\right]\cdot\left[g_J\wedge\delta_q(\omega_G)\right](\delta_{s'}).\end{cases}$$

In $<dx,y>$, we have $q+s'-\boldsymbol{g}=p+q'-r'$. The definition of the dual of a homomorphism gives that

$$\left[(\bigwedge^{q+s'-\boldsymbol{g}}X^*)(\gamma_J)\right][f_I]=\gamma_J\left[(\bigwedge^{q+s'-\boldsymbol{g}}X)(f_I)\right];$$

thus, $<dx,y>$ is equal to

$$\begin{cases}\delta_{p'\,0}B_0\chi(-p+q-q'+r'+s'=\boldsymbol{g})\chi(p+r+q'+t'=\boldsymbol{f})(-1)^{r+p+q'r'-q}\\ \qquad\sum\limits_{|I|=p+q'-r'}b_p\left[[\varphi_I(b_{q'})](\alpha_{r'})\right]\cdot\left[(\bigwedge^{p+q'-r'}X)(f_I)\wedge\delta_q(\omega_G)\right](\delta_{s'}),\end{cases}$$

which is equal to $<x,dy>$, and (2.13) holds for case (2).

Case (4). Take $x=\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}(p,q,r)\subseteq\mathbb{T}^{(z)}$ and

$$y=\alpha_{p'}\otimes c_{q'}\otimes\lambda^{(r')}\in\mathbb{T}(p',q',r')\subseteq\mathbb{T}^{(\boldsymbol{g}-\boldsymbol{f}-z)}.$$

We see that $<x,dy>$ is equal to

$$\begin{cases}\sum\limits_{\substack{0\le t\\|I|=q'+r'-t}}\sigma_{\boldsymbol{g}-\boldsymbol{f}-z}(p',q',r',t)\chi(p=q'+r'-t)\chi(q=\boldsymbol{g}-\boldsymbol{f}+p'+r'-t)\chi(p+q+r+t=\boldsymbol{f}-1)\\ \qquad\alpha_p(f_I)\cdot c_q\left(\left[(\bigwedge^{\boldsymbol{f}-p'+t-q'-r'}X)\left((\varphi_I\wedge\alpha_{p'})[\omega_F]\right)\wedge c_{q'}\right](\omega_{G^*})\right)\end{cases}$$

$$=\begin{cases}\sum\limits_{|I|=p}\sigma_{\boldsymbol{g}-\boldsymbol{f}-z}(p',q',r',q'+r'-p)\chi(p\le q'+r')\chi(q=\boldsymbol{g}-\boldsymbol{f}+p'+p-q')\\ \qquad\chi(q+r+q'+r'=\boldsymbol{f}-1)\alpha_p(f_I)\cdot c_q\left(\left[(\bigwedge^{\boldsymbol{f}-p'-p}X)\left((\varphi_I\wedge\alpha_{p'})[\omega_F]\right)\wedge c_{q'}\right](\omega_{G^*})\right).\end{cases}$$

Notice that $\chi(p\le q'+r')=\chi(p+q+r\le\boldsymbol{f}-1)=1$; so, $<x,dy>$ is equal to

$$\begin{cases}\sigma_{\boldsymbol{g}-\boldsymbol{f}-z}(p',q',r',q'+r'-p)\chi(-p-p'+q+q'=\boldsymbol{g}-\boldsymbol{f})\\ \qquad\chi(q+r+q'+r'=\boldsymbol{f}-1)c_q\left(\left[(\bigwedge^{\boldsymbol{f}-p'-p}X)\left((\alpha_p\wedge\alpha_{p'})[\omega_F]\right)\wedge c_{q'}\right](\omega_{G^*})\right).\end{cases}$$

On the other hand, $<dx,y>$ is equal to

$$\begin{cases}\sum\limits_{\substack{0\le t\\|I|=q+r-t}}\sigma_z(p,q,r,t)\chi(q+r-t=p')\chi(\boldsymbol{g}-\boldsymbol{f}+p+r-t=q')\\ \qquad\chi(p+q+2r-t+r'=2\boldsymbol{f}-\boldsymbol{g}-1)\\ \qquad f_I(\alpha_{p'})\cdot\left(\left[(\bigwedge^{\boldsymbol{f}-p+t-q-r}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\right)(c_{q'})\end{cases}$$

$$=\begin{cases}\sum\limits_{|I|=p'}\sigma_z(p,q,r,q+r-p')\chi(\boldsymbol{g}-\boldsymbol{f}=-p-p'+q+q')\chi(p'\le q+r)\\ \qquad\chi(q+q'+r+r'=\boldsymbol{f}-1)\\ \qquad f_I(\alpha_{p'})\cdot\left(\left[(\bigwedge^{\boldsymbol{f}-p-p'}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\right)(c_{q'}).\end{cases}$$

In $<dx,y>$, $\chi(p'\le q+r)=\chi(p'+q'+r'\le\boldsymbol{f}-1)=1$; thus, $<dx,y>$ is equal to

$$\begin{cases}(-1)^{pp'+q(\boldsymbol{f}-p'-p)+q'(q+\boldsymbol{f}-p'-p)}\sigma_z(p,q,r,q+r-p')\chi(\boldsymbol{g}-\boldsymbol{f}=-p-p'+q+q')\\ \qquad\chi(q+q'+r+r'=\boldsymbol{f}-1)c_q\left(\left[(\bigwedge^{\boldsymbol{f}-p-p'}X)\left((\alpha_p\wedge\alpha_{p'})[\omega_F]\right)\wedge c_{q'}\right](\omega_{G^*})\right).\end{cases}$$

The proof of (2.13) for case (4) is complete because

$$(-1)^{pp'+q(\boldsymbol{f}-p'-p)+q'(q+\boldsymbol{f}-p'-p)}\sigma_z(p,q,r,q+r-p')=-\theta_{\boldsymbol{f}+\boldsymbol{g}}\sigma_{\boldsymbol{g}-\boldsymbol{f}-z}(p',q',r',q'+r'-p).\quad\square$$

3. Properties of the complexes $\mathbb{I}^{(z)}$.

The zeroth homology of the complex $\mathbb{I}^{(z)}$ is identified in Corollary 3.5. In Observation 3.3, we establish homomorphisms from $H_0(\mathbb{I}^{(z)})$ to ideals of $H(\mathbb{I}^{(0)})$. (These maps are shown to be isomorphisms in section 8.) Proposition 3.6 contains the short exact sequence of complexes for $\mathbb{I}^{(z)}$ which was promised in (0.1); this exact sequence is critical to the proof, in Theorem 8.6, that $\mathbb{I}^{(z)}$ is acyclic.

Definition 3.1. Adopt Data 1.2. Let K be the $R-$ideal

$$K = I_{\boldsymbol{f}}(X) + I_1(X^*(u)) + I_1(X(v)),$$

N be the $R-$module which is defined by the exact sequence:

$$G^* \oplus \bigwedge^2 F^* \oplus (F^* \otimes F) \xrightarrow{[X^* \quad v \quad 1\otimes X^*(u)]} F^* \to N \to 0,$$

and $\mathfrak{N}$ be the $R-$module defined by the exact sequence:

$$\left[(F^* \otimes \bigwedge^{\boldsymbol{f}-1} G^*) \oplus \bigwedge^{\boldsymbol{f}-2} G^* \oplus (F \otimes \bigwedge^{\boldsymbol{f}} G^*)\right] \xrightarrow{h} \begin{array}{c} R/K \\ \oplus \\ \bigwedge^{\boldsymbol{f}-1} G^* \end{array} \to \mathfrak{N} \to 0,$$

where

$$h(\alpha_1 \otimes \delta_{\boldsymbol{f}-1}) = \begin{bmatrix} \left[(\bigwedge^{\boldsymbol{f}-1} X)[\alpha_1(\omega_F)]\right](\delta_{\boldsymbol{f}-1}) \\ v(\alpha_1)\cdot \delta_{\boldsymbol{f}-1} \end{bmatrix},\quad h(\delta_{\boldsymbol{f}-2}) = \begin{bmatrix} 0 \\ u \wedge \delta_{\boldsymbol{f}-2}\end{bmatrix}, \text{ and}$$

$$h(b_1 \otimes \delta_{\boldsymbol{f}}) = \begin{bmatrix} 0 \\ [X(b_1)](\delta_{\boldsymbol{f}}) \end{bmatrix}.$$

Remark 3.2. It is not difficult to see that N and $\mathfrak{N}$ are modules over R/K. The only interesting part of this argument is the proof that the element

$$(\bigwedge^r X)(b_r)](\delta_{\boldsymbol{f}}) \wedge \delta_{r-1}$$

of $\bigwedge^{\boldsymbol{f}-1} G^*$ is in the image of h for $1 \le r \le \boldsymbol{f}$. This proof proceeds by induction on r. If $2 \le r$, then

$$\begin{aligned} &h\left(b_1 \otimes \left[(\bigwedge^{r-1} X)(b_{r-1})\right](\delta_{\boldsymbol{f}}) \wedge \delta_{r-1}\right) \\ &= [(\bigwedge^r X)(b_1 \wedge b_{r-1})](\delta_{\boldsymbol{f}}) \wedge \delta_{r-1} + \text{an element of } \operatorname{im} h. \end{aligned}$$

Observation 3.3. *If the notation of Theorem 0.3, Convention 1.4, and Definition 3.1 are adopted, then there are $R/K-$module surjections $N \twoheadrightarrow \mathfrak{b}_3$, $N \twoheadrightarrow \mathfrak{a}_2$, and $\mathfrak{N} \twoheadrightarrow \mathfrak{p}_2$.*

Proof. It is clear that the map $F^* \to R$, which is given by $\alpha_1 \mapsto v(\alpha_1)$, induces a surjection of N onto $\mathfrak{b}_3$. Let $\delta_{\boldsymbol{f}-1}$ be the element $\gamma^{[1]} \wedge \ldots \wedge \gamma^{[\boldsymbol{f}-1]}$ of $\bigwedge^{\boldsymbol{f}-1} G^*$. It is not difficult to see that the map $m\colon F^* \to R$, which is given by $m(\alpha_1)$ is equal to $\left[(\bigwedge^{\boldsymbol{f}-1} X^*)(\delta_{\boldsymbol{f}-1})\right][\alpha_1(\omega_F)]$, induces a surjection of N onto $\mathfrak{a}_2$. For example, if $\alpha_2 \in \bigwedge^2 F^*$, then Proposition 1.1 shows that $m(v(\alpha_2))$ is equal to

$$\begin{aligned}\left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X^*)(\delta_{\boldsymbol{f}-1})\right]\left[[v(\alpha_2)](\omega_F)\right] &= \left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X^*)(\delta_{\boldsymbol{f}-1})\right]\left[v \wedge \alpha_2(\omega_F)\right]\\ &= (-1)^{\boldsymbol{f}}\,[\alpha_2(\omega_F)]\left(v\left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X^*)(\delta_{\boldsymbol{f}-1})\right]\right)\\ &= (-1)^{\boldsymbol{f}}\,[\alpha_2(\omega_F)]\left[(\textstyle\bigwedge^{\boldsymbol{f}-2} X^*)\left([X(v)](\delta_{\boldsymbol{f}-1})\right)\right] \in K.\end{aligned}$$

If $b_{\boldsymbol{f}-1}$ is the element $f^{[2]} \wedge \ldots \wedge f^{[\boldsymbol{f}]}$ of $\bigwedge^{\boldsymbol{f}-1} F$, then it is not difficult to see that the map

$$m_2\colon R/K \oplus \textstyle\bigwedge^{\boldsymbol{f}-1} G^* \to R/K,$$

which is given by

$$m_2\begin{bmatrix} r\\ \delta_{\boldsymbol{f}-1}\end{bmatrix} = v(b_{\boldsymbol{f}-1}[\omega_{F^*}])\cdot r - \left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X)(b_{\boldsymbol{f}-1})\right](\delta_{\boldsymbol{f}-1}),$$

induces a surjection of $\mathfrak{N}$ onto $\mathfrak{p}_2$. For example, if $\alpha_1 \in F^*$ and $\delta_{\boldsymbol{f}-1} \in \bigwedge^{\boldsymbol{f}-1} G^*$, then $m_2 \circ h(\alpha_1 \otimes \delta_{\boldsymbol{f}-1})$ is equal to

$$\begin{aligned}&v(b_{\boldsymbol{f}-1}[\omega_{F^*}])\cdot\left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X)\,(\alpha_1[\omega_F])\right](\delta_{\boldsymbol{f}-1}) - v(\alpha_1)\cdot\left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X)(b_{\boldsymbol{f}-1})\right](\delta_{\boldsymbol{f}-1})\\ &= \left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X)\left(\left(v\,[b_{\boldsymbol{f}-1}(\omega_{F^*}) \wedge \alpha_1]\right)[\omega_F]\right)\right](\delta_{\boldsymbol{f}-1})\\ &= \left[(\textstyle\bigwedge^{\boldsymbol{f}-1} X)\left(v \wedge [b_{\boldsymbol{f}-1}(\omega_{F^*}) \wedge \alpha_1]\,[\omega_F]\right)\right](\delta_{\boldsymbol{f}-1})\\ &= \left[X(v) \wedge (\textstyle\bigwedge^{\boldsymbol{f}-2} X)\left([b_{\boldsymbol{f}-1}(\omega_{F^*}) \wedge \alpha_1]\,[\omega_F]\right)\right](\delta_{\boldsymbol{f}-1}) \in K. \quad \square\end{aligned}$$

Let $\mathbb{I}_i^{(z)}$ denote the submodule of $\mathbb{I}^{(z)}$ in position i.

Lemma 3.4.

(a) *Every non-zero summand of $\mathbb{I}^{(z)}$ of the form $\mathbb{L}(p,q,r,s,t)$ has position at least* 0.

(b) *If $\mathbb{L}(p,q,r,s,t)$ is a non-zero summand of $\mathbb{I}_0^{(z)}$, then (p,q,r,s,t) is equal to $(z-1,1,0,\boldsymbol{g},0)$.*

(c) *If $\mathbb{L}(p,q,r,s,t)$ is a non-zero summand of $\mathbb{I}_1^{(z)}$, then (p,q,r,s,t) is equal to $(z-2,1,0,\boldsymbol{g}-1,1)$, $(z-1,2,0,\boldsymbol{g},0)$, $(z-1,1,1,\boldsymbol{g},0)$, or $(z-2,2,0,\boldsymbol{g},0)$.*

(d) *Every non-zero summand of $\mathbb{I}^{(z)}$ of the form $\mathbb{U}(p,q,r)$ has position at least z.*

(e) *If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{I}_z^{(z)}$, then (p,q,r) is equal to $(0,z,0)$.*

(f) *If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{I}_{z+1}^{(z)}$, then (p,q,r) is equal to $(1,z,0)$, $(0,z+1,0)$, or $(0,z-1,1)$.*

(g) *Every non-zero summand of $\mathbb{I}^{(z)}$ of the form $\mathbb{T}(p,q,r)$ has position at least $z+1$.*

(h) *If* $\mathbb{T}(p,q,r)$ *is a non-zero summand of* $\mathbb{I}^{(z)}_{z+1}$, *then* (p,q,r) *is equal to* $(0,\boldsymbol{g}-\boldsymbol{f}-z,\boldsymbol{f}-\boldsymbol{g}+z)$.

(i) *Every non-zero summand of* $\mathbb{I}^{(z)}$ *of the form* $\mathbb{W}(p,q,r,s,t)$ *has position at least* $z+2$.

(j) *If* $\mathbb{W}(p,q,r,s,t)$ *is a non-zero summand of* $\mathbb{I}^{(z)}_{z+2}$, *then* $p=0$, $q=\boldsymbol{f}$, $r\le \boldsymbol{f}-1$, $r+s=2\boldsymbol{f}+z$, *and* $t=1-\boldsymbol{f}$.

(k) *If* $-1\le z$ *and* j *is negative, then* $\mathbb{I}^{(z)}_j=0$.

Proof. If $\mathbb{L}(p,q,r,s,t)$ is a non-zero summand of $\mathbb{I}^{(z)}$, then

$$\begin{aligned}0 &\le (s+t-\boldsymbol{g})+(z-1-p-t)+(p+q+t-z)+r+t=q+r+s+2t-1-\boldsymbol{g}\\ &= \text{ the position of } \mathbb{L}(p,q,r,s,t) \text{ in } \mathbb{I}^{(z)}.\end{aligned}$$

If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{I}^{(z)}$, then

$$z\le (q+r-z)+p+r+z=p+q+2r= \text{ the position of } \mathbb{U}(p,q,r) \text{ in } \mathbb{I}^{(z)}.$$

If $\mathbb{T}(p,q,r)$ is a non-zero summand of $\mathbb{I}^{(z)}$, then

$$z+1\le p+(q+r)+(r-\boldsymbol{f}+\boldsymbol{g}-z)+z+1=p+q+2r+\boldsymbol{g}-\boldsymbol{f}+1= \text{ the position of } \mathbb{T}(p,q,r) \text{ in } \mathbb{I}^{(z)}.$$

If $\mathbb{W}(p,q,r,s,t)$ is a non-zero summand of $\mathbb{I}^{(z)}$, then

$$\begin{aligned}z+2 &\le (p+q+t-1)+(q+r+s+t-2\boldsymbol{f}-z-1)+(\boldsymbol{f}-q)+p+z+2\\ &=2p+q+r+s+2t-\boldsymbol{f}= \text{ the position of } \mathbb{W}(p,q,r,s,t) \text{ in } \mathbb{I}^{(z)}. \quad\square\end{aligned}$$

Corollary 3.5. *Adopt the notation of Definition 3.1 with* $1\le \boldsymbol{f}$. *Then,*

$$H_0(\mathbb{I}^{(z)})=\begin{cases}\mathfrak{N}, & \textit{if } -1=z,\\ R/K, & \textit{if } 0=z, \textit{ and}\\ S_z(N), & \textit{if } 1\le z.\end{cases}$$

Proof. Lemma 3.4 and Definition 2.3 yield that

$$\Big[\chi(2\le \boldsymbol{f})\mathbb{U}(0,1,0)\oplus\chi(2\le \boldsymbol{f})\mathbb{U}(1,0,0)\oplus\mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f},\boldsymbol{f}-\boldsymbol{g})\Big]\xrightarrow{d_1}\mathbb{U}(0,0,0)\to H_0(\mathbb{I}^{(0)})\to 0$$

is exact, where

$$\begin{aligned}d_1(1\otimes\delta_1\otimes\mu^{(0)})&=1\otimes[X(v)](\delta_1)\otimes\mu^{(0)},\\ d_1(b_1\otimes 1\otimes\mu^{(0)})&=[X^*(u)](b_1)\otimes 1\otimes\mu^{(0)}, \text{ and}\\ d_1(1\otimes c_{\boldsymbol{g}-\boldsymbol{f}}\otimes\lambda^{(\boldsymbol{f}-\boldsymbol{g})})&=(-1)^{\boldsymbol{g}+\boldsymbol{g}\boldsymbol{f}}\theta_{\boldsymbol{g}-\boldsymbol{f}}\otimes[(\textstyle\bigwedge^{\boldsymbol{f}}X)(\omega_F)\wedge c_{\boldsymbol{g}-\boldsymbol{f}}](\omega_{G^*})\otimes\mu^{(0)}.\end{aligned}$$

The calculation of $H_0(\mathbb{I}^{(1)})$ is due to the exact sequence

$$\Big[\mathbb{L}(0,2,0,\boldsymbol{g},0)\oplus\chi(2\le \boldsymbol{f})\mathbb{L}(0,1,1,\boldsymbol{g},0)\oplus\mathbb{U}(0,1,0)\Big]\xrightarrow{d_1}\mathbb{L}(0,1,0,\boldsymbol{g},0)\to H_0(\mathbb{I}^{(1)})\to 0,$$

where

$$\begin{aligned}
d_1(1\otimes\alpha_2\otimes 1\otimes\omega_G\otimes\nu^{(0)}) &= 1\otimes v(\alpha_2)\otimes 1\otimes\omega_G\otimes\nu^{(0)},\\
d_1(1\otimes\alpha_1\otimes b_1\otimes\omega_G\otimes\nu^{(0)}) &= -1\otimes\alpha_1\otimes[X^*(u)](b_1)\otimes\omega_G\otimes\nu^{(0)},\ \text{and}\\
d_1(1\otimes\delta_1\otimes\mu^{(0)}) &= -1\otimes X^*(\delta_1)\otimes 1\otimes\omega_G\otimes\nu^{(0)}.
\end{aligned}$$

Fix z with $2\le z$. We have the exact sequence

$$\mathbb{I}_1^{(z)}\xrightarrow{d_1}\mathbb{L}(z-1,1,0,\boldsymbol{g},0)\to H_0(\mathbb{I}^{(z)})\to 0,$$

where $\mathbb{I}_1^{(z)}$ is equal to

$\mathbb{L}(z-2,1,0,\boldsymbol{g}-1,1)\oplus\mathbb{L}(z-1,2,0,\boldsymbol{g},0)\oplus\chi(2\le\boldsymbol{f})\mathbb{L}(z-1,1,1,\boldsymbol{g},0)\oplus\mathbb{L}(z-2,2,0,\boldsymbol{g},0)$,

$$\begin{aligned}
d_1(A_{z-2}\otimes\alpha_1\otimes 1\otimes\delta_1(\omega_G)\otimes\nu^{(1)}) &= X^*(\delta_1)\cdot A_{z-2}\otimes\alpha_1\otimes 1\otimes\omega_G\otimes\nu^{(0)},\\
d_1(A_{z-1}\otimes\alpha_2\otimes 1\otimes\omega_G\otimes\nu^{(0)}) &= A_{z-1}\otimes v(\alpha_2)\otimes 1\otimes\omega_G\otimes\nu^{(0)},\\
d_1(A_{z-1}\otimes\alpha_1\otimes b_1\otimes\omega_G\otimes\nu^{(0)}) &= -A_{z-1}\otimes\alpha_1\otimes[X^*(u)](b_1)\otimes\omega_G\otimes\nu^{(0)},\ \text{and}\\
d_1(A_{z-2}\otimes\alpha_2\otimes 1\otimes\omega_G\otimes\nu^{(0)}) &= \sum_{|I|=1}\varphi_I\cdot A_{z-2}\otimes f_I(\alpha_2)\otimes 1\otimes\omega_G\otimes\nu^{(0)}.
\end{aligned}$$

Notice that $d_1\colon\mathbb{L}(z-2,2,0,\boldsymbol{g},0)\to\mathbb{L}(z-1,1,0,\boldsymbol{g},0)$ is the Koszul complex map which is induced by the identity map on F^*. Since

$$S_{z-2}F^*\otimes\textstyle\bigwedge^2F^*\xrightarrow{\partial}S_{z-1}F^*\otimes\bigwedge^1F^*\xrightarrow{\partial}S_zF^*\to 0$$

is an exact sequence (see Remark 1.16); it follows that

$$\begin{array}{c}
S_{z-1}F^*\otimes G^*\\
\oplus\\
S_{z-1}F^*\otimes\bigwedge^2F^*\\
\oplus\\
S_zF^*\otimes F
\end{array}\longrightarrow S_zF^*\to H_0(\mathbb{I}^{(z)})\to 0$$

is exact, where the map is given by

$$\begin{aligned}
A_{z-1}\otimes\delta_1 &\mapsto A_{z-1}\cdot X^*(\delta_1),\\
A_{z-1}\otimes\alpha_2 &\mapsto A_{z-1}\cdot v(\alpha_2),\ \text{and}\\
A_z\otimes b_1 &\mapsto A_z\cdot[X^*(u)](b_1).
\end{aligned}$$

Finally, we compute $H_0(\mathbb{I}^{(-1)})$. If $\boldsymbol{f}=1$, then it is not difficult to see that $H_O(\mathbb{I}^{(-1)})$ and $\mathfrak{N}$ are both equal to R/K. Henceforth, we take $2\le\boldsymbol{f}$. Identify $\mathbb{I}_0^{(-1)}$ with $R\oplus\bigwedge^{\boldsymbol{f}-1}G^*$ by way of

$$n_0\colon\mathbb{U}(0,0,0)\oplus\mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f}+1,\boldsymbol{f}-\boldsymbol{g}-1)\to R\oplus\textstyle\bigwedge^{f-1}G^*,$$

where

$$n_0(1\otimes 1\otimes \mu^{(0)}) = \begin{bmatrix}1\\0\end{bmatrix} \quad \text{and} \quad n_0(1\otimes c_{\boldsymbol{g}-\boldsymbol{f}+1}\otimes \lambda^{(\boldsymbol{f}-\boldsymbol{g}-1)}) = \begin{bmatrix}0\\ c_{\boldsymbol{g}-\boldsymbol{f}+1}(\omega_{G^*})\end{bmatrix}.$$

Identify $\mathbb{I}_1^{(-1)}$ with the $R-$module M, which is equal to

$$\chi(3\le \boldsymbol{f})G^*\oplus\chi(3\le\boldsymbol{f})F\oplus(F^*\otimes{\textstyle\bigwedge}^{\boldsymbol{f}-1}G^*)\oplus{\textstyle\bigwedge}^{\boldsymbol{f}-2}G^*\oplus{\textstyle\bigwedge}^{\boldsymbol{f}}G^*\oplus\bigoplus_{r\le\boldsymbol{f}-1}({\textstyle\bigwedge}^r F^*\otimes{\textstyle\bigwedge}^{2\boldsymbol{f}-1-r}G^*),$$

by using the isomorphism $n_1\colon M\to\mathbb{I}_1^{(-1)}$, which is given by

$$\begin{aligned}
n_1(\delta_1) &= 1\otimes\delta_1\otimes\mu^{(0)}\in\chi(3\le\boldsymbol{f})\mathbb{U}(0,1,0),\\
n_1(b_1) &= b_1\otimes 1\otimes\mu^{(0)}\in\chi(3\le\boldsymbol{f})\mathbb{U}(1,0,0),\\
n_1(\alpha_1\otimes\delta_{\boldsymbol{f}-1}) &= \alpha_1\otimes\delta_{\boldsymbol{f}-1}(\omega_G)\otimes\lambda^{(\boldsymbol{f}-\boldsymbol{g}-1)}\in\mathbb{T}(1,\boldsymbol{g}-\boldsymbol{f}+1,\boldsymbol{f}-\boldsymbol{g}-1),\\
n_1(\delta_{\boldsymbol{f}-2}) &= 1\otimes\delta_{\boldsymbol{f}-2}(\omega_G)\otimes\lambda^{(\boldsymbol{f}-\boldsymbol{g}-1)}\in\mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f}+2,\boldsymbol{f}-\boldsymbol{g}-1),\\
n_1(\delta_{\boldsymbol{f}}) &= 1\otimes\delta_{\boldsymbol{f}}(\omega_G)\otimes\lambda^{(\boldsymbol{f}-\boldsymbol{g})}\in\mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f},\boldsymbol{f}-\boldsymbol{g}),\\
n_1(\alpha_r\otimes\delta_{2\boldsymbol{f}-1-r}) &= 1\otimes\omega_F\otimes\alpha_r\otimes\delta_{2\boldsymbol{f}-1-r}\otimes\xi^{(1-\boldsymbol{f})}\in\mathbb{W}(0,\boldsymbol{f},r,2\boldsymbol{f}-1-r,1-\boldsymbol{f}).
\end{aligned}$$

Let $h'=n_0\circ d_1\circ n_1$. We have an exact sequence

$$M\xrightarrow{h'}\begin{matrix}R\\\oplus\\{\textstyle\bigwedge}^{\boldsymbol{f}-1}G^*\end{matrix}\to H_0(\mathbb{I}^{(-1)})\to 0,$$

where

$$h'(\delta_1)=\begin{bmatrix}[X(v)](\delta_1)\\0\end{bmatrix},\qquad h'(b_1)=\begin{bmatrix}[X^*(u)](b_1)\\0\end{bmatrix},$$

$$h'(\alpha_1\otimes\delta_{\boldsymbol{f}-1})=\begin{bmatrix}(-1)^{\boldsymbol{f}+\boldsymbol{g}+1+\boldsymbol{f}\boldsymbol{g}}\theta_{\boldsymbol{g}-\boldsymbol{f}+1}\cdot\left[({\textstyle\bigwedge}^{\boldsymbol{f}-1}X)[\alpha_1(\omega_F)]\right](\delta_{\boldsymbol{f}-1})\\ v(\alpha_1)\cdot\delta_{\boldsymbol{f}-1}\end{bmatrix},$$

$$h'(\delta_{\boldsymbol{f}-2})=\begin{bmatrix}0\\(-1)^{\boldsymbol{g}+\boldsymbol{f}}u\wedge\delta_{\boldsymbol{f}-2}\end{bmatrix},$$

$$h'(\delta_{\boldsymbol{f}})=\begin{bmatrix}(-1)^{\boldsymbol{g}+1+\boldsymbol{f}\boldsymbol{g}}\theta_{\boldsymbol{g}-\boldsymbol{f}}\cdot(\boldsymbol{f}-1)\left[({\textstyle\bigwedge}^{\boldsymbol{f}}X)(\omega_F)\right](\delta_{\boldsymbol{f}})\\(-1)^{\boldsymbol{g}-\boldsymbol{f}}[X(v)](\delta_{\boldsymbol{f}})\end{bmatrix},\ \text{and}$$

$$h'(\alpha_r\otimes\delta_{2\boldsymbol{f}-1-r})=\begin{bmatrix}0\\\pm\left[({\textstyle\bigwedge}^{\boldsymbol{f}-r}X)(\alpha_r(\omega_F))\right][\delta_{2\boldsymbol{f}-1-r}]\end{bmatrix}.$$

Recall the map h from Definition 3.1. We make two observations. First, observe that

$$h\colon F\otimes{\textstyle\bigwedge}^{\boldsymbol{f}}G^*\to R\oplus{\textstyle\bigwedge}^{\boldsymbol{f}-1}G^*\ \text{ and }\ h'\colon\bigoplus_{r\le\boldsymbol{f}-1}{\textstyle\bigwedge}^r F^*\otimes{\textstyle\bigwedge}^{2\boldsymbol{f}-1-r}G^*\to R\oplus{\textstyle\bigwedge}^{\boldsymbol{f}-1}G^*$$

have the same image because

$$h'\left((b_1\wedge b_{\boldsymbol{f}-1-r})(\omega_{F^*})\otimes\delta_{2\boldsymbol{f}-1-r}\right)=\pm h\left(b_1\otimes\left[(\textstyle\bigwedge^{\boldsymbol{f}-r-1}X)(b_{\boldsymbol{f}-r-1})\right](\delta_{2\boldsymbol{f}-1-r})\right).$$

For our second observation, use Lemma 1.9.c to see that

$$\sum_{|K|=1}h'\left(X^*(\gamma_K)\otimes g_K(\delta_{\boldsymbol{f}})\right)=\begin{bmatrix}(-1)^{\boldsymbol{g}+\boldsymbol{f}\boldsymbol{g}}\theta_{\boldsymbol{g}-\boldsymbol{f}+1}\boldsymbol{f}\delta_{\boldsymbol{f}}\left((\textstyle\bigwedge^{\boldsymbol{f}}X)(\omega_F)\right)\\ [X(v)](\delta_{\boldsymbol{f}})\end{bmatrix};$$

hence, it follows that

$$\begin{bmatrix}[(\textstyle\bigwedge^{\boldsymbol{f}}X)(\omega_F)](\delta_{\boldsymbol{f}})\\ 0\end{bmatrix}=(-1)^{\boldsymbol{g}+\boldsymbol{f}\boldsymbol{g}}\theta_{\boldsymbol{g}-\boldsymbol{f}}\left(h'(\delta_{\boldsymbol{f}})+(-1)^{\boldsymbol{g}+\boldsymbol{f}+1}\sum_{|K|=1}h'\left(X^*(\gamma_K)\otimes g_K(\delta_{\boldsymbol{f}})\right)\right).$$

At this point it is not difficult to see that $H_0(\mathbb{I}^{(-1)})\cong\mathfrak{N}$. $\square$

Proposition 3.6. *Adopt the data of 1.2 and 1.4. Let* $\widetilde{G}=\bigoplus_{i=1}^{\boldsymbol{g}-1}g^{[i]}$, $\widetilde{G}^*=\bigoplus_{i=1}^{\boldsymbol{g}-1}\gamma^{[i]}$, $\widetilde{X}\colon F\to\widetilde{G}$ *be the composition* $F\xrightarrow{X}G=\widetilde{G}\oplus Rg^{[\boldsymbol{g}]}\xrightarrow{\text{proj}}\widetilde{G}$, $\widetilde{u}$ *be the restriction of* u *to* $\widetilde{G}$, *and* $\overline{u}$ *be the element*

$$\overline{u}=[\,\widetilde{u}\quad 0\,]:G=\widetilde{G}\oplus Rg^{[\boldsymbol{g}]}\to R$$

of G^*. *Form the complexes* $(\widetilde{\mathbb{I}}^{(z)},\widetilde{d})$ *and the modules* $\widetilde{K}$, $\widetilde{N}$, *and* $\widetilde{\mathfrak{N}}$ *using the data* $\widetilde{X}\colon F\to\widetilde{G}$, $\widetilde{u}\in\widetilde{G}^*$, *and* $v\in F$. *Form the complexes* $(\overline{\mathbb{I}},\overline{d})$ *and the modules* $\overline{K}$ *and* $\overline{N}$ *using the data* $X\colon F\to G$, $\overline{u}\in G^*$, *and* $v\in F$. *Then, for every integer* z, *there is a short exact sequence of complexes*

$$0\to\widetilde{\mathbb{I}}^{(z)}\to\overline{\mathbb{I}}^{(z)}\to\widetilde{\mathbb{I}}^{(z-1)}[-1]\to 0.$$

Furthermore, these short exact sequences induce the following long exact sequences of homology:

$$\cdots\to H_1(\widetilde{\mathbb{I}}^{(-1)})\to H_1(\widetilde{\mathbb{I}}^{(0)})\to H_1(\overline{\mathbb{I}}^{(0)})\to\widetilde{\mathfrak{N}}\to\overline{K}/\widetilde{K}\to 0,$$

when $0=z$, *and*

$$\cdots\to H_1(\widetilde{\mathbb{I}}^{(z-1)})\to H_1(\widetilde{\mathbb{I}}^{(z)})\to H_1(\overline{\mathbb{I}}^{(z)})\to S_{z-1}\widetilde{N}\xrightarrow{X^*(\gamma^{[\boldsymbol{g}]})}S_z\widetilde{N}\to S_z\overline{N}\to 0$$

when $1\le z$, *where* $S_0\widetilde{N}$ *is taken to mean* $R/\widetilde{K}$.

Proof. The modules $\widetilde{\mathbb{L}}(p,q,r,s,t),\dots,\widetilde{\mathbb{W}}(p,q,r,s,t)$ of Definition 2.1 are formed using $\widetilde{G}$ in place of G. The modules $\overline{\mathbb{L}}(p,q,r,s,t),\dots,\overline{\mathbb{W}}(p,q,r,s,t)$ are exactly the same as $\mathbb{L}(p,q,r,s,t),\dots,\mathbb{W}(p,q,r,s,t)$. Define

$$\Phi\colon\widetilde{\mathbb{L}}(p,q,r,s-1,t)\to\overline{\mathbb{L}}(p,q,r,s,t),\qquad\Phi\colon\widetilde{\mathbb{U}}(p,q,r)\to\overline{\mathbb{U}}(p,q,r),$$

$$\Phi\colon\widetilde{\mathbb{T}}(p,q-1,r+1)\to\overline{\mathbb{T}}(p,q,r),\ \text{and}\quad\Phi\colon\widetilde{\mathbb{W}}(p,q,r,s,t)\to\overline{\mathbb{W}}(p,q,r,s,t)$$

by

$$\Phi(A_p\otimes\alpha_q\otimes b_r\otimes c_{s-1}\otimes\nu^{(t)})=A_p\otimes\alpha_q\otimes b_r\otimes c_{s-1}\wedge g^{[\boldsymbol{g}]}\otimes\nu^{(t)},$$

$$\Phi(b_p\otimes\delta_q\otimes\mu^{(r)})=b_p\otimes\delta_q\otimes\mu^{(r)},$$

$$\Phi(\alpha_p\otimes c_{q-1}\otimes\lambda^{(r+1)})=(-1)^{\boldsymbol{f}}\alpha_p\otimes g^{[\boldsymbol{g}]}\wedge c_{q-1}\otimes\lambda^{(r)},\text{ and}$$

$$\Phi(B_p\otimes b_q\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t)})=(-1)^{\boldsymbol{f}+\boldsymbol{g}}B_p\otimes b_q\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t)}.$$

Define

$$\begin{array}{ll}\Psi\colon \overline{\mathbb{L}}(p,q,r,s,t)\to\widetilde{\mathbb{L}}(p,q,r,s,t-1), & \Psi\colon\overline{\mathbb{U}}(p,q,r)\to\widetilde{\mathbb{U}}(p,q-1,r),\\ \Psi\colon\overline{\mathbb{T}}(p,q,r)\to\widetilde{\mathbb{T}}(p,q,r),\text{ and} & \Psi\colon\overline{\mathbb{W}}(p,q,r,s,t)\to\widetilde{\mathbb{W}}(p,q,r,s-1,t)\end{array}$$

by

$$\Psi(A_p\otimes\alpha_q\otimes b_r\otimes c_s\otimes\nu^{(t)})=\begin{cases}A_p\otimes\alpha_q\otimes b_r\otimes c_s\otimes\nu^{(t-1)}, & \text{if } c_s\in\bigwedge^s\widetilde{G},\\ 0, & \text{if } c_s=c_{s-1}\wedge g^{[\boldsymbol{g}]},\ c_{s-1}\in\bigwedge^{s-1}\widetilde{G},\end{cases}$$

$$\Psi(b_p\otimes\delta_q\otimes\mu^{(r)})=\begin{cases}0, & \text{if } \delta_q\in\bigwedge^q\widetilde{G}^*,\\ (-1)^{\boldsymbol{g}}b_p\otimes\delta_{q-1}\otimes\mu^{(r)}, & \text{if } \delta_q=\delta_{q-1}\wedge\gamma^{[\boldsymbol{g}]},\ \delta_{q-1}\in\bigwedge^{q-1}\widetilde{G}^*,\end{cases}$$

$$\Psi(\alpha_p\otimes c_q\otimes\lambda^{(r)})=\begin{cases}\alpha_p\otimes c_q\otimes\lambda^{(r)}, & \text{if } c_q\in\bigwedge^q\widetilde{G},\\ 0, & \text{if } c_q=c_{q-1}\wedge g^{[\boldsymbol{g}]},\ c_{q-1}\in\bigwedge^{q-1}\widetilde{G},\text{ and}\end{cases}$$

$$\Psi(B_p\otimes b_q\otimes\alpha_r\otimes\delta_s\otimes\xi^{(t)})=\begin{cases}0, & \text{if } \delta_s\in\bigwedge^s\widetilde{G}^*,\\ B_p\otimes b_q\otimes\alpha_r\otimes\delta_{s-1}\otimes\xi^{(t)}, & \text{if } \delta_s=\gamma^{[\boldsymbol{g}]}\wedge\delta_{s-1},\ \delta_{s-1}\in\bigwedge^{s-1}\widetilde{G}^*.\end{cases}$$

It is clear that

$$0\to\widetilde{\mathbb{L}}(p,q,r,s-1,t)\xrightarrow{\Phi}\overline{\mathbb{L}}(p,q,r,s,t)\xrightarrow{\Psi}\widetilde{\mathbb{L}}(p,q,r,s,t-1)\to 0$$

is a short exact sequence of $R-$modules for all integers p, q, r, s, and t. It is also clear that

$$(p,q,r,s-1,t)\in\widetilde{T}_{\mathbb{L}}^{(z)}\iff(p,q,r,s,t)\in\overline{T}_{\mathbb{L}}^{(z)}\iff(p,q,r,s,t-1)\in\widetilde{T}_{\mathbb{L}}^{(z-1)}.$$

If $i=q+r+s+2t-1-\boldsymbol{g}$, then $\overline{\mathbb{L}}(p,q,r,s,t)$ has position i in $\overline{\mathbb{I}}^{(z)}$; $\widetilde{\mathbb{L}}(p,q,r,s-1,t)$ has position i in $\widetilde{\mathbb{I}}^{(z)}$; and $\widetilde{\mathbb{L}}(p,q,r,s,t-1)$ has position $i-1$ in $\widetilde{\mathbb{I}}^{(z-1)}$. (Of course, this means that $\widetilde{\mathbb{L}}(p,q,r,s,t-1)$ has position i in $\widetilde{\mathbb{I}}^{(z-1)}[-1]$.) The analogous statements about

$$0\to\widetilde{\mathbb{U}}(p,q,r)\xrightarrow{\Phi}\overline{\mathbb{U}}(p,q,r)\xrightarrow{\Psi}\widetilde{\mathbb{U}}(p,q-1,r)\to 0,$$

$$0\to\widetilde{\mathbb{T}}(p,q-1,r+1)\xrightarrow{\Phi}\overline{\mathbb{T}}(p,q,r)\xrightarrow{\Psi}\widetilde{\mathbb{T}}(p,q,r)\to 0,\text{ and}$$

$$0\to\widetilde{\mathbb{W}}(p,q,r,s,t)\xrightarrow{\Phi}\overline{\mathbb{W}}(p,q,r,s,t)\xrightarrow{\Psi}\widetilde{\mathbb{W}}(p,q,r,s-1,t)\to 0$$

are also easy to establish. A straightforward calculation shows that Φ and Ψ are maps of complexes. Some of the facts which are used in this calculation are:

(a) $X^*(\overline{u}) = \widetilde{X}^*(\widetilde{u}) \in F^*$,
(b) $\overline{u}(g^{[\boldsymbol{g}]}) = 0$,
(c) $\overline{u} = \widetilde{u}$ as elements of G^*,
(d) $\omega_G = \omega_{\widetilde{G}} \wedge g^{[\boldsymbol{g}]}$, $\omega_{G^*} = \gamma^{[\boldsymbol{g}]} \wedge \omega_{\widetilde{G}^*}$,
(e) if δ_q is in $\bigwedge^q \widetilde{G}^*$, then $[X(v)](\delta_q) = [\widetilde{X}(v)](\delta_q)$ and $\delta_q(\omega_G) = \delta_q(\omega_{\widetilde{G}}) \wedge g^{[\boldsymbol{g}]}$, and
(f) $\overline{\sigma}_z(p,q,r,t) = (-1)^{f+q+1}\widetilde{\sigma}_z(p,q-1,r+1,t) = -\widetilde{\sigma}_{z-1}(p,q,r,t)$.

Now that we have established that

$$0 \to \widetilde{\mathbb{I}}^{(z)} \xrightarrow{\Phi} \overline{\mathbb{I}}^{(z)} \xrightarrow{\Psi} \widetilde{\mathbb{I}}^{(z-1)}[-1] \to 0$$

is a short exact sequence of complexes, we consider the induced long exact sequence

$$\cdots \to H_1(\widetilde{\mathbb{I}}^{(z)}) \to H_1(\overline{\mathbb{I}}^{(z)}) \to H_0(\widetilde{\mathbb{I}}^{(z-1)}) \xrightarrow{\partial} H_0(\widetilde{\mathbb{I}}^{(z)}) \xrightarrow{\Phi_*} H_0(\overline{\mathbb{I}}^{(z)}) \to 0$$

of homology for non-negative z. Use Corollary 3.5 to evaluate H_0. It is clear that $\Phi_*\colon S_z\widetilde{N} \to S_z\overline{N}$ is the natural quotient map. For positive z, we use the snake lemma to verify that the connecting homomorphism $\partial\colon S_{z-1}\widetilde{N} \to S_z\widetilde{N}$ is multiplication by the element $X^*(\gamma^{[\boldsymbol{g}]})$ of the symmetric algebra $S_\bullet^{R/\widetilde{K}}(\widetilde{N})$. $\square$

4. The complex $\mathbb{M}^{(z)}$.

In this section we split off a split exact summand $\mathbb{N}^{(z)}$ from the complex $\mathbb{I}^{(z)}$ of section 2. The resulting quotient is isomorphic to the complex we call $(\mathbb{M}^{(z)}, m)$. The module structure of $\mathbb{M}^{(z)}$ is given in Theorem 4.5. The differential m is given in Theorem 4.8. The theorems are proved in section 7. The present section concludes with numerical information about, and examples of, the complexes $\mathbb{M}^{(z)}$.

Definition 4.1. Adopt Data 1.2. For all integers p, q, r, and z, let

$$\begin{aligned}\mathbb{V}(p,q,r,z) &= \mathcal{M}(p,q,r)[F^*] \otimes \textstyle\bigwedge^{\boldsymbol{g}-z+1+p} G, \text{ and}\\ \mathbb{S}(p,q,r,z) &= \mathcal{K}(p,q,r)[F] \otimes \textstyle\bigwedge^{p+\boldsymbol{f}+z+1} G^*,\end{aligned}$$

where the functors $\mathcal{M}$ and $\mathcal{K}$ are defined in Definitions 5.2 and 5.4.

Definition 4.2. Adopt the notation of Definitions 4.1 and 2.1. For each integer z, define the graded $R-$module $\mathbb{M}^{(z)}$ by

$$\mathbb{M}^{(z)} = \bigoplus_{S_{\mathbb{S}}^{(z)}} \mathbb{S}(p,q,r,z) \oplus \bigoplus_{S_{\mathbb{T}}^{(z)}} \mathbb{T}(p,q,r) \oplus \bigoplus_{S_{\mathbb{U}}^{(z)}} \mathbb{U}(p,q,r) \oplus \bigoplus_{S_{\mathbb{V}}^{(z)}} \mathbb{V}(p,q,r,z),$$

where

$$\begin{aligned}S_{\mathbb{S}}^{(z)} &= \{(p,q,r) \mid 0 \le p \le \boldsymbol{g}-\boldsymbol{f}-z-1, \quad 1 \le q \le \boldsymbol{f}, \quad 1 \le r \le \boldsymbol{f}\},\\ S_{\mathbb{T}}^{(z)} &= \{(p,q,r) \mid 0 \le p, \quad \boldsymbol{g}-\boldsymbol{f}-z \le q, \quad \boldsymbol{f}-\boldsymbol{g}+z \le r, \quad p+q+r \le \boldsymbol{f}-1\},\\ S_{\mathbb{U}}^{(z)} &= \{(p,q,r) \mid 0 \le p, \quad z \le q, \quad 0 \le r, \quad p+q+r \le \boldsymbol{f}-1+z\},\\ S_{\mathbb{V}}^{(z)} &= \{(p,q,r) \mid 0 \le p \le z-1, \quad 1 \le q \le \boldsymbol{f}, \quad 1 \le r \le \boldsymbol{f}\},\end{aligned}$$

(a) the position of $\mathbb{V}(p,q,r,z)$ in $\mathbb{M}^{(z)}$ is $\boldsymbol{f}+z-2-p+q-r$,
(b) the position of $\mathbb{U}(p,q,r)$ in $\mathbb{M}^{(z)}$ is $p+q+2r$,
(c) the position of $\mathbb{T}(p,q,r)$ in $\mathbb{M}^{(z)}$ is $p+q+2r+\boldsymbol{g}-\boldsymbol{f}+1$, and
(d) the position of $\mathbb{S}(p,q,r,z)$ in $\mathbb{M}^{(z)}$ is $\boldsymbol{f}+z+1+p-q+r$.

Observation 4.3. *The graded modules*

$$\mathbb{M}^{(z)} \quad \text{and} \quad \left(\mathbb{M}^{(\boldsymbol{g}-\boldsymbol{f}-z)}\right)^*[-(\boldsymbol{g}+\boldsymbol{f}-1)],$$

of Definition 4.2, are isomorphic for all integers z.

Proof. It is clear that the sets $S_{\mathbb{S}}^{(z)}$ and $S_{\mathbb{V}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}$ are equal. Also,

$$(p,q,r)\in S_{\mathbb{T}}^{(z)} \iff (p,q,\boldsymbol{f}-1-p-q-r)\in S_{\mathbb{U}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}.$$

The ideas of Propositions 2.12 and 5.5 produce isomorphisms

$$\begin{aligned}
&\Big[\bigoplus_{S_{\mathbb{U}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}} \mathbb{U}(p,q,r)\Big]^*[-(\boldsymbol{g}+\boldsymbol{f}-1)] \cong \bigoplus_{S_{\mathbb{T}}^{(z)}} \mathbb{T}(p,q,r) \text{ and}\\
&\Big[\bigoplus_{S_{\mathbb{V}}^{(\boldsymbol{g}-\boldsymbol{f}-z)}} \mathbb{V}(p,q,r,\boldsymbol{g}-\boldsymbol{f}-z)\Big]^*[-(\boldsymbol{g}+\boldsymbol{f}-1)] \cong \bigoplus_{S_{\mathbb{S}}^{(z)}} \mathbb{S}(p,q,r,z)
\end{aligned} \tag{4.4}$$

of graded $R-$modules. $\square$

Theorem 4.5. *Adopt Data 1.2. For each integer z, let $\mathbb{I}^{(z)}$ be the complex of Definition 2.3 and $\mathbb{M}^{(z)}$ be the graded $R-$module of Definition 4.2. Then there exists a split exact subcomplex $\mathbb{N}^{(z)}$ of $\mathbb{I}^{(z)}$ such that the graded modules $\mathbb{I}^{(z)}/\mathbb{N}^{(z)}$ and $\mathbb{M}^{(z)}$ are isomorphic.*

Before we are able to describe the differential m of $\mathbb{M}^{(z)}$ in Theorem 4.8 we must identify graded-module maps $\sigma\colon \mathbb{M}^{(z)}\to\mathbb{I}^{(z)}$ and $\tau\colon \mathbb{I}^{(z)}\to\mathbb{M}^{(z)}$. This project is accomplished in the next two definitions.

Definition 4.6. Define

$$\begin{aligned}
&\text{quot}\colon S_pF^*\otimes \textstyle\bigwedge^q F^*\otimes\bigwedge^r F^*\otimes\bigwedge^{\boldsymbol{g}-z+1+p} G\to\mathbb{V}(p,q,r,z),\\
&s\colon \mathbb{V}(p,q,r,z)\to S_pF^*\otimes \textstyle\bigwedge^q F^*\otimes\bigwedge^r F^*\otimes\bigwedge^{\boldsymbol{g}-z+1+p} G,\\
&\text{incl}\colon \mathbb{S}(p,q,r,z)\to D_pF\otimes \textstyle\bigwedge^q F\otimes\bigwedge^r F\otimes\bigwedge^{p+\boldsymbol{f}+z+1} G^*, \text{ and}\\
&t\colon D_pF\otimes \textstyle\bigwedge^q F\otimes\bigwedge^r F\otimes\bigwedge^{p+\boldsymbol{f}+z+1} G^*\to\mathbb{S}(p,q,r,z)
\end{aligned}$$

by: "quot" is the natural quotient map, s is a fixed splitting of "quot", "incl" is the natural inclusion map, and t is a fixed splitting of "incl".

Note. Theorem 5.11 and Proposition 5.5 ensure the existence of s and t.

Definition 4.7. Define the map of graded modules $\sigma\colon \mathbb{M}^{(z)} \to \mathbb{I}^{(z)}$ by

$$\begin{aligned}
&\mathbb{V}(p,q,r,z) \xrightarrow{s} S_pF^* \otimes \textstyle\bigwedge^q F^* \otimes \bigwedge^r F^* \otimes \bigwedge^{\boldsymbol{g}-z+1+p} G\\
&\qquad \xrightarrow{\text{nat}} \mathbb{L}(p,q,\boldsymbol{f}-r,\boldsymbol{g}-z+1+p,z-1-p) \hookrightarrow \mathbb{I}^{(z)},\\
&\mathbb{U}(p,q,r) \hookrightarrow \mathbb{I}^{(z)}, \qquad \mathbb{T}(p,q,r) \hookrightarrow \mathbb{I}^{(z)}, \quad \text{and}\\
&\mathbb{S}(p,q,r,z) \xrightarrow{\text{incl}} D_pF \otimes \textstyle\bigwedge^q F \otimes \bigwedge^r F \otimes \bigwedge^{p+\boldsymbol{f}+z+1} G^*\\
&\qquad \xrightarrow{\text{nat}} \mathbb{W}(p,q,\boldsymbol{f}-r,p+\boldsymbol{f}+z+1,r-p-q) \hookrightarrow \mathbb{I}^{(z)}.
\end{aligned}$$

Define the map of graded modules $\tau\colon \mathbb{I}^{(z)} \to \mathbb{M}^{(z)}$ by

$$\begin{aligned}
&\mathbb{L}(p,q,r,s,t) \xrightarrow{\chi(p+t=z-1)\cdot\chi(\boldsymbol{g}=s+t)\cdot\text{nat}} S_pF^* \otimes \textstyle\bigwedge^q F^* \otimes \bigwedge^{\boldsymbol{f}-r} F^* \otimes \bigwedge^{\boldsymbol{g}-z+1+p} G\\
&\qquad \xrightarrow{\text{quot}} \mathbb{V}(p,q,\boldsymbol{f}-r,z) \hookrightarrow \mathbb{M}^{(z)},\\
&\mathbb{U}(p,q,r) \xrightarrow{\chi(z\le q)\cdot\text{id}} \mathbb{M}^{(z)}, \qquad \mathbb{T}(p,q,r) \xrightarrow{\chi(\boldsymbol{g}-\boldsymbol{f}-z\le q)\cdot\text{id}} \mathbb{M}^{(z)}, \quad \text{and}\\
&\mathbb{W}(p,q,r,s,t) \xrightarrow{\chi(2\boldsymbol{f}+z+1=q+r+s+t)\cdot\chi(p+q+r+t=\boldsymbol{f})\cdot\text{nat}}\\
&\qquad D_pF \otimes \textstyle\bigwedge^q F \otimes \bigwedge^{\boldsymbol{f}-r} F \otimes \bigwedge^s G^* \xrightarrow{t} \mathbb{S}(p,q,\boldsymbol{f}-r,z) \hookrightarrow \mathbb{M}^{(z)}.
\end{aligned}$$

Note. The map "nat" is the natural isomorphism which is induced by $\alpha_r \mapsto \alpha_r(\omega_F)$ for $\alpha_r \in \bigwedge^r F^*$ and $b_r \mapsto b_r(\omega_{F^*})$ for $b_r \in \bigwedge^r F$.

Theorem 4.8. *There exists an $R-$module homomorphism $P\colon \mathbb{I}^{(z)} \to \mathbb{I}^{(z)}$, with $P(\mathbb{I}^{(z)}_i) \subseteq \mathbb{I}^{(z)}_{i+1}$, such that the following statements hold.*

(a) *The complex $\mathbb{I}^{(z)}/\mathbb{N}^{(z)}$ of Theorem 4.5 is isomorphic to $(\mathbb{M}^{(z)}, m)$, where the differential $m\colon \mathbb{M}^{(z)} \to \mathbb{M}^{(z)}$ is the composition*

$$\mathbb{M}^{(z)} \xrightarrow{\sigma} \mathbb{I}^{(z)} \xrightarrow{d} \mathbb{I}^{(z)} \xrightarrow{1-d\circ P} \mathbb{I}^{(z)} \xrightarrow{\tau} \mathbb{M}^{(z)}.$$

(b) *If $\psi\colon \mathbb{I}^{(z)} \to \mathbb{M}^{(z)}$ is given by $\psi = \tau\circ(1-d\circ P)$, then ψ is a map of complexes.*
(c) *If $\rho\colon \mathbb{M}^{(z)} \to \mathbb{I}^{(z)}$ is given by $\rho = (1-P\circ d)\circ\sigma$, then ρ is a map of complexes and $\psi\circ\rho$ is the identity map on $\mathbb{M}^{(z)}$.*
(d) *The image of m is contained in $[I_1(u)+I_1(v)+I_1(X)]\,\mathbb{M}^{(z)}$.*

Remark 4.9. If $\mathbb{I}^{(z)}$ is a homogeneous complex, in the sense of Remark 2.4, then the map $1-d\circ P$ of Theorem 4.8 is a homogeneous map of degree zero. If $0 \le z$, then $\mathbb{M}^{(z)}$ is a homogeneous complex with degree zero maps, provided

(a) the shift of $\mathbb{V}(p,q,r,z)$ is $-p+q-2r+2\boldsymbol{f}-2+z$,
(b) the shift of $\mathbb{U}(p,q,r)$ is $2p+2q+3r-z$,
(c) the shift of $\mathbb{T}(p,q,r)$ is $p+q+3r+2\boldsymbol{g}-\boldsymbol{f}-z$, and
(d) the shift of $\mathbb{S}(p,q,r,z)$ is $p-q+2r+2\boldsymbol{f}-1+z$.

If $z=-1$, then the appropriate grading on $\mathbb{M}^{(z)}$ is obtained by subtracting 1 from each shift in (a)–(d).

Next we record numerical information about the complex $\mathbb{M}^{(z)}$. The shifts in $\mathbb{M}^{(z)}$ are given in Remark 4.9; the rank of $\mathbb{V}(p,q,r,z)$ and $\mathbb{S}(p,q,r,z)$ may be computed using the note which follows Theorem 5.11. In Proposition 4.10 we record the first and last contributions of $\mathbb{S}$, $\mathbb{T}$, $\mathbb{U}$, and $\mathbb{V}$, respectively, to $\mathbb{M}^{(z)}$. Corollary 4.11 gives the left and right ends of $\mathbb{M}^{(z)}$. We let $\mathbb{M}_i^{(z)}$ denote the submodule of $\mathbb{M}^{(z)}$ in position i.

Proposition 4.10. *Adopt Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be positive integers with $\boldsymbol{f}-1\le\boldsymbol{g}$.*

(**a**) *If $\mathbb{V}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_i^{(z)}$, then* $0\le i\le\min\{2\boldsymbol{f}+z-3,\boldsymbol{g}+2\boldsymbol{f}-2\}$.

(**b**) *If $\mathbb{V}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_0^{(z)}$, then $(p,q,r)=(z-1,1,\boldsymbol{f})$.*

(**c**) *If $\mathbb{V}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_1^{(z)}$, then (p,q,r) is equal to $(z-2,1,\boldsymbol{f})$, $(z-1,2,\boldsymbol{f})$, or $(z-1,1,\boldsymbol{f}-1)$.*

(**d**) *If $\mathbb{V}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_{2\boldsymbol{f}+z-3}^{(z)}$, then $(p,q,r)=(0,\boldsymbol{f},1)$.*

(**e**) *If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{M}_i^{(z)}$, then* $z\le i\le\min\{2\boldsymbol{f}+z-2,2\boldsymbol{f}+\boldsymbol{g}-2\}$.

(**f**) *If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{M}_z^{(z)}$, then $(p,q,r)=(0,z,0)$.*

(**g**) *If $\mathbb{U}(p,q,r)$ is a non-zero summand of $\mathbb{M}_{2\boldsymbol{f}-2+z}^{(z)}$, then $(p,q,r)=(0,z,\boldsymbol{f}-1)$.*

(**h**) *If $\mathbb{T}(p,q,r)$ is a non-zero summand of $\mathbb{M}_i^{(z)}$, then* $z+1\le i\le\min\{2\boldsymbol{f}+z-1,\boldsymbol{f}+\boldsymbol{g}-1\}$.

(**i**) *If $\mathbb{T}(p,q,r)$ is a non-zero summand of $\mathbb{M}_{z+1}^{(z)}$, then (p,q,r) is equal to $(0,\boldsymbol{g}-\boldsymbol{f}-z,\boldsymbol{f}-\boldsymbol{g}+z)$.*

(**j**) *If $\mathbb{T}(p,q,r)$ is a non-zero summand of $\mathbb{M}_{2\boldsymbol{f}+z-1}^{(z)}$, then $z\le\boldsymbol{g}-\boldsymbol{f}$ and (p,q,r) is equal to $(0,\boldsymbol{g}-\boldsymbol{f}-z,2\boldsymbol{f}-\boldsymbol{g}+z-1)$.*

(**k**) *If $\mathbb{T}(p,q,r)$ is a non-zero summand of $\mathbb{M}_{\boldsymbol{f}+\boldsymbol{g}-1}^{(z)}$, then $\boldsymbol{g}-\boldsymbol{f}\le z$ and (p,q,r) is equal to $(0,0,\boldsymbol{f}-1)$.*

(**l**) *If $\boldsymbol{g}<\boldsymbol{f}+z+1$, then the summand $\bigoplus\mathbb{S}(p,q,r,z)$ in $\mathbb{M}^{(z)}$ is zero.*

(**m**) *If $\mathbb{S}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_i^{(z)}$, then $z+2\le i\le\boldsymbol{f}+\boldsymbol{g}-1$.*

(**n**) *If $\mathbb{S}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_{z+2}^{(z)}$, then $z\le\boldsymbol{g}-\boldsymbol{f}-1$ and (p,q,r) is equal to $(0,\boldsymbol{f},1)$.*

(**o**) *If $\mathbb{S}(p,q,r,z)$ is a non-zero summand of $\mathbb{M}_{\boldsymbol{f}+\boldsymbol{g}-1}^{(z)}$, then $z\le\boldsymbol{g}-\boldsymbol{f}-1$ and (p,q,r) is equal to $(\boldsymbol{g}-\boldsymbol{f}-z-1,1,\boldsymbol{f})$.*

Proof. The position of $\mathbb{V}(p,q,r,z)$ in $\mathbb{M}^{(z)}$ is $(z-1-p)+(q-1)+(\boldsymbol{f}-r)$, with

$$0\le z-1-p\le\min\{\boldsymbol{g},z-1\},\quad 0\le q-1\le\boldsymbol{f}-1,\quad\text{and}\quad 0\le\boldsymbol{f}-r\le\boldsymbol{f}-1.$$

Assertions (a)–(d) are now obvious. Definition 4.2 also gives that

$$\begin{aligned} z\le\ \text{the position of }\mathbb{U}(p,q,r)\ &=p+q+2r=2(p+q+r)-p-q\\ &\le 2(\boldsymbol{f}-1+z)-z=2\boldsymbol{f}-2+z.\end{aligned}$$

Also, $p+q+2r \le 2(p+r)+q \le 2(\boldsymbol{f}-1)+\boldsymbol{g}$. Assertions (e)–(g) follow. Use

$$\begin{aligned} z+1 = (\boldsymbol{g}-\boldsymbol{f}-z)+2(\boldsymbol{f}-\boldsymbol{g}+z)+\boldsymbol{g}-\boldsymbol{f}+1 &\le \text{ the position of } \mathbb{T}(p,q,r) \\ = p+q+2r+\boldsymbol{g}-\boldsymbol{f}+1 &= 2(p+q+r)-p-q+\boldsymbol{g}-\boldsymbol{f}+1 \\ \le 2(\boldsymbol{f}-1)+\min\{\boldsymbol{f}+z-\boldsymbol{g},0\}+(\boldsymbol{g}-\boldsymbol{f}+1) &= \min\{2\boldsymbol{f}-1+z, \boldsymbol{g}+\boldsymbol{f}-1\} \end{aligned}$$

for (h)–(k). The same type of argument also establishes (l)–(o). $\square$

Corollary 4.11. *Let $\boldsymbol{f}$ and $\boldsymbol{g}$ be integers with $2 \le \boldsymbol{f}$ and $\boldsymbol{f}-1 \le \boldsymbol{g}$.*

(**a**) *If $i \le -1$ and $-1 \le z$, then $\mathbb{M}_i^{(z)} = 0$.*

(**b**) *The module $\mathbb{M}_0^{(z)}$ is equal to*

$$\begin{cases} \mathbb{U}(0,0,0) \oplus \mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f}+1,\boldsymbol{f}-\boldsymbol{g}-1) \cong R \oplus \bigwedge^{\boldsymbol{f}-1} G^* & \text{if } -1 = z, \\ \mathbb{U}(0,0,0) \cong R & \text{if } 0 = z, \\ \mathbb{V}(z-1,1,\boldsymbol{f},z) \cong S_z F^* & \text{if } 1 \le z. \end{cases}$$

If the hypotheses of Remark 2.4 are in effect, then

$$\mathbb{M}_0^{(z)} = \begin{cases} R[0] \oplus R[-(f-2)]^{\binom{\boldsymbol{g}}{\boldsymbol{f}-1}} & \text{if } -1 = z, \\ R[0]^{\binom{\boldsymbol{f}+z-1}{z}} & \text{if } 0 \le z. \end{cases}$$

(**c**) *The module $\mathbb{M}_1^{(z)}$ is equal to*

$$\begin{cases} \left\{ \begin{matrix} \mathbb{U}(0,1,0) \oplus \mathbb{U}(1,0,0) \oplus \mathbb{T}(1,\boldsymbol{g}-\boldsymbol{f}+1,\boldsymbol{f}-\boldsymbol{g}-1) \\ \oplus \mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f}+2,\boldsymbol{f}-\boldsymbol{g}-1) \oplus \mathbb{S}(0,\boldsymbol{f},1,z) \end{matrix} \right\} & \text{if } -1 = z \text{ and } 3 \le \boldsymbol{f}, \\ \mathbb{U}(0,1,0) \oplus \mathbb{U}(1,0,0) \oplus \mathbb{T}(0,\boldsymbol{g}-\boldsymbol{f},\boldsymbol{f}-\boldsymbol{g}) & \text{if } 0 = z, \\ \mathbb{V}(0,2,\boldsymbol{f},z) \oplus \mathbb{V}(0,1,\boldsymbol{f}-1,z) \oplus \mathbb{U}(0,1,0) & \text{if } z = 1, \\ \mathbb{V}(z-2,1,\boldsymbol{f},z) \oplus \mathbb{V}(z-1,2,\boldsymbol{f},z) \oplus \mathbb{V}(z-1,1,\boldsymbol{f}-1,z) & \text{if } 2 \le z. \end{cases}$$

(**d**) *If $-1 \le z \le \boldsymbol{g}-\boldsymbol{f}+1$, then $\mathbb{M}_i^{(z)} = 0$, whenever $\boldsymbol{g}+\boldsymbol{f} \le i$, and $\mathbb{M}_{\boldsymbol{g}+\boldsymbol{f}-1}^{(z)}$ is equal to*

$$\begin{cases} \mathbb{S}(\boldsymbol{g}-\boldsymbol{f}-z-1,1,\boldsymbol{f},z) \cong D_{\boldsymbol{g}-\boldsymbol{f}-z} F & \text{if } -1 \le z \le \boldsymbol{g}-\boldsymbol{f}-1, \\ \mathbb{T}(0,0,\boldsymbol{f}-1) \cong R & \text{if } \boldsymbol{g}-\boldsymbol{f} = z, \\ \mathbb{T}(0,0,\boldsymbol{f}-1) \oplus \mathbb{U}(0,\boldsymbol{g}-\boldsymbol{f}+1,\boldsymbol{f}-1) \cong R \oplus \bigwedge^{\boldsymbol{f}-1} G & \text{if } \boldsymbol{g}-\boldsymbol{f}+1 = z. \end{cases}$$

If the hypotheses of Remark 2.4 are in effect, then

$$\mathbb{M}_{\boldsymbol{g}+\boldsymbol{f}-1}^{(z)} = \begin{cases} R[-(\boldsymbol{g}+3\boldsymbol{f}-4)]^{\binom{\boldsymbol{g}}{\boldsymbol{f}-1}} & \text{if } -1 = z, \\ R[-(\boldsymbol{g}+3\boldsymbol{f}-3)]^{\binom{\boldsymbol{g}-z-1}{\boldsymbol{f}-1}} & \text{if } 0 \le z \le \boldsymbol{g}-\boldsymbol{f}, \\ R[-(\boldsymbol{g}+3\boldsymbol{f}-4)]^1 \oplus R[-(2\boldsymbol{f}+\boldsymbol{g}-2)]^{\binom{\boldsymbol{g}}{\boldsymbol{f}-1}} & \text{if } \boldsymbol{g}-\boldsymbol{f}+1 = z. \end{cases}$$

(**e**) *If $\boldsymbol{g}-\boldsymbol{f}+2 \le z \le \boldsymbol{g}$, then $\mathbb{M}_i^{(z)} = 0$ for $2\boldsymbol{f}+z-1 \le i$ and*

$$\mathbb{M}_{2\boldsymbol{f}+z-2}^{(z)} = \mathbb{U}(0,z,\boldsymbol{f}-1) \cong \bigwedge^z G^* \cong R[-(3\boldsymbol{f}-3+z)]^{\binom{\boldsymbol{g}}{z}}.$$

(**f**) *If $\boldsymbol{g}+1 \le z$, then $\mathbb{M}_i^{(z)} = 0$ for $\boldsymbol{g}+2\boldsymbol{f}-1 \le i$ and*

$$\mathbb{M}_{\boldsymbol{g}+2\boldsymbol{f}-2}^{(z)} = \mathbb{V}(z-1-\boldsymbol{g},\boldsymbol{f},1,z) \cong S_{z-\boldsymbol{g}} F^* \cong R[-(3\boldsymbol{f}+\boldsymbol{g}-3)]^{\binom{z-\boldsymbol{g}+\boldsymbol{f}-1}{\boldsymbol{f}-1}}.$$

Proof. Use Proposition 4.10, Example 5.3, and Remark 4.9. $\square$

Example 4.12. Fix $\boldsymbol{f} = 3$ and $\boldsymbol{g} = 6$. We record the graded module $\mathbb{M}^{(z)}$ for $-1 \le z \le 6$. We use the notation of the command "numinfo" from the computer system Macaulay. In other words, $2\!:\!1\ \ 3\!:\!42\ \ 4\!:\!21$ in position 2 of $\mathbb{M}^{(1)}$ means that $\mathbb{M}^{(1)}_2$ is equal to $R(-2) \oplus R(-3)^{42} \oplus R(-4)^{21}$.

$\mathbb{M}^{(-1)}$ is

position	degrees		
0	0:1	1:15	
1	2:120		
2	3:315	4:75	
3	4:405	5:351	6:20
4	5:309	6:565	7:120
5	6:125	7:471	8:216
6	7:21	8:201	9:190
7	9:35	10:84	
8	11:15		

$\mathbb{M}^{(0)}$ is

position	degrees		
0	0:1		
1	2:9	3:20	
2	3:1	4:156	
3	5:276	6:65	
4	6:191	7:258	8:15
5	7:84	8:261	9:83
6	8:15	9:127	10:99
7	10:24	11:51	
8	12:10		

$\mathbb{M}^{(1)}$ is

position	degrees		
0	0:3		
1	1:9	2:8	
2	2:1	3:42	4:21
3	4:9	5:174	
4	6:210	7:33	
5	7:57	8:127	9:6
6	8:10	9:69	10:36
7	10:15	11:26	
8	12:6		

$\mathbb{M}^{(2)}$ is

position	degrees		
0	0:6		
1	1:26	2:15	
2	2:36	3:69	4:10
3	3:6	4:127	5:57
4	5:33	6:210	
5	7:174	8:9	
6	8:21	9:42	10:1
7	10:8	11:9	
8	12:3		

$\mathbb{M}^{(3)}$ is

position	degrees		
0	0:10		
1	1:51	2:24	
2	2:99	3:127	4:15
3	3:83	4:261	5:84
4	4:15	5:258	6:191
5	6:65	7:276	
6	8:156	9:1	
7	9:20	10:9	
8	12:1		

$\mathbb{M}^{(4)}$ is

position	degrees		
0	0:15		
1	1:84	2:35	
2	2:190	3:201	4:21
3	3:216	4:471	5:125
4	4:120	5:565	6:309
5	5:20	6:351	7:405
6	7:75	8:315	
7	9:120		
8	10:15	11:1	

$\mathbb{M}^{(5)}$ is

position	degrees		
0	0 : 21		
1	1 : 125	2 : 48	
2	2 : 309	3 : 291	4 : 28
3	3 : 405	4 : 737	5 : 174
4	4 : 295	5 : 999	6 : 456
5	5 : 111	6 : 765	7 : 650
6	6 : 15	7 : 314	8 : 540
7	8 : 51	9 : 261	
8	10 : 65		
9	11 : 6		

$\mathbb{M}^{(6)}$ is

position	degrees		
0	0 : 28		
1	1 : 174	2 : 63	
2	2 : 456	3 : 397	4 : 36
3	3 : 650	4 : 1,059	5 : 231
4	4 : 540	5 : 1,545	6 : 631
5	5 : 258	6 : 1,325	7 : 951
6	6 : 64	7 : 663	8 : 855
7	7 : 6	8 : 177	9 : 461
8	9 : 19	10 : 141	
9	11 : 21		
10	12 : 1		

Example 4.13. Fix $\boldsymbol{f} = 6$ and $\boldsymbol{g} = 9$. We record the graded module $\mathbb{M}^{(0)}$.

pos.	degrees						
0	0 : 1						
1	2 : 15	6 : 84					
2	3 : 1	4 : 105	7 : 846				
3	5 : 15	6 : 455	8 : 3,591	9 : 624			
4	6 : 1	7 : 105	8 : 1,365	9 : 8,570	10 : 4,500	11 : 720	
5	8 : 15	9 : 455	10 : 15,828	11 : 14,061	12 : 5,349	13 : 540	
6	9 : 1	10 : 105	11 : 14,028	12 : 24,824	13 : 17,256	14 : 3,996	15 : 216
7	11 : 15	12 : 3,941	13 : 27,099	14 : 31,512	15 : 12,865	16 : 1,611	17 : 36
8	12 : 1	13 : 1,113	14 : 12,564	15 : 35,624	16 : 23,556	17 : 5,196	18 : 270
9	14 : 141	15 : 3,864	16 : 19,629	17 : 27,012	18 : 9,490	19 : 876	
10	15 : 1	16 : 504	17 : 6,264	18 : 16,094	19 : 10,956	20 : 1,596	
11	18 : 840	19 : 5,265	20 : 6,816	21 : 1,849			
12	20 : 720	21 : 2,270	22 : 1,182				
13	22 : 315	23 : 399					
14	24 : 56						

5. The functor $\mathcal{M}(p, q, r)$.

In this section we introduce a family of functors $\{\mathcal{M}(p, q, r)\}$, which are universally free in the sense of [1, Def. 2.1]. Given the data of 1.2, many of the summands of the minimal complex $\mathbb{M}^{(z)}$ of section 4 have the form $\mathcal{M}(p, q, r)[F^*] \otimes \bigwedge^s G$, for some integers p, q, r, and s.

Data 5.1. Let F be a free module of rank $\boldsymbol{f}$ over the commutative noetherian ring R, and let $B(p, q, r)$ be the $R-$module $S_pF \otimes \bigwedge^q F \otimes \bigwedge^r F$.

Definition 5.2. Adopt Data 5.1. For all integers p, q, and r, define

$$\mathcal{M}(p,q,r)[F] = \begin{cases} \dfrac{B(p,q,r)}{\operatorname{im}(\partial_h + \partial_v)} & \text{if } 1 \le p, \\[2ex] \dfrac{B(p,q,r)}{\operatorname{im}\Delta} & \text{if } 0 = p, \\[2ex] 0 & \text{if } p < 0, \text{ where} \end{cases}$$

$$\partial_h \colon B(p-1,q+1,r) \to B(p,q,r) \quad \text{and} \quad \partial_v \colon B(p-1,q,r+1) \to B(p,q,r)$$

are the Koszul maps, in the sense of (1.15), induced by the identity map on F, and

$$\Delta \colon \bigwedge^{q+r} F \to \bigwedge^q F \otimes \bigwedge^r F = B(0,q,r)$$

is the co-multiplication map.

Example 5.3. We use the Schur functor notation of [2, Def. II.1.3] or [3, pg. 466]. The hook $\lambda = (q, 1^p)$ represents the partition

$$\lambda = (q, \underbrace{1, \dots, 1}_{p \text{ times}}).$$

Apply Remark 1.16 to see that

(a) if $q \le 0$ or $r \le 0$, then $\mathcal{M}(p,q,r)[F] = 0$,
(b) if $0 \le p$, then $\mathcal{M}(p,1,\boldsymbol{f})[F] \cong \mathcal{M}(p,\boldsymbol{f},1)[F] \cong S_{p+1}F$,
(c) if $0 \le p$, then $\mathcal{M}(p,\boldsymbol{f},\boldsymbol{f})[F] \cong S_p F$,
(d) if $1 \le p, q$, then $\mathcal{M}(p,q,\boldsymbol{f})[F]$ and $\mathcal{M}(p,\boldsymbol{f},q)[F]$ are both isomorphic to the Schur functor $L_{(q,1^p)}F$.
(e) if $1 \le q$, then $\mathcal{M}(0,q,1)[F]$ and $\mathcal{M}(0,1,q)[F]$ are both isomorphic to the Schur functor $L_{(q,1)}F$.

Definition 5.4. Let F be a free module of rank $\boldsymbol{f}$ over the commutative noetherian ring R, and let $B'(p,q,r)$ be the $R-$module $D_pF \otimes \bigwedge^q F \otimes \bigwedge^r F$. For all integers p, q, and r, define $\mathcal{K}(p,q,r)[F]$ to be

$$\begin{cases} \operatorname{Ker}\left(B'(p,q,r) \xrightarrow{\begin{bmatrix}\delta_h \\ \delta_v\end{bmatrix}} B'(p-1,q+1,r) \oplus B'(p-1,q,r+1) \right) & \text{if } 1 \le p, \\ \operatorname{Ker}\left(B'(0,q,r) \xrightarrow{\mu} \bigwedge^{q+r} F \right) & \text{if } 0 = p, \\ 0 & \text{if } p < 0, \end{cases}$$

where

$$\delta_h(B_p \otimes b_q \otimes b'_r) = \sum_{|I|=1} \varphi_I(B_p) \otimes b_q \wedge f_I \otimes b'_r,$$

$$\delta_v(B_p \otimes b_q \otimes b'_r) = \sum_{|I|=1} \varphi_I(B_p) \otimes b_q \otimes b'_r \wedge f_I,$$

and μ is exterior multiplication.

The following observation is an immediate consequence of the definitions.

Proposition 5.5. *The modules $\mathcal{K}(p,q,r)[F]$ and $(\mathcal{M}(p,q,r)[F^*])^*$ are naturally isomorphic for all free $R-$modules F.*

The main results in this section are Theorem 5.7, where we resolve $\mathcal{M}(p,q,r)[F]$, and Theorem 5.11, where we prove that $\mathcal{M}(p,q,r)[F]$ is a free $R-$module.

Definition 5.6. Adopt Data 5.1. For non-negative integers p, q, and r, define $\mathbb{B}(p,q,r) = \mathbb{B}(p,q,r)(F)$ to be the total complex of the following (extended) double complex:

$$\begin{tikzcd}
\bigwedge^{p+q+r} F \arrow[r, "\pm\Delta"] \arrow[d, "\pm\Delta"'] \arrow[dr] & {} & & B(0,q,p+r) \arrow[d, "\partial_v"'] \\
{} & {} & \vdots \arrow[d, "\partial_v"'] & \vdots \arrow[d, "\partial_v"'] \\
 & \cdots \arrow[r, "\partial_h"] & B(p-2,q+1,r+1) \arrow[r, "\partial_h"] \arrow[d, "\partial_v"'] & B(p-1,q,r+1) \arrow[d, "\partial_v"'] \\
B(0,p+q,r) \arrow[r, "\partial_h"] & \cdots \arrow[r, "\partial_h"] & B(p-1,q+1,r) \arrow[r, "\partial_h"] & B(p,q,r).
\end{tikzcd}$$

The module $B(a,b,c)$ has position $b+c$ in $\mathbb{B}(p,q,r)$, the module $\bigwedge^{p+q+r} F$ has position $p+q+r+1$. If the module $\bigwedge^{p+q+r} F$ is ignored, for the time being, then the rest of the diagram is a double complex. The horizontal map

$$\partial_h \colon B(a,b,c) \to B(a+1,b-1,c)$$

is the Koszul complex map associated to the identity map on F. The vertical map map $\partial_v \colon B(a,b,c) \to B(a+1,b,c-1)$ is $(-1)^b$ times the Koszul complex map associated to the identity map on F. The module $\bigwedge^{p+q+r} F$ maps to each module of the form $B(0,b,c)$, where $b+c = p+q+r$ and $q \le b \le p+q$. The map

$$\bigwedge^{p+q+r} F \to B(0,b,c)$$

is equal to $(-1)^b$ times the co-multiplication map $\bigwedge^{p+q+r} F \to \bigwedge^b F \otimes \bigwedge^c F$.

Theorem 5.7. *Adopt Data 5.1. If p, q, and r are non-negative integers, then the complex $\mathbb{B}(p,q,r)$ of Definition 5.6 is a resolution of $\mathcal{M}(p,q,r)[F]$.*

Proof. It is clear that $H_{q+r}(\mathbb{B}(p,q,r)) = \mathcal{M}(p,q,r)[F]$. We prove that $H_i(\mathbb{B}(p,q,r))$ is zero for $q+r+1 \le i$. Let $\mathbb{B}'(p,q,r)$ be the subcomplex

$$\mathbb{B}'(p,q,r) = \bigoplus_{\substack{a+b+c=p+q+r \\ b \le q-1 \text{ or } c \le r-1}} B(a,b,c)$$

of $\mathbb{B}(p+q+r,0,0)$. Observe that

$$0 \to \mathbb{B}'(p,q,r) \to \mathbb{B}(p+q+r,0,0) \to \mathbb{B}(p,q,r) \to 0$$

is a short exact sequence of complexes. The long exact sequence of homology completes the proof as soon as we show

(a) $\mathbb{B}(M,0,0)$ is split exact for all integers M, and

(b) $H_i(\mathbb{B}'(p,q,r)) = 0$, whenever $q+r \le i$.

We first prove (a). Let $\mathbb{K}$ be the Koszul complex associated to the $R-$module map

$$F \oplus F \xrightarrow{\begin{bmatrix}\mathrm{id} & 0\end{bmatrix}} F.$$

It is well known that the homology of $\mathbb{K}$ is given by $H_i(\mathbb{K}) = \bigwedge^i F$ for all integers i. In particular, the non-zero homology of the graded strand

$$\mathbb{K}(M): \quad 0 \to S_0F \otimes \bigwedge^M(F \oplus F) \to \cdots \to S_MF \otimes \bigwedge^0(F \oplus F) \to 0$$

of $\mathbb{K}$, is concentrated in position M. We kill the cycles of $\mathbb{K}(M)$; thereby creating the split exact complex $\widetilde{\mathbb{K}}(M)$:

$$0 \to \bigwedge^M F \xrightarrow{\delta} S_0F \otimes \bigwedge^M(F \oplus F) \to \cdots \to S_MF \otimes \bigwedge^0(F \oplus F) \to 0,$$

where δ sends $\bigwedge^M F$ onto the summand $\bigwedge^0 F \otimes \bigwedge^M F$ of

$$\bigwedge^M(F \oplus F) = \sum_{\ell=0}^{M} \bigwedge^\ell F \otimes \bigwedge^{M-\ell} F.$$

It is easy to see that the commutative diagram

$$\begin{array}{ccc} F \oplus F & \xrightarrow{\begin{bmatrix}\mathrm{id} & 0\end{bmatrix}} & F \\ {\scriptstyle \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}}\Big\downarrow & & {\scriptstyle \mathrm{id}}\Big\downarrow \\ F \oplus F & \xrightarrow{\begin{bmatrix}\mathrm{id} & \mathrm{id}\end{bmatrix}} & F \end{array}$$

induces an isomorphism from $\widetilde{\mathbb{K}}(M)$ to $\mathbb{B}(M, 0, 0)$.

Now we prove (b). Fix an integer i, with $q + r \le i$. Let $M = p + q + r$, and let z be an $i-$cycle in $\mathbb{B}'(p, q, r)$. Decompose z as

$$z = \sum_{k=0}^{q-1} x_k + \sum_{k=i-r+1}^{i} x_k$$

where $x_k \in B(M - i, k, i - k)$. Suppose we have found $y \in \mathbb{B}'(p, q, r)$ such that

$$z - d(y) = \sum_{k=0}^{N} x'_k + \sum_{k=i-r+1}^{i} x_k$$

for some $N \le q - 1$, where d is the differential in $\mathbb{B}'(p, q, r)$. Apply d to the cycle $z - d(y)$ in order to see that

$$\partial_v(x'_N) = 0 \in B(M - i + 1, N, i - N - 1).$$

(This is the key point in the argument. It is essential to notice that $\partial_h(x_{i-r+1})$ is not an element of $B(M - i + 1, N, i - N - 1)$, because $r - 1 < i - N - 1$.) Recall, from Remark 1.16, that the vertical maps

$$B(M - i - 1, N, i - N + 1) \xrightarrow{\partial_v} B(M - i, N, i - N) \xrightarrow{\partial_v} B(M - i + 1, N, i - N - 1)$$

form an exact complex. Thus, there is an element

$$y' \in B(M-i-1, N, i-N+1) \subseteq \mathbb{B}'(p,q,r)$$

such that $\partial_v(y') = x'_N$. We now have

$$z - d(y+y') = \sum_{k=0}^{N-1} x''_k + \sum_{k=i-r+1}^{i} x_k$$

for some $x''_k \in B(M-i, k, i-k)$. Repeat the procedure to see that $z = d(Y)$ for some $Y \in \mathbb{B}'(p,q,r)$. $\square$

Our first goal in this section has been accomplished. We complete the section by finding a basis for $\mathcal{M}(p,q,r)[F]$, thereby ensuring that $\mathcal{M}(p,q,r)[F]$ is a free module. Some notation is needed.

Remark 5.8. Let $f^{[1]}, \dots, f^{[\boldsymbol{f}]}$ be the basis for F of Convention 1.4. If $a_1, \dots, a_p$, $b_1, \dots, b_q, c_1, \dots, c_r$ are integers, between 1 and $\boldsymbol{f}$, then we let

$$a_1 \cdots a_p \otimes b_1 \wedge \dots \wedge b_q \otimes c_1 \wedge \dots \wedge c_r$$

represent the element

$$f^{[a_1]} \dots f^{[a_p]} \otimes f^{[b_1]} \wedge \dots \wedge f^{[b_q]} \otimes f^{[c_1]} \wedge \dots \wedge f^{[c_r]}$$

of $B(p,q,r)$. Whenever possible, we insist that

$$a_1 \le \cdots \le a_p, \quad b_1 < \cdots < b_q, \quad \text{and} \quad c_1 < \cdots < c_r. \tag{5.9}$$

Definition 5.10. If p, q, r, and $\boldsymbol{f}$ are integers, then define

$$R(p,q,r,\boldsymbol{f}) = \sum_{i=1}^{r} \sum_{k=1}^{\boldsymbol{f}} \sum_{\ell=1}^{k} \binom{p+\ell-1}{p} \binom{\ell-1}{i-1} \binom{k-1}{q-1} \binom{\boldsymbol{f}-k}{r-i}.$$

Theorem 5.11. *For all integers p, q, and r, the $R-$module $\mathcal{M}(p,q,r)[F]$ of Definition 5.2 is free of rank $R(p,q,r,\boldsymbol{f})$.*

Note. As soon as the proof of Theorem 5.11 (including the proof of Theorem 6.7) is complete, then Theorem 5.7 shows that if p, q, r, and $\boldsymbol{f}$ are non-negative integers, then $R(p,q,r,\boldsymbol{f})$ is equal to

$$\sum_{\{(a,b) \mid 0 \le a,b \text{ and } a+b \le p\}} (-1)^{a+b} \binom{\boldsymbol{f}-1+p-a-b}{\boldsymbol{f}-1} \binom{\boldsymbol{f}}{q+a} \binom{\boldsymbol{f}}{r+b} + (-1)^{p+1} \binom{\boldsymbol{f}}{p+q+r};$$

moreover, the rank of $\mathcal{M}(p,q,r)[F]$ is equal to this common value.

Proof of Theorem 5.11. If $p \le -1$, $q \le 0$, $r \le 0$, or $\boldsymbol{f} \le 0$, then the module $B(p,q,r)$ and the integer $R(p,q,r,\boldsymbol{f})$ are both equal to zero. Henceforth, we assume that $0 \le p$ and $1 \le q, r, \boldsymbol{f}$.

We next take $p=0$. Define $s\colon B(0,q,r)\to \bigwedge^{q+r} F$ by

$$s(b_1\wedge\ldots\wedge b_q\otimes c_1\wedge\ldots\wedge c_r)=\begin{cases}0 & \text{if } c_1\le b_q\\ f^{[b_1]}\wedge\ldots\wedge f^{[b_q]}\wedge f^{[c_1]}\wedge\ldots\wedge f^{[c_r]} & \text{if } b_q<c_1.\end{cases}$$

(The conventions of Remark 5.8, including the inequalities of (5.9), are in effect.) It is clear that $s\circ\Delta$ is equal to the identity on $\bigwedge^{q+r} F$. It follows that

$$B(0,q,r)=\operatorname{im}\Delta\oplus\ker s;$$

and therefore, $\mathcal{M}(0,q,r)[F]$ is isomorphic to $\ker s$. Pick a basis $\mathcal{B}_1$ for $\bigwedge^{q+r} F$. Let

$$S_1=\{\Delta(x_1)\in B(0,q,r)\mid x_1\in\mathcal{B}_1\},\quad \text{and}$$

$$S_2=\{b_1\wedge\ldots\wedge b_q\otimes c_1\wedge\ldots\wedge c_r\in B(0,q,r)\mid (5.9)\text{ holds and } c_1\le b_q\}.$$

It is clear that $\ker s$ is generated by S_2 and that $B(0,q,r)$ is generated by $S_1\cup S_2$. Observe that

$$|S_1|+|S_2|=\binom{\boldsymbol{f}}{q+r}+\left[\binom{\boldsymbol{f}}{q}\binom{\boldsymbol{f}}{r}-\binom{\boldsymbol{f}}{q+r}\right]=\binom{\boldsymbol{f}}{q}\binom{\boldsymbol{f}}{r}=\operatorname{rank}B(0,q,r).$$

It follows that $S_1\cup S_2$ is a basis for $B(0,q,r)$; and therefore, $\ker s$ is a free module of rank $\binom{\boldsymbol{f}}{q}\binom{\boldsymbol{f}}{r}-\binom{\boldsymbol{f}}{q+r}$, which, according to Proposition 6.6, is the same as $R(0,q,r,\boldsymbol{f})$.

Henceforth, we take $1\le p$. The proof is by induction on p. The Koszul complex

$$0\to B(0,p+q,r)\to\ldots\xrightarrow{\partial_h}B(p-1,q+1,r)\xrightarrow{\partial_h}B(p,q,r)\to\cdots\to B(p+q,0,r)\to 0 \tag{5.12}$$

is split exact; hence, $B(p-1,q+1,r)/\operatorname{im}\partial_h$ is a free module of rank

$$\sum_{\ell=0}^{p-1}(-1)^\ell\binom{\boldsymbol{f}+p-\ell-2}{\boldsymbol{f}-1}\binom{\boldsymbol{f}}{q+1+\ell}\binom{\boldsymbol{f}}{r}. \tag{5.13}$$

The induction hypothesis ensures that $\mathcal{M}(p-1,q,r+1)[F]$ is a free module of rank

$$R(p-1,q,r+1,\boldsymbol{f}). \tag{5.14}$$

Pick bases

$$\mathcal{B}_1 \text{ for } \frac{B(p-1,q+1,r)}{\operatorname{im}\partial_h}\quad\text{and}\quad \mathcal{B}_2\text{ for }\mathcal{M}(p-1,q,r+1)[F].$$

Let

$$S_1=\{\partial_h(x_1)\in B(p,q,r)\mid x_1\in\mathcal{B}_1\},$$
$$S_2=\{\partial_v(x_2)\in B(p,q,r)\mid x_2\in\mathcal{B}_2\},\quad\text{and}$$
$$S_3=\{a_1\cdots a_p\otimes b_1\wedge\ldots\wedge b_q\otimes c_1\wedge\ldots\wedge c_r\in B(p,q,r)\mid (5.9)\text{ holds and } a_p\le c_i\le b_q \text{ for some } i\}.$$

We will prove that

(a) $|S_3|=R(p,q,r,\boldsymbol{f})$,
(b) $|S_1|+|S_2|+|S_3|=\operatorname{rank}B(p,q,r)$,
(c) $\operatorname{im}(\partial_h+\partial_v)=RS_1+RS_2$, and
(d) $B(p,q,r)=RS_1+RS_2+RS_3$.

Once assertions (a) — (d) are established, then it is clear that $S_1 \cup S_2 \cup S_3$ is a basis for $B(p,q,r)$ and that $\mathcal{M}(p,q,r)[F]$ is a free module of rank $R(p,q,r,\boldsymbol{f})$.

Observe that S_3 is the disjoint union $\bigcup\limits_{i=1}^{r} T_i$, where T_i is equal to

$$\{a_1 \cdots a_p \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_r \mid (5.9) \text{ holds and } a_p \le c_i \le b_q < c_{i+1}\}$$

for $1 \le i \le r-1$, and

$$T_r = \{a_1 \cdots a_p \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_r \mid (5.9) \text{ holds and } a_p \le c_r \le b_q\}\,.$$

Observe that

$$|T_i| = \sum_{k=1}^{\boldsymbol{f}} \sum_{\ell=1}^{k} \binom{p+\ell-1}{p}\binom{\ell-1}{i-1}\binom{k-1}{q-1}\binom{\boldsymbol{f}-k}{r-i}.$$

Indeed, if b_q is represented by k and c_i is represented by ℓ, then there are $\binom{p+\ell-1}{p}$ ways to choose $a_1 \le \cdots \le a_p$, with $a_p \le c_i$; $\binom{\ell-1}{i-1}$ ways to choose $c_1 < \cdots < c_{i-1}$, with $c_{i-1} < c_i$; $\binom{k-1}{q-1}$ ways to choose $b_1 < \cdots < b_{q-1}$, with $b_{q-1} < b_q$; and $\binom{\boldsymbol{f}-k}{r-i}$ ways to choose $c_{i+1} < \cdots < c_r$, with $b_q < c_{i+1}$. It is now clear that $|S_3|$ is equal to $R(p,q,r,\boldsymbol{f})$, and (a) is established. The values of $|S_1|$ and $|S_2|$ are given in (5.13) and (5.14), respectively. Theorem 6.7 yields (b). It is clear, from the fact that (5.12) is a complex, that $RS_1 = \operatorname{im}\partial_h$. We know that the diagrams

$$\begin{array}{ccc} B(p-2,q+1,r+1) & \xrightarrow{\partial_h} & B(p-1,q,r+1) \\ {\scriptstyle \partial_v}\big\downarrow & & {\scriptstyle \partial_v}\big\downarrow \\ B(p-1,q+1,r) & \xrightarrow{\partial_h} & B(p,q,r) \end{array} \qquad \text{for } 2 \le p, \text{ and}$$

$$\begin{array}{ccc} \bigwedge^{q+r+1} F & \xrightarrow{(-1)^q\Delta} & B(0,q,r+1) \\ {\scriptstyle (-1)^{q+1}\Delta}\big\downarrow & & {\scriptstyle \partial_v}\big\downarrow \\ B(0,q+1,r) & \xrightarrow{\partial_h} & B(1,q,r) \end{array}$$

commute; see Theorem 5.7, if necessary. It follows that $\operatorname{im}\partial_v \subseteq RS_2 + \operatorname{im}\partial_h$; and therefore, (c) is established.

We now prove (d). Let $x = a_1 \cdots a_p \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_r$ be an element of $B(p,q,r)$ which satisfies (5.9).

Claim 1. If $\max\{a_p, c_r\} \le b_q$, then $x \in RS_3 + \operatorname{im}(\partial_v + \partial_h)$.

Proof. If $a_p \le c_r$, then $x \in S_3$. If $c_r < a_p$, then

$$\partial_v(a_1 \cdots a_{p-1} \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_r \wedge a_p) = y + (-1)^r x,$$

for some element y of RS_3.

Claim 2. If $\max\{b_q, c_r\} \le a_p$, then $x \in RS_3 + \operatorname{im}(\partial_v + \partial_h)$.

Proof. Apply Claim 1 to see that

$$\partial_h(a_1 \cdots a_{p-1} \otimes b_1 \wedge \ldots \wedge b_q \wedge a_p \otimes c_1 \wedge \ldots \wedge c_r) = y + (-1)^q x,$$

for some element y of $RS_3 + \operatorname{im}(\partial_h + \partial_v)$.

At this point we have

$$c_r \le b_q \quad \text{or} \quad c_r \le a_p \implies x \in RS_3 + \operatorname{im}(\partial_h + \partial_v).$$

The proof of (d) proceeds by induction. The induction hypothesis ensures that

$$c_i \le b_q \quad \text{or} \quad c_i \le a_p \implies x \in RS_3 + \operatorname{im}(\partial_h + \partial_v),$$

for some i, with $1 \le i \le r$. Henceforth, we assume that $\max\{a_p, b_q\} < c_i$. We prove

$$c_{i-1} \le b_q \quad \text{or} \quad c_{i-1} \le a_p \implies x \in RS_3 + \operatorname{im}(\partial_h + \partial_v).$$

(In this discussion, we may treat c_0 as 0.)

Claim 3. If $\max\{a_p, c_{i-1}\} \le b_q$, then $x \in RS_3 + \operatorname{im}(\partial_v + \partial_h)$.

Proof. If $a_p \le c_{i-1}$, then $x \in S_3$. If $c_{i-1} < a_p$, then

$$\partial_v(a_1 \cdots a_{p-1} \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_{i-1} \wedge a_p \wedge c_i \wedge \cdots \wedge c_r) = y + (-1)^{i-1} x + y',$$

where

$$y = \sum_{k=1}^{i-1} (-1)^{k+1} a_1 \cdots a_{p-1} \cdot c_k \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge \widehat{c_k} \wedge \cdots \wedge c_{i-1} \wedge a_p \wedge c_i \wedge \cdots \wedge c_r, \text{ and}$$

$$y' = \sum_{k=i}^{r} (-1)^k a_1 \cdots a_{p-1} \cdot c_k \otimes b_1 \wedge \ldots \wedge b_q \otimes c_1 \wedge \ldots \wedge c_{i-1} \wedge a_p \wedge c_i \wedge \cdots \wedge \widehat{c_k} \wedge \cdots \wedge c_r.$$

The induction hypothesis ensures that $y' \in RS_3 + \operatorname{im}(\partial_v + \partial_h)$. Each term of y is in S_3.

Claim 4. If $\max\{b_q, c_{i-1}\} \le a_p$, then $x \in RS_3 + \operatorname{im}(\partial_v + \partial_h)$.

Proof. Apply Claim 3 to see that

$$\partial_h(a_1 \cdots a_{p-1} \otimes b_1 \wedge \ldots \wedge b_q \wedge a_p \otimes c_1 \wedge \ldots \wedge c_r) = y + (-1)^q x,$$

for some element y of $RS_3 + \operatorname{im}(\partial_h + \partial_v)$.

So, (d) is established, and the proof is complete. $\square$

6. Binomial coefficients.

In this section we prove the binomial coefficient identities, Proposition 6.6 and Theorem 6.7, which were used in the proof of Theorem 5.11. We begin by recalling some elementary facts about binomial coefficients. More details may be found in [14, 15]; in particular, identity (e) is Lemma 1.3 of [14].

Observation 6.1. *The following statements hold for all integers a, b, and c:*

(*a*) *if* $0 \le a < b$, *then* $\binom{a}{b} = 0$,

(*b*) $\binom{a}{b-1} + \binom{a}{b} = \binom{a+1}{b}$,

(*c*) *if* $0 \le a$, *then* $\binom{a}{b} = \binom{a}{a-b}$,

(*d*) $\binom{a}{b} = (-1)^b \binom{b-a-1}{b}$, *and*

(*e*) *if* $0 \le a$, *then* $\sum\limits_{k\in\mathbb{Z}} (-1)^k \binom{b+k}{c+k}\binom{a}{k} = (-1)^a \binom{b}{a+c}$.

Lemma 6.2. *Let a, b, and c be integers.*

(*a*) *If* $0 \le a$, *then* $\sum\limits_{k=0}^{a} \binom{a}{k}\binom{b}{c+k} = \binom{b+a}{c+a}$.

(*b*) *If a and c are non-negative, then* $\sum\limits_{k=0}^{b} (-1)^k \binom{a}{k}\binom{b-k}{c} = \binom{b-a}{c-a} + (-1)^{b+c}\binom{a-c-1}{a-b-1}$.

Proof. The proof of (a) is by induction on a. If $a = 0$, then both sides of the proposed identity are equal to $\binom{b}{c}$. Henceforth, we assume that $0 \le a$. Decompose the first binomial coefficient to see that

$$\sum_{k=0}^{a+1} \binom{a+1}{k}\binom{b}{c+k} = \sum_{k=0}^{a} \binom{a}{k}\binom{b}{c+k} + \sum_{k=0}^{a} \binom{a}{k}\binom{b}{c+k+1}.$$

The induction hypothesis completes the proof.

The proof of (b) is by induction on a. If a is zero, then use parts (d) and (c) of Observation 6.1 to see that the right side of the proposed identity is

$$\begin{cases} \binom{b}{c} & \text{if } 0 \le b \\ (-1)^c\binom{c-b-1}{c} + (-1)^{c+1}\binom{c-b-1}{-b-1} = 0 & \text{if } b < 0, \end{cases}$$

which is equal to the left side. Henceforth, we assume that $0 \le a$. Decompose the first binomial coefficient to see that

$$\sum_{k=0}^{b} (-1)^k \binom{a+1}{k}\binom{b-k}{c} = \sum_{k=0}^{b} (-1)^k \binom{a}{k}\binom{b-k}{c} + \sum_{k=0}^{b-1} (-1)^{k+1}\binom{a}{k}\binom{b-1-k}{c}.$$

Once again, the induction hypothesis completes the proof. $\square$

Lemma 6.3. *Let a, b, and c be integers with $0 \le a$. If, either $0 \le b$, or else, $b+c+1 \ne 0$, then*

$$\sum_{\ell=0}^{a} \binom{c+\ell}{c}\binom{\ell}{b} = \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{a+c+1}{c+1-\ell}\binom{a+\ell}{b+\ell}.$$

Proof. If c is negative, then both sides of the proposed identity are 0. Henceforth, we assume that $0 \le c$. If b is negative, then the left side of the proposed identity is zero. It is easy to see that the right side is

$$\sum_{\ell\in\mathbb{Z}} (-1)^{\ell+1}\binom{a+c+1}{c+1-\ell}\binom{a+\ell}{b+\ell} = \sum_{\ell\in\mathbb{Z}} (-1)^{\ell+1}\binom{a+c+1}{a+\ell}\binom{a+\ell}{b+\ell}$$
$$= \sum_{k\in\mathbb{Z}} (-1)^{a+k+1}\binom{a+c+1}{k}\binom{k}{b-a+k},$$

which, according to Observation 6.1.e, is equal to

$$(-1)^c\binom{0}{b+c+1} = 0.$$

Henceforth, we assume that $0 \le b$.

The proof proceeds by induction on a. When $a = 0$, then the left side of the proposed identity is

$$\binom{0}{b} = \begin{cases} 0 & \text{if } 1 \le b \\ 1 & \text{if } 0 = b, \end{cases}$$

and the right side of the proposed identity is

$$\begin{cases} 0 & \text{if } 1 \le b \\ \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{c+1}{c+1-\ell} = 1 & \text{if } 0 = b. \end{cases}$$

Henceforth, we assume that $0 \le a$. We must prove that

$$\sum_{\ell=0}^{a+1} \binom{c+\ell}{c}\binom{\ell}{b} = \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{a+c+2}{c+1-\ell}\binom{a+1+\ell}{b+\ell}. \tag{6.4}$$

Observe that the induction hypothesis gives that the left side of (6.4) is equal to $T_1 + T_2$, where

$$T_1 = \binom{a+1+c}{c}\binom{a+1}{b} \quad \text{and} \quad T_2 = \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{a+c+1}{c+1-\ell}\binom{a+\ell}{b+\ell}.$$

Decompose the second binomial coefficient to write the right side of (6.4) as T_3+T_4, where

$$T_3 = \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{a+c+2}{c+1-\ell}\binom{a+\ell}{b+\ell} \quad \text{and} \quad T_4 = \sum_{\ell=1}^{c+1} (-1)^{\ell+1}\binom{a+c+2}{c+1-\ell}\binom{a+\ell}{b+\ell-1}.$$

Decompose the first binomial coefficient to write T_4 and perform routine manipulations in order to see that $T_4 = T_1 + (T_2 - T_3)$. □

Note. Recall that the numbers $R(p, q, r, \boldsymbol{f})$ are defined in 5.10.

Lemma 6.5. *If p, q, r, and $\boldsymbol{f}$ are integers, then*

$$R(p,q,r,\boldsymbol{f}+1) = R(p,q,r,\boldsymbol{f}) + R(p,q,r-1,\boldsymbol{f}) + \sum_{\ell=1}^{\boldsymbol{f}+1} \binom{p+\ell-1}{p}\binom{\ell-1}{r-1}\binom{\boldsymbol{f}}{q-1}.$$

Proof. If $r \le 0$, then both sides of the proposed identity are zero. Henceforth, we assume $1 \le r$. Separate the middle summation into $1 \le k \le \boldsymbol{f}$ and $k = \boldsymbol{f}+1$ in order to write $R(p,q,r,\boldsymbol{f}+1) = T_1 + T_2$, where

$$T_1 = \sum_{i=1}^{r}\sum_{k=1}^{\boldsymbol{f}}\sum_{\ell=1}^{k} \binom{p+\ell-1}{p}\binom{\ell-1}{i-1}\binom{k-1}{q-1}\binom{\boldsymbol{f}+1-k}{r-i}$$

and T_2 is the last term in the proposed identity. Decompose the fourth binomial coefficient to write $T_1 = R(p,q,r,\boldsymbol{f}) + R(p,q,r-1,\boldsymbol{f})$. $\square$

Proposition 6.6. *If q, r, and $\boldsymbol{f}$ are non-negative integers, then*

$$R(0,q,r,\boldsymbol{f}) = \binom{\boldsymbol{f}}{q}\binom{\boldsymbol{f}}{r} - \binom{\boldsymbol{f}}{q+r}.$$

Proof. If $r=0$, $q=0$, or $\boldsymbol{f}=0$, then both sides of the proposed identity are zero. Henceforth, we assume that $1 \le q,r,\boldsymbol{f}$. We induct on $\boldsymbol{f}$. Apply Lemma 6.5 to see that

$$R(0,q,r,\boldsymbol{f}+1) = R(0,q,r,\boldsymbol{f}) + R(0,q,r-1,\boldsymbol{f}) + T, \quad \text{where}$$

$$T = \sum_{\ell=1}^{\boldsymbol{f}+1} \binom{\ell-1}{r-1}\binom{\boldsymbol{f}}{q-1} = \binom{\boldsymbol{f}+1}{r}\binom{\boldsymbol{f}}{q-1}.$$

(The last equality is well known. One could also use Lemma 6.3, with $c = 0$.) The induction hypothesis, together with some routine manipulations of binomial coefficients, yields the result. $\square$

Theorem 6.7. *If p, q, r, and $\boldsymbol{f}$ are integers with $1 \le p$ and $0 \le q$, then*

$$R(p,q,r,\boldsymbol{f}) + R(p-1,q,r+1,\boldsymbol{f}) = \tbinom{\boldsymbol{f}-1+p}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q}\tbinom{\boldsymbol{f}}{r} - \sum_{\ell=0}^{p-1}(-1)^{\ell}\tbinom{\boldsymbol{f}+p-\ell-2}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q+1+\ell}\tbinom{\boldsymbol{f}}{r}.$$

Proof. Fix integers p and q, with $1 \le p$ and $0 \le q$. The proof proceeds by induction on $\boldsymbol{f}$ and r. If $\boldsymbol{f} \le 0$ or $r \le -1$, then both sides of the proposed identity are zero. We next consider the case $\boldsymbol{f} = 1$ with $0 \le r$. It is easy to see that

$$R(p,q,r,1) = \begin{cases} 1 & \text{if } r = q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that both sides of the proposed identity are equal to

$$\begin{cases} 1 & \text{if } r = q = 1, \\ 1 & \text{if } r = 0 \text{ and } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Henceforth, we fix integers $\boldsymbol{f}$ and r, with $1 \le \boldsymbol{f}$ and $0 \le r$. The induction hypothesis ensures that the proposed identity holds for $(p,q,r,\boldsymbol{f})$ and $(p,q,r-1,\boldsymbol{f})$. We will show that

$$R(p,q,r,\boldsymbol{f}+1) + R(p-1,q,r+1,\boldsymbol{f}+1) = T_1 + T_2,$$

where

$$T_1 = \binom{\boldsymbol{f}+p}{\boldsymbol{f}}\binom{\boldsymbol{f}+1}{q}\binom{\boldsymbol{f}+1}{r}$$

and

$$T_2 = -\sum_{\ell=0}^{p-1}(-1)^\ell\binom{\boldsymbol{f}+p-\ell-1}{\boldsymbol{f}}\binom{\boldsymbol{f}+1}{q+1+\ell}\binom{\boldsymbol{f}+1}{r}.$$

Apply Lemma 6.5 to write

$$\begin{aligned} R(p,q,r,\boldsymbol{f}+1) &= R(p,q,r,\boldsymbol{f}) + R(p,q,r-1,\boldsymbol{f}) + T_3, \quad \text{and} \\ R(p-1,q,r+1,\boldsymbol{f}+1) &= R(p-1,q,r+1,\boldsymbol{f}) + R(p-1,q,r,\boldsymbol{f}) + T_4, \qquad \text{where} \end{aligned}$$

$$\begin{aligned} T_3 &= \sum_{\ell=1}^{\boldsymbol{f}+1}\binom{p+\ell-1}{p}\binom{\ell-1}{r-1}\binom{\boldsymbol{f}}{q-1}, \quad \text{and} \\ T_4 &= \sum_{\ell=1}^{\boldsymbol{f}+1}\binom{p+\ell-2}{p-1}\binom{\ell-1}{r}\binom{\boldsymbol{f}}{q-1}. \end{aligned}$$

The induction hypothesis yields

$$R(p,q,r,\boldsymbol{f}) + R(p-1,q,r+1,\boldsymbol{f}) = T_5 + T_6$$

and

$$R(p,q,r-1,\boldsymbol{f}) + R(p-1,q,r,\boldsymbol{f}) = T_7 + T_8,$$

where

$$\begin{aligned} T_5 &= \tbinom{\boldsymbol{f}-1+p}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q}\tbinom{\boldsymbol{f}}{r}, \\ T_6 &= -\sum_{\ell=0}^{p-1}(-1)^\ell\tbinom{\boldsymbol{f}+p-\ell-2}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q+1+\ell}\tbinom{\boldsymbol{f}}{r}, \\ T_7 &= \tbinom{\boldsymbol{f}-1+p}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q}\tbinom{\boldsymbol{f}}{r-1}, \quad \text{and} \\ T_8 &= -\sum_{\ell=0}^{p-1}(-1)^\ell\tbinom{\boldsymbol{f}+p-\ell-2}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q+1+\ell}\tbinom{\boldsymbol{f}}{r-1}. \end{aligned}$$

We will prove that $T_1+T_2 = T_3+T_4+T_5+T_6+T_7+T_8$. It is easy to see that

$$\begin{aligned} T_5+T_7 &= \tbinom{\boldsymbol{f}-1+p}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q}\tbinom{\boldsymbol{f}+1}{r}, \quad \text{and} \\ T_6+T_8 &= -\sum_{\ell=0}^{p-1}(-1)^\ell\tbinom{\boldsymbol{f}+p-\ell-2}{\boldsymbol{f}-1}\tbinom{\boldsymbol{f}}{q+1+\ell}\tbinom{\boldsymbol{f}+1}{r}. \end{aligned}$$

Decompose the middle binomial coefficient to write T_2, and perform some routine calculations, in order to see that

$$-T_2+T_6+T_8 = \binom{\boldsymbol{f}+p-1}{\boldsymbol{f}}\binom{\boldsymbol{f}}{q}\binom{\boldsymbol{f}+1}{r}.$$

It is now clear that

$$-T_1 - T_2 + T_5 + T_6 + T_7 + T_8 = -\binom{\boldsymbol{f}+p}{\boldsymbol{f}}\binom{\boldsymbol{f}}{q-1}\binom{\boldsymbol{f}+1}{r}.$$

Apply Lemma 6.3 to write

$$T_3 = \sum_{k=-1}^{p-1} (-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\binom{\boldsymbol{f}}{q-1}\binom{\boldsymbol{f}+k+2}{r+1+k}, \quad \text{and}$$
$$T_4 = \sum_{\ell=1}^{p}(-1)^{\ell+1}\binom{\boldsymbol{f}+p}{p-\ell}\binom{\boldsymbol{f}}{q-1}\binom{\boldsymbol{f}+\ell}{r+\ell}.$$

Observe that $-T_1 - T_2 + T_3 + T_5 + T_6 + T_7 + T_8$ is equal to

$$\binom{\boldsymbol{f}+p}{p-1}\binom{\boldsymbol{f}}{q-1}\binom{\boldsymbol{f}+1}{r} + \sum_{k=0}^{p-1}(-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\binom{\boldsymbol{f}}{q-1}\binom{\boldsymbol{f}+k+2}{r+1+k}.$$

We complete the proof by showing that $S_1 + S_2 + S_3 = 0$, where

$$S_1 = \binom{\boldsymbol{f}+p}{p-1}\binom{\boldsymbol{f}+1}{r}, \quad S_2 = \sum_{k=0}^{p-1}(-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\binom{\boldsymbol{f}+k+2}{r+1+k}, \quad \text{and}$$

$$S_3 = \sum_{\ell=1}^{p}(-1)^{\ell+1}\binom{\boldsymbol{f}+p}{p-\ell}\binom{\boldsymbol{f}+\ell}{r+\ell}.$$

Lemma 6.2.a gives that S_2 is

$$\begin{aligned}
&= \sum_{k=0}^{p-1}(-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\sum_{m=0}^{k+1}\binom{\boldsymbol{f}+1}{r+m}\binom{k+1}{m} \\
&= \sum_{k=-1}^{p-1}(-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\sum_{m=0}^{k+1}\binom{\boldsymbol{f}+1}{r+m}\binom{k+1}{m} - \binom{\boldsymbol{f}+p+1}{p}\binom{\boldsymbol{f}+1}{r} \\
&= \sum_{m=0}^{p}\left[\sum_{k=m-1}^{p-1}(-1)^{k+1}\binom{\boldsymbol{f}+p+1}{p-1-k}\binom{k+1}{m}\right]\binom{\boldsymbol{f}+1}{r+m} - \binom{\boldsymbol{f}+p+1}{p}\binom{\boldsymbol{f}+1}{r}.
\end{aligned}$$

Apply Lemma 6.2.b to see that the expression inside the brackets is equal to

$$\sum_{\ell=0}^{p-m}(-1)^{p+\ell}\tbinom{\boldsymbol{f}+p+1}{\ell}\tbinom{p-\ell}{m} = \sum_{\ell=0}^{p}(-1)^{p+\ell}\tbinom{\boldsymbol{f}+p+1}{\ell}\tbinom{p-\ell}{m} = (-1)^m\tbinom{\boldsymbol{f}+p-m}{\boldsymbol{f}}.$$

It follows that

$$S_2 = \sum_{m=0}^{p}(-1)^m\binom{\boldsymbol{f}+p-m}{\boldsymbol{f}}\binom{\boldsymbol{f}+1}{r+m} - \binom{\boldsymbol{f}+p+1}{p}\binom{\boldsymbol{f}+1}{r}.$$

Parts (a) and (b) of Lemma 6.2 give

$$\begin{aligned} S_3 &= \sum_{\ell=1}^{p}\sum_{m\in\mathbb{Z}}(-1)^{\ell+1}\binom{\boldsymbol{f}+p}{p-\ell}\binom{\ell-1}{m}\binom{\boldsymbol{f}+1}{r+1+m} \\ &= \sum_{m=0}^{p-1}\sum_{\ell=1}^{p}(-1)^{\ell+1}\binom{\boldsymbol{f}+p}{p-\ell}\binom{\ell-1}{m}\binom{\boldsymbol{f}+1}{r+1+m} \\ &= \sum_{m=0}^{p-1}(-1)^{m}\binom{\boldsymbol{f}+p-1-m}{\boldsymbol{f}}\binom{\boldsymbol{f}+1}{r+1+m}. \end{aligned}$$

It is now easy to see that $S_1+S_2+S_3=0$, and the proof is complete. $\square$

7. The proof of Theorems 4.5 and 4.8.

Let $(\mathbb{F},f)$ be a complex of $R-$modules. Suppose that each module $\mathbb{F}_r$ decomposes as $\mathbb{F}_r=\mathbb{L}_r\oplus\mathbb{M}_r\oplus\widehat{\mathbb{N}}_r$ and each map $f_r\colon\mathbb{F}_r\to\mathbb{F}_{r-1}$ decomposes as

$$f_r=\begin{bmatrix}0&0&\sigma_r\\0&m_r&0\\0&0&0\end{bmatrix}\colon\mathbb{L}_r\oplus\mathbb{M}_r\oplus\widehat{\mathbb{N}}_r\to\mathbb{L}_{r-1}\oplus\mathbb{M}_{r-1}\oplus\widehat{\mathbb{N}}_{r-1}. \tag{7.1}$$

If every map $\sigma_r\colon\widehat{\mathbb{N}}_r\to\mathbb{L}_{r-1}$ is an isomorphism, then it is clear that $(\mathbb{M},m)$ is a complex which is quasi-isomorphic to $(\mathbb{F},f)$. Unfortunately, when one has a real example in mind for $\mathbb{F}$, the process of splitting off the split exact summands of $\mathbb{F}$ in order to obtain a minimal complex is more complicated. Indeed, even if one has good candidates for $\mathbb{L}$, $\mathbb{M}$, and $\widehat{\mathbb{N}}$, a significant amount of linear algebra is required before each f_r has the form of (7.1). The next result describes $\mathbb{F}$ after $\oplus\widehat{\mathbb{N}}_r$ has been split off, even if each f_r only looks like

$$\begin{bmatrix}*&*&\text{isomorphism}\\ *&*&*\\ *&*&*\end{bmatrix}.$$

Proposition 7.2. *Let $(\mathbb{F},f)$ be a complex of free $R-$modules. Suppose that each module $\mathbb{F}_r$ decomposes as $\mathbb{F}_r=\mathbb{L}_r\oplus\mathbb{M}_r\oplus\widehat{\mathbb{N}}_r$. Let $\pi_r^{\mathbb{A}}\colon\mathbb{F}_r\to\mathbb{A}_r$, for $\mathbb{A}$ equal to $\mathbb{L}$, $\mathbb{M}$, and $\widehat{\mathbb{N}}$, be the projection maps which are induced by this decomposition. Suppose that the composition*

$$\widehat{\mathbb{N}}_{r+1}\xrightarrow{\text{incl}}\mathbb{F}_{r+1}\xrightarrow{f_{r+1}}\mathbb{F}_r\xrightarrow{\pi_r^{\mathbb{L}}}\mathbb{L}_r \tag{7.3}$$

is an isomorphism for all r. Let $\theta_r\colon\mathbb{L}_r\to\widehat{\mathbb{N}}_{r+1}$ be the inverse of (7.3). Define $(\mathbb{N},n)$ to be the subcomplex of $(\mathbb{F},f)$ with $\mathbb{N}_r=\widehat{\mathbb{N}}_r+f_{r+1}(\widehat{\mathbb{N}}_{r+1})$ and n_r equal to the restriction of f_r to $\mathbb{N}_r$. For each integer r, define $\psi_r\colon\mathbb{F}_r\to\mathbb{M}_r$ by $\psi_r=\pi_r^{\mathbb{M}}\circ(1-f_{r+1}\circ\theta_r\circ\pi_r^{\mathbb{L}})$, $\rho_r\colon\mathbb{M}_r\to\mathbb{F}_r$ by $\rho_r=\operatorname{incl}_r-\theta_{r-1}\circ\pi_{r-1}^{\mathbb{L}}\circ f_r$, and $m_r\colon\mathbb{M}_r\to\mathbb{M}_{r-1}$ to be the composition

$$\mathbb{M}_r\xrightarrow{\text{incl}_r}\mathbb{F}_r\xrightarrow{f_r}\mathbb{F}_{r-1}\xrightarrow{\psi_{r-1}}\mathbb{M}_{r-1}.$$

Then the following statements all hold.

(a) *The complex* $(\mathbb{N}, n)$ *is split exact.*
(b) *The modules and maps* $\{m_r : \mathbb{M}_r \to \mathbb{M}_{r-1}\}$ *form a complex, which we denote* $(\mathbb{M}, m)$*.*
(c) *The maps* $\{\psi_r : \mathbb{F}_r \to \mathbb{M}_r\}$ *form a map of complexes; furthermore,*

$$0 \to (\mathbb{N}, n) \xrightarrow{\text{incl}} (\mathbb{F}, f) \xrightarrow{\psi} (\mathbb{M}, m) \to 0$$

is a short exact sequence of complexes.
(d) *The maps* $\{\rho_r : \mathbb{M}_r \to \mathbb{F}_r\}$ *form a map of complexes; furthermore,* $\psi_r \circ \rho_r$ *is the identity map on* $\mathbb{M}_r$*.*

Proof. It is not difficult to adapt the proof of [16, Prop. 3.14] to the present situation. □

We apply Proposition 7.2, by way of Proposition 7.5, to prove Theorem 4.5.

Definition 7.4. A collection of $R-$module maps $\{b_r : \mathbb{B}_r \to \mathbb{B}_{r-1} \mid r \in \mathbb{Z}\}$ is called *replaceable* if there is a finite, totally ordered, set T and there are submodules $\mathbb{B}_r^{(t)}$ of $\mathbb{B}_r$ for all $r \in \mathbb{Z}$ and $t \in T$ such that the following conditions hold.

(1) Each module $\mathbb{B}_r$ is equal to the direct sum $\bigoplus \mathbb{B}_r^{(t)}$, where the sum is taken over all $t \in T$.
(2) Each b_r is a non-increasing map in the sense that, if $x \in \mathbb{B}_r^{(t)}$, then $b_r(x)$ is an element of $\sum \mathbb{B}_{r-1}^{(t')}$, where the sum is taken over all $t' \in T$, with $t' \le t$.
(3) If $b_r^{(t)} : \mathbb{B}_r^{(t)} \to \mathbb{B}_{r-1}^{(t)}$ is defined to be the composition

$$\mathbb{B}_r^{(t)} \xrightarrow{\text{incl}} \mathbb{B}_r \xrightarrow{b_r} \mathbb{B}_{r-1} \xrightarrow{\text{proj}} \mathbb{B}_{r-1}^{(t)},$$

then the maps $\{b_r^{(t)} : \mathbb{B}_r^{(t)} \to \mathbb{B}_{r-1}^{(t)} \mid r \in \mathbb{Z}\}$ form a complex, which we denote $\mathbb{B}^{(t)}$,
(4) For each $t \in T$, there is an integer N_t such that either
 (a) $\mathbb{B}_i^{(t)} = 0$ for all i with $i < N_t$ and the augmented complex

$$\dots \xrightarrow{b_{N_t+2}^{(t)}} \mathbb{B}_{N_t+1}^{(t)} \xrightarrow{b_{N_t+1}^{(t)}} \mathbb{B}_{N_t}^{(t)} \xrightarrow{\text{aug}^{(t)}} H_{N_t}(\mathbb{B}^{(t)}) \to 0$$

 is split exact; or else,
 (b) $\mathbb{B}_i^{(t)} = 0$ for all i with $N_t < i$ and the augmented complex

$$0 \to H_{N_t}(\mathbb{B}^{(t)}) \xrightarrow{\text{aug}^{(t)}} \mathbb{B}_{N_t}^{(t)} \xrightarrow{b_{N_t}^{(t)}} \mathbb{B}_{N_t-1}^{(t)} \xrightarrow{b_{N_t-1}^{(t)}} \dots$$

 is split exact.

If the maps $\{b_r\}$ are replaceable, then we let $(\mathbb{B}, b^{[0]})$ represent the direct sum $\bigoplus_{t \in T} \mathbb{B}^{(t)}$ of the complexes $\mathbb{B}^{(t)}$ from (3).

Note. In practice, the maps $\{b_r\}$ of Definition 7.4 will **not** from a complex; the relevant complex $(\mathbb{B}, b^{[0]})$ is obtained by considering only the parts of b_r which preserve the $T-$grading. Proposition 7.5 shows how to replace the maps $\{b_r\}$ with the homology of the complex $(\mathbb{B}, b^{[0]})$.

Proposition 7.5. *Let $(\mathbb{F}, f)$ be a complex of $R-$modules. Suppose that each module $\mathbb{F}_r$ of $\mathbb{F}$ decomposes as $\mathbb{F}_r = \mathbb{A}_r \oplus \mathbb{B}_r$. Let $b_r\colon \mathbb{B}_r \to \mathbb{B}_{r-1}$ be the composition*

$$\mathbb{B}_r \xrightarrow{\text{incl}} \mathbb{F}_r \xrightarrow{f_r} \mathbb{F}_{r-1} \xrightarrow{\text{proj}} \mathbb{B}_{r-1}.$$

If the maps $\{b_r\colon \mathbb{B}_r \to \mathbb{B}_{r-1} \mid r \in \mathbb{Z}\}$ are replaceable in the sense of Definition 7.4 and $\mathbb{B}$ is the complex $(\mathbb{B}, b^{[0]})$ which is created in Definition 7.4, then there exists a split exact subcomplex $\mathbb{N}$ of $\mathbb{F}$ such that $\mathbb{N}$ is a direct summand of $\mathbb{F}$ and $\mathbb{F}_r/\mathbb{N}_r \cong H_r(\mathbb{B}) \oplus \mathbb{A}_r$.

Proof. Adopt the notation of Definition 7.4. For each $t \in T$, let $\{h^{(t)}\colon \mathbb{B}_r^{(t)} \to \mathbb{B}_{r+1}^{(t)}\}$ be a homotopy on the augmented, split exact, complex of hypothesis (4). It follows that

$$\mathbb{B}_r^{(t)} = \begin{cases} \operatorname{Im}\left(h_{r-1}^{(t)} \circ b_r^{(t)}\right) \oplus \operatorname{Im} b_{r+1}^{(t)} & \text{if } N_t \neq r, \\ \operatorname{Im}\left(h_{r-1}^{(t)} \circ \operatorname{aug}^{(t)}\right) \oplus \operatorname{Im} b_{r+1}^{(t)} & \text{if } N_t = r \text{ in case (a), and} \\ \operatorname{Im}\left(h_{r-1}^{(t)} \circ b_r^{(t)}\right) \oplus \operatorname{Im}\left(\operatorname{aug}^{(t)}\right) & \text{if } N_t = r \text{ in case (b).} \end{cases}$$

For every $r \in \mathbb{Z}$ and $t \in T$, define $\mathbb{L}_r^{(t)} = \operatorname{Im} b_{r+1}^{(t)}$, $\widehat{\mathbb{N}}_r^{(t)} = \operatorname{Im}\left(h_{r-1}^{(t)} \circ b_r^{(t)}\right)$, and

$$\widehat{\mathbb{M}}_r^{(t)} = \begin{cases} \operatorname{Im}\left(h_{r-1}^{(t)} \circ \operatorname{aug}^{(t)}\right) & \text{if } N_t = r \text{ in case (a),} \\ \operatorname{Im}\left(\operatorname{aug}^{(t)}\right) & \text{if } N_t = r \text{ in case (b), and} \\ 0 & \text{if } N_t \neq r. \end{cases}$$

It is clear that $\mathbb{B}_r^{(t)} = \mathbb{L}_r^{(t)} \oplus \widehat{\mathbb{M}}_r^{(t)} \oplus \widehat{\mathbb{N}}_r^{(t)}$ for all r and t. It is also clear that the restriction of $b_{r+1}^{(t)}$ to $\widehat{\mathbb{N}}_{r+1}^{(t)}$ gives an isomorphism $b_{r+1}^{(t)}\colon \widehat{\mathbb{N}}_{r+1}^{(t)} \to \mathbb{L}_r^{(t)}$ for all r and t. Define submodules

$$\mathbb{L}_r = \bigoplus_{t\in T} \mathbb{L}_r^{(t)}, \quad \widehat{\mathbb{M}}_r = \bigoplus_{t\in T} \widehat{\mathbb{M}}_r^{(t)}, \quad \mathbb{M}_r = \widehat{\mathbb{M}}_r \oplus \mathbb{A}_r, \quad \text{and} \quad \widehat{\mathbb{N}}_r = \bigoplus_{t\in T} \widehat{\mathbb{N}}_r^{(t)}$$

of $\mathbb{F}_r$. Observe that $\mathbb{F}_r = \mathbb{L}_r \oplus \mathbb{M}_r \oplus \widehat{\mathbb{N}}_r$ and that the map of (7.3) sends $y \in \widehat{\mathbb{N}}_{r+1}^{(t)}$ to $b_{r+1}^{(t)}(y) + y'$ for some $y' \in \sum \mathbb{L}_r^{(t')}$, with $t' < t$. If the map of (7.3) is expressed as a matrix, then it is a triangular matrix with isomorphisms on the main diagonal; thus, it is an isomorphism. Apply Proposition 7.2 to see that $\mathbb{N} = \widehat{\mathbb{N}} + f(\widehat{\mathbb{N}})$ is a split exact subcomplex of $\mathbb{F}$ with $\mathbb{N}$ a direct summand of $\mathbb{F}$ and

$$\mathbb{F}_r/\mathbb{N}_r \cong \mathbb{M}_r = \widehat{\mathbb{M}}_r \oplus \mathbb{A}_r \cong H_r(\mathbb{B}) \oplus \mathbb{A}_r. \quad \square$$

Proof of Theorem 4.5. Submodules $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{I}^{(z)}$ are introduced in Definition 7.8. It is not difficult to see that $\mathbb{I}^{(z)} = \mathbb{A} \oplus \mathbb{B}$. Let $b\colon \mathbb{B} \to \mathbb{B}$ be the composition

$$\mathbb{B} \xrightarrow{\text{incl}} \mathbb{I}^{(z)} \xrightarrow{d} \mathbb{I}^{(z)} \xrightarrow{\text{proj}} \mathbb{B}. \tag{7.6}$$

Lemma 7.9 shows that the map $b\colon \mathbb{B} \to \mathbb{B}$ decomposes as the direct sum

$$\mathbb{B}' \oplus \mathbb{B}'' \xrightarrow{\begin{bmatrix} b' & 0 \\ 0 & b'' \end{bmatrix}} \mathbb{B}' \oplus \mathbb{B}''. \tag{7.7}$$

Thus, in light of Proposition 7.5, it suffices to show that

(a) the maps of $b'\colon \mathbb{B}' \to \mathbb{B}'$ are replaceable with the homology of $(\mathbb{B}', b'^{[0]})$ equal to $\bigoplus \mathbb{V}(p,q,r,z)$, where the sum is taken over $S_{\mathbb{V}}^{(z)}$; and

(b) the maps of $b''\colon \mathbb{B}'' \to \mathbb{B}''$ are replaceable with the homology of $(\mathbb{B}'', b''^{[0]})$ equal to $\bigoplus \mathbb{S}(p,q,r,z)$, where the sum is taken over $S_{\mathbb{S}}^{(z)}$.

We first prove (a). The finite, totally ordered set T which gives the grading of Definition 7.4 for $\mathbb{B}'$ is given in Definition 7.11, together with Proposition 7.14.c. Condition (1) of Definition 7.4 is established in Proposition 7.12; condition (2) in Lemma 7.13; condition (3) in Proposition 7.14; and condition (4.a) in Corollary 7.19. The homology of $\mathbb{B}'$ is recorded in Proposition 7.20. Thus (a) is established. Assertion (b) follows from the duality of Proposition 2.12. In other words, the maps of $b''\colon \mathbb{B}'' \to \mathbb{B}''$ are isomorphic to

$$\big[\text{the maps of } b'\colon \mathbb{B}' \to \mathbb{B}' \text{ from } \mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}-z)}\big]^* [-(\boldsymbol{g}+\boldsymbol{f}-1)].$$

It follows that the maps of b'' are replaceable. (Condition (4.b) holds in place of (4.a).) The observation of (4.4) completes the proof by showing that the homology of $(\mathbb{B}'', b''^{[0]})$ is equal to $\bigoplus \mathbb{S}(p,q,r,z)$, where the sum is taken over $S_{\mathbb{S}}^{(z)}$. $\square$

Definition 7.8. Adopt the notation of Definitions 2.3 and 4.2. Define submodules $\mathbb{A}$ and $\mathbb{B} = \mathbb{B}' \oplus \mathbb{B}''$ of $\mathbb{I}^{(z)}$ by

$$\mathbb{A} = \bigoplus_{S_{\mathbb{U}}^{(z)}} \mathbb{U}(p,q,r) \oplus \bigoplus_{S_{\mathbb{T}}^{(z)}} \mathbb{T}(p,q,r), \quad \mathbb{B}' = \mathbb{L}^{(z)} \oplus \bigoplus_{\overline{S}_{\mathbb{U}}^{(z)}} \mathbb{U}(p,q,r), \quad \text{and}$$

$$\mathbb{B}'' = \bigoplus_{\overline{S}_{\mathbb{T}}^{(z)}} \mathbb{T}(p,q,r) \oplus \mathbb{W}^{(z)}, \quad \text{where}$$

$$\overline{S}_{\mathbb{U}}^{(z)} = \{(p,q,r) \in T_{\mathbb{U}}^{(z)} \mid q \le z-1\} \quad \text{and} \quad \overline{S}_{\mathbb{T}}^{(z)} = \{(p,q,r) \in T_{\mathbb{T}}^{(z)} \mid q \le \boldsymbol{g}-\boldsymbol{f}-z-1\}.$$

Lemma 7.9. *Let $\mathbb{B} = \mathbb{B}' \oplus \mathbb{B}''$ be the submodule of $\mathbb{I}^{(z)}$ which is given in Definition 7.8, and let $b\colon \mathbb{B} \to \mathbb{B}$ be the map of (7.6). Then, the compositions*

$$\mathbb{B}' \xrightarrow{\text{incl}} \mathbb{B} \xrightarrow{b} \mathbb{B} \xrightarrow{\text{proj}} \mathbb{B}'' \quad \textit{and} \quad \mathbb{B}'' \xrightarrow{\text{incl}} \mathbb{B} \xrightarrow{b} \mathbb{B} \xrightarrow{\text{proj}} \mathbb{B}'$$

are both zero.

Proof. The only interesting case involves the map $\mathbb{T}(p,q,r) \to \mathbb{B}'$ for $(p,q,r) \in \overline{S}_{\mathbb{T}}^{(z)}$. Definition 2.1 shows that the curcial part is

$$\alpha_p \otimes c_q \otimes \lambda^{(r)} \mapsto \text{proj}\left(\text{an element of } \sum_{0 \le t \le q+r} \mathbb{U}(q-t+r, \boldsymbol{g}-\boldsymbol{f}+p+r-t, t)\right). \tag{7.10}$$

The hypothesis on (p,q,r) guarantees that $q \le \boldsymbol{g}-\boldsymbol{f}-z-1$ and this ensures that the middle parameter in the image of (7.10) is at least $z+1$; hence, (7.10) is the zero map. $\square$

Definition 7.11. Let $(T,<)$ be the partially ordered set whose elements are

$$T=\{(p,q,r)\in\mathbb{Z}^3 \mid 0\le p,\quad 1\le q,\quad \text{and}\quad 1\le r\}.$$

If (p,q,r) and (p',q',r') are elements of T, then $(p',q',r')<(p,q,r)$ provided either

$$\begin{cases} q'-r'-p'<q-r-p;\text{ or else,}\\ q'-r'-p'=q-r-p \text{ and } p'<p.\end{cases}$$

Consider any total order (also denoted by "$<$") on T which extends the above partial order. For $(p,q,r)\in T$, let

$$\mathbb{D}[[p,q,r]]=\bigoplus\mathbb{L}(a,b,\boldsymbol{f}+a+b-p-q-r,\boldsymbol{g}-z+1+p,z-1+q-a-b)\oplus\mathbb{U}(\boldsymbol{f}-r-p-q,z-1-p,p+q),$$

where the first sum is taken over $\{(a,b)\mid q\le b \text{ and } a+b\le q+p\}$.

Proposition 7.12. *There exists a direct sum decomposition $\mathbb{B}'=\bigoplus\mathbb{D}[[p,q,r]]$, where the sum is taken over all $(p,q,r)\in T$.*

Proof. If $\mathbb{L}(a,b,c,d,e)$ is a non-zero summand of $\mathbb{B}'$, then $\mathbb{L}(a,b,c,d,e)$ is a summand $\mathbb{D}[[z-1-\boldsymbol{g}+d,a+b+e+1-z,\boldsymbol{f}+\boldsymbol{g}-c-d-e]]$. If $\mathbb{U}(a,b,c)$ is a non-zero summand of $\mathbb{B}'$, then $\mathbb{U}(a,b,c)$ is a summand of $\mathbb{D}[[z-1-b,b+c+1-z,\boldsymbol{f}-a-c]]$. $\square$

Lemma 7.13. *The map $b'\colon\mathbb{B}'\to\mathbb{B}'$, which is defined in (7.7), is non-increasing, in the sense of Definition 7.4.*

Proof. If $x'\in\mathbb{D}[[p',q',r']]$ and $x\in\mathbb{D}[[p,q,r]]$, with $(p',q',r')<(p,q,r)$ in T, then we write $x'<x$. The proof of Proposition 7.12 makes it possible for us to calculate this order quickly. Indeed, if $x\in\mathbb{L}(a,b,c,d,e)$, then

$$\text{“}q-r-p\text{ for }x\text{”}=a+b+c+2e+2-2z-\boldsymbol{f}\quad\text{and}\quad\text{“}p\text{ for }x\text{”}=d-\boldsymbol{g}+z-1;$$

and if x is a non-zero element of the summand $\mathbb{U}(a,b,c)$ of $\mathbb{B}'$, then

$$\text{“}q-r-p\text{ for }x\text{”}=a+2b+2c+2-2z-\boldsymbol{f}\quad\text{and}\quad\text{“}p\text{ for }x\text{”}=-b+z-1.$$

Define $d^{[0]}\colon\mathbb{D}[[p,q,r]]\to\mathbb{D}[[p,q,r]]$ to be the composition

$$\mathbb{D}[[p,q,r]]\xrightarrow{\text{incl}}\mathbb{I}^{(z)}\xrightarrow{d}\mathbb{I}^{(z)}\xrightarrow{\text{proj}}\mathbb{D}[[p,q,r]].$$

If x is the element $A_a\otimes\alpha_b\otimes b_c\otimes c_d\otimes\nu^{(e)}$ of $\mathbb{L}(a,b,c,d,e)$, then Definition 2.3 yields $d(x)=x'+d^{[0]}(x)$, where $x'<x$, and

$$d^{[0]}(x)=\begin{cases}\chi(a+e\le z-2)\sum\limits_{|I|=1}\varphi_I\cdot A_a\otimes f_I(\alpha_b)\otimes b_c\otimes c_d\otimes\nu^{(e)}\\[2ex] +(-1)^b\chi(\boldsymbol{g}+1\le d+e)\sum\limits_{|J|=1}\varphi_J\cdot A_a\otimes\alpha_b\otimes f_J\wedge b_c\otimes c_d\otimes\nu^{(e-1)}.\end{cases}$$

Also, if $x=b_a\otimes\delta_b\otimes\mu^{(c)}\in\mathbb{U}(a,b,c)$, then $d(x)=x'+d^{[0]}(x)$, where $x'<x$, and

$$d^{[0]}(x)=\sum_{\substack{b\le t\le z-1\\|I|=c+b-t}}(-1)^{t+a}1\otimes\varphi_I\otimes f_I\wedge b_a\otimes\delta_b(\omega_G)\otimes\nu^{(t)}.\quad\square$$

Proposition 7.14. *Let (p,q,r) be an element of T, and $N = 2 - \boldsymbol{f} - z + p + 2r$. Recall the complex $\mathbb{B}(p,q,r) = \mathbb{B}(p,q,r)(F^*)$ of Definition 5.6 and the map $d^{[0]}$ from the proof of Lemma 7.13. Then*

(a) *the maps and modules $(\mathbb{D}[[p,q,r]], d^{[0]})$ form a complex,*
(b) *the complexes $(\mathbb{D}[[p,q,r]], d^{[0]})$ and $\mathbb{B}(p,q,r)[N] \otimes \bigwedge^{\boldsymbol{g}+p-z+1} G$ are isomorphic,*
(c) *the set $\{(p',q',r') \in T \mid \mathbb{D}[[p',q',r']] \neq 0\}$ is finite, and*
(d) *the module $\mathbb{D}[[p,q,r]]_i$ is equal to*

$$\begin{cases} 0 & \text{if } i \le \boldsymbol{f} + z - 3 - p + q - r, \\ \mathbb{L}(p,q,\boldsymbol{f}-r,\boldsymbol{g}-z+1+p,z-1-p) & \text{if } i = \boldsymbol{f} + z - 2 - p + q - r. \end{cases}$$

Proof. Let $\mathbb{E}$ denote $\mathbb{B}(p,q,r)[N] \otimes \bigwedge^{\boldsymbol{g}+p-z+1} G$. For integers a, b, c, and d, let

$$\Phi\colon \mathbb{L}(a,b,c,d,z-1+q-a-b) \to B(a,b,\boldsymbol{f}-c) \otimes {\textstyle\bigwedge^d} G \quad \text{and} \tag{7.15}$$

$$\Phi\colon \mathbb{U}(a,b,p+q) \to {\textstyle\bigwedge^{\boldsymbol{f}-a}} F^* \otimes {\textstyle\bigwedge^{\boldsymbol{g}-b}} G \tag{7.16}$$

be the isomorphisms which are given by

$$\Phi(A_a \otimes \alpha_b \otimes b_c \otimes c_d \otimes \nu^{(z-1+q-a-b)}) = A_a \otimes \alpha_b \otimes b_c[\omega_{F^*}] \otimes c_d \quad \text{and}$$
$$\Phi(b_a \otimes \delta_b \otimes \mu^{(p+q)}) = (-1)^{\boldsymbol{f}+r+p+z+1} b_a[\omega_{F^*}] \otimes \delta_b[\omega_G],$$

respectively. A straightforward calculation shows that the above maps induce an isomorphism of graded modules

$$\Phi\colon \mathbb{D}[[p,q,r]] \to \mathbb{E}. \tag{7.17}$$

Indeed, the domain of (7.15) is in $\mathbb{D}[[p,q,r]]$ if and only if the range of (7.15) is in $\mathbb{E}$; furthermore, each module has position $\boldsymbol{f} + z - 2 - a + q - r$ in its respective complex. Also, the domain of (7.16) is in $\mathbb{D}[[p,q,r]]$ if and only if the range of (7.16) is in $\mathbb{E}$; furthermore, the position of each module is $\boldsymbol{f} + z - 1 + q - r$ in its respective complex. A short calculation now yields that (7.17) is a map of complexes, thereby completing the proof of (b) and (a). Assertion (c) is clear because $\mathbb{B}(p,q,r) \otimes \bigwedge^{\boldsymbol{g}+p-z+1} G$ is the zero complex if $z \le p$, or $\boldsymbol{f} + 1 \le q$, or $\boldsymbol{f} + 1 \le r$. Assertion (d) is also clear because $\mathbb{B}(p,q,r)_i = 0$, if $i < q + r$, and $\mathbb{B}(p,q,r)_{q+r} = B(p,q,r)$. □

Definition 7.18. Let (p,q,r) be in T, and let $n = \boldsymbol{f} + z - 2 - p + q - r$. Define $\operatorname{aug}\colon \mathbb{D}[[p,q,r]]_n \to \mathbb{V}(p,q,r,z)$ to be the composition

$$\mathbb{D}[[p,q,r]]_n = \mathbb{L}(p,q,\boldsymbol{f}-r,\boldsymbol{g}-z+1+p,z-1-p)$$

$$\xrightarrow{\text{nat}} S_pF^* \otimes {\textstyle\bigwedge^q} F^* \otimes {\textstyle\bigwedge^r} F^* \otimes {\textstyle\bigwedge^{\boldsymbol{g}-z+1+p}} G \xrightarrow{\text{quot}} \mathbb{V}(p,q,r,z).$$

See Definition 4.7 for the meaning of “quot” and “nat”.

Corollary 7.19. *If* $(p,q,r) \in T$, *then*

$$H_i\Big(\mathbb{D}[[p,q,r]]\Big) = \begin{cases} \mathbb{V}(p,q,r,z) & \text{if } i = \boldsymbol{f} + z - 2 - p + q - r, \\ 0 & \text{if } i \neq \boldsymbol{f} + z - 2 - p + q - r. \end{cases}$$

In particular, the augmented complex aug: $\mathbb{D}[[p,q,r]] \to \mathbb{V}(p,q,r,z)$ *is split exact.*

Proof. Combine Proposition 7.14 and Theorems 5.7 and 5.11. $\square$

Proposition 7.20. *The homology of* $(\mathbb{B}', b'^{[0]})$ *is isomorphic to* $\bigoplus \mathbb{V}(p,q,r,z)$, *where the sum is taken over the set* $S_{\mathbb{V}}^{(z)}$ *of Definition 4.2.*

Proof. We know that $(\mathbb{B}', b'^{[0]})$ is the direct sum of the complexes $(\mathbb{D}[[p,q,r]], d^{[0]})$ as (p,q,r) varies over the elements of T; furthermore, $\mathbb{D}[[p,q,r]]$ is the zero complex when (p,q,r) is in T but not $S_{\mathbb{V}}^{(z)}$. $\square$

Proof of Theorem 4.8. Theorem 4.5 was proved by applying Proposition 7.2; thus, during the proof of Theorem 4.5, we decomposed $\mathbb{I}^{(z)}$ as $\mathbb{L} \oplus \mathbb{M} \oplus \widehat{\mathbb{N}}$, where $\mathbb{M}$ is isomorphic to $\mathbb{M}^{(z)}$ and the composition

$$\widehat{\mathbb{N}}_{i+1} \xrightarrow{\text{incl}} \mathbb{I}_{i+1}^{(z)} \xrightarrow{d} \mathbb{I}_i^{(z)} \xrightarrow{\text{proj}} \mathbb{L}_i \tag{7.21}$$

is an isomorphism. Let θ be the inverse of (7.21) and let P be the composition

$$\mathbb{I}_i^{(z)} \xrightarrow{\text{proj}} \mathbb{L}_i \xrightarrow{\theta_i} \widehat{\mathbb{N}}_{i+1} \xrightarrow{\text{incl}} \mathbb{I}_{i+1}^{(z)}.$$

Notice that the maps σ and τ of Definition 4.7 are compositions

$$\mathbb{M}^{(z)} \xrightarrow{\varepsilon} \mathbb{M} \xrightarrow{\text{incl}} \mathbb{I}^{(z)} \quad \text{and} \quad \mathbb{I}^{(z)} \xrightarrow{\text{proj}} \mathbb{M} \xrightarrow{\varepsilon^{-1}} \mathbb{M}^{(z)}, \tag{7.22}$$

respectively, for a fixed isomorphism ε from $\mathbb{M}^{(z)}$ to $\mathbb{M}$. Proposition 7.2 shows that the differential on $\mathbb{M}^{(z)}$ is the composition

$$\mathbb{M}^{(z)} \xrightarrow{\varepsilon} \mathbb{M} \xrightarrow{\text{incl}} \mathbb{I}^{(z)} \xrightarrow{d} \mathbb{I}^{(z)} \xrightarrow{1-d\circ P} \mathbb{I}^{(z)} \xrightarrow{\text{proj}} \mathbb{M} \xrightarrow{\varepsilon^{-1}} \mathbb{M}^{(z)};$$

which, in light of (7.22), establishes (a). Assertions (b) and (c) may be read from Proposition 7.2 in a similar manner.

We now prove (d). Take "$\equiv$" to mean congruent mod $[I_1(u) + I_1(v) + I_1(X)]\,\mathbb{I}^{(z)}$. We prove that the image of $d \circ \sigma \colon \mathbb{M}^{(z)} \to \mathbb{I}^{(z)}$ is congruent to zero. There are four cases. First, we take $x \in \mathbb{V}(p,q,r,z)$ for some (p,q,r) in $S_{\mathbb{V}}^{(z)}$. It follows that $\sigma(x)$ is in $\mathbb{L}(p,q,\boldsymbol{f}-r,s,t)$, where $p+t = z-1$ and $s+t = \boldsymbol{g}$. Definition 2.3 shows that $d \circ \sigma(x) \equiv 0$. For the second case, we take $x \in \mathbb{U}(p,q,r)$ for some (p,q,r) in $S_{\mathbb{U}}^{(z)}$. It is clear that $d \circ \sigma(x)$ is congruent to an element of $\sum \mathbb{L}(*,*,*,*,t)$, with $q \le t \le z-1$. The hypothesis $z \le q$ ensures that $d \circ \sigma(x) \equiv 0$. For the third case, we take $x \in \mathbb{T}(p,q,r)$ for some (p,q,r) in $S_{\mathbb{T}}^{(z)}$. We see that $d \circ \sigma(x)$ is congruent to an element of $\mathbb{U}(*,*,t)$ with $\boldsymbol{f} - p + t - q - r = 0$. The hypothesis on (p,q,r) yields that $t \le -1$; hence $d \circ \sigma(x) \equiv 0$ in $\mathbb{I}^{(z)}$. Finally, we take $x \in \mathbb{S}(p,q,r,z)$ for some (p,q,r) in $S_{\mathbb{S}}^{(z)}$. The definition of $\mathbb{S}$ in 4.1 and 5.4 shows that $\sigma(x)$ is killed by the part of d which does not involve u, v, or X. $\square$

8. Exactness.

The ultimate proof of exactness occurs in Theorem 8.6. It is an induction on $\boldsymbol{g}$ and is derived from the short exact sequence of complexes of Proposition 3.6. This part of the argument closely follows the proof of [13, Thm. 7.36]. The base case, $\boldsymbol{g}=\boldsymbol{f}-1$, requires a substantial calculation, which is the content of section 9. If $\boldsymbol{g}\le\boldsymbol{f}-2$, then Example 8.16 shows that $\mathbb{I}^{(0)}$ is not acyclic.

Theorem 8.1. *Adopt Data 1.2. If $\boldsymbol{g}=\boldsymbol{f}-1$, $-1\le z$, and the data (u,X,v) is generic (in the sense of Convention 1.5.b), then $\mathbb{I}^{(z)}$ is acyclic.*

Proof. The proof is by induction on $\operatorname{rank} F$. If $\boldsymbol{f}=1$, then $\mathbb{I}^{(z)}$ is not very interesting, but it is acyclic; see Example 2.5. If $\boldsymbol{f}=2$ and $-1\le z\le 1$, then Examples 2.7 and 2.8 show that $\mathbb{I}^{(z)}$ is acyclic. For $2\le z$, $\mathbb{I}^{(z)}$ is homologically equivalent to $\mathbb{M}^{(z)}$, which has length three. Thus, according to the acyclicity lemma, it suffices to show that $\mathbb{I}^{(z)}_P$ is acyclic for each prime ideal P of grade 2. The ideal $I_1(X)+I_1(v)$ has grade 4; thus, either $I_1(X)\not\subseteq P$ (in which case, the main argument applies), or $I_1(v)\not\subseteq P$, (in which case, Lemma 8.2 applies).

Henceforth, we take $3\le\boldsymbol{f}$; and, we assume, by induction, that the result holds for $\operatorname{rank} F=\boldsymbol{f}-1$. The complexes $\mathbb{I}^{(z)}$ and $\mathbb{M}^{(z)}$ are homologically equivalent. Corollary 4.11 shows that the length of $\mathbb{M}^{(z)}$ is at most

$$\boldsymbol{g}+2\boldsymbol{f}-2=3(\boldsymbol{f}-1)\le\boldsymbol{f}(\boldsymbol{f}-1)=\operatorname{grade} I_1(X).$$

By the acyclicity lemma [7, Cor. 4.2], it suffices to show that $\mathbb{I}^{(z)}_x$ is acyclic for each entry x of X. Fix such an x. There exists R_x-module isomorphisms $\psi_1\colon F_x\to F_x$ and $\psi_2\colon G_x\to G_x$ such that $\psi_2^{-1}\circ X\circ\psi_1$ is equal to $\left[\begin{smallmatrix}X'&0\\0&1\end{smallmatrix}\right]$ for some $\boldsymbol{g}-1\times\boldsymbol{f}-1$ matrix X'. Let $\widetilde{\mathbb{I}}$ denote $\mathbb{I}^{(z)}[\psi_2^*(u),\psi_2^{-1}\circ X\circ\psi_1,\psi_1^{-1}(v)]$. Observe that there exists a subring R_1 of R_x such that R_x is a polynomial ring with coefficient ring equal to R_1 and indeterminates given by the the entries of $\psi_2^*(u)$, X', and $\psi_1^{-1}(v)$ (including the last entry from each of $\psi_2^*(u)$ and $\psi_1^{-1}(v)$). Lemma 8.3 shows that $\mathbb{I}^{(z)}_x$ is isomorphic to $\widetilde{\mathbb{I}}$. If the entries of $\psi_2^*(u)$ are $u_1',\dots,u_{\boldsymbol{g}}'$ and the entries of $\psi_1^{-1}(v)$ are $v_1',\dots,v_{\boldsymbol{f}}'$, then let

$$u'=[\,u_1',\quad\dots,\quad u_{\boldsymbol{g}-1}'\,]\quad\text{and}\quad v'=\begin{bmatrix}v_1'\\ \vdots\\ v_{\boldsymbol{f}-1}'\end{bmatrix}.$$

Let $\mathbb{I}'$ represent the complex $\mathbb{I}^{(z)}[u',X',v']$. The induction hypotheses ensures that $\mathbb{I}'$ is acyclic. Theorem 9.1 shows that $\widetilde{\mathbb{I}}$ is homologically equivalent to the tensor product

$$\mathbb{I}'\otimes\left(0\to R_x\xrightarrow{\begin{bmatrix}-v_{\boldsymbol{f}}'\\ u_{\boldsymbol{g}}'\end{bmatrix}}R_x\oplus R_x\xrightarrow{[\,u_{\boldsymbol{g}}'\quad v_{\boldsymbol{f}}'\,]}R_x\right).$$

The indeterminates $u_{\boldsymbol{g}}',v_{\boldsymbol{f}}'$ form a regular sequence on $H_0(\mathbb{I}')$; hence, $\widetilde{\mathbb{I}}$ is acyclic. $\square$

Lemma 8.2. *Adopt Data 1.2 with $\boldsymbol{f}=2$, $\boldsymbol{g}=1$, and $2\le z$. If $I_1(X^*(u))+I_1(X(v))$ is an ideal of grade 2 and $I_1(v)=R$, then $\mathbb{I}^{(z)}$ is exact.*

Proof. We use the notation of Conventions 1.4 and 1.5.a. No harm is done if we assume that $v=f^{[2]}$. Under this assumption, $v(\omega_{F^*})=\varphi^{[1]}$. First we fix z, with $3\le z$. Recall the index set $T_{\mathbb{L}}^{(z)}$ from Definition 2.3. Observe that $(p,1,r,s,t)$ is in $T_{\mathbb{L}}^{(z)}$ if and only if (p,r,s,t) is taken from the following list:

$$(z-3,0,0,2),\ (z-2,0,1,1),\ (z-2,1,0,1),\ (z-1,1,1,0),\ (z-2,0,0,1),\quad \text{and}\quad (z-1,0,1,0).$$

For each such (p,r,s,t), let

$$\mathbb{C}[p,r,s,t]=\mathbb{L}(p-1,2,r,s,t)\oplus\mathbb{L}(p,2,r,s,t)\oplus\mathbb{L}(p,1,r,s,t).$$

Observe that $\mathbb{I}^{(z)}=\bigoplus\mathbb{C}[p,r,s,t]$, where the sum is taken over the above list. Filter $\mathbb{I}^{(z)}$ by taking $(p',q',r',s',t')<(p,q,r,s,t)$, whenever

$$t'<t;\quad \text{or else,}\quad t'=t \text{ and } r'+s'<r+s.$$

Observe that d is a non-increasing map. Let $d^{[0]}$ be the component of d which is homogeneous with respect to the above filtration. We see that $d^{[0]}$ carries each $\mathbb{C}[p,r,s,t]$ to itself and that the map

$$\mathbb{L}(p-1,2,r,s,t)\oplus\mathbb{L}(p,2,r,s,t)\xrightarrow{d^{[0]}}\mathbb{L}(p,1,r,s,t)$$

is one-to-one and has cokernel $\varphi^{[2]p}\otimes\varphi^{[2]}\otimes\bigwedge^r F\otimes\bigwedge^s G\otimes\nu^{(t)}$. Let $\widehat{\mathbb{N}}$ represent $\bigoplus\mathbb{L}(p,2,r,s,t)$, where the sum is taken over all (p,r,s,t) such that $(p,2,r,s,t)$ is in $T_{\mathbb{L}}^{(z)}$, and let $\mathbb{N}$ be the subcomplex $\widehat{\mathbb{N}}+d(\widehat{\mathbb{N}})$ of $\mathbb{I}^{(z)}$. It is easy to see that $d(\widehat{\mathbb{N}})\subseteq \operatorname{im} d^{[0]}+\widehat{\mathbb{N}}$; thus, $\mathbb{N}$ is equal to $\widehat{\mathbb{N}}+\operatorname{im} d^{[0]}$; and therefore, $\mathbb{N}$ is split exact. Let $\overline{}$ represent mod $\mathbb{N}$. We see that $\overline{\mathbb{I}}^{(z)}$ looks like

$$0\to\overline{\mathbb{L}}(z-3,1,0,0,2)\xrightarrow{\overline{d}_3}\begin{matrix}\overline{\mathbb{L}}(z-2,1,1,0,1)\\ \oplus\\ \overline{\mathbb{L}}(z-2,1,0,1,1)\end{matrix}\xrightarrow{\overline{d}_2}\begin{matrix}\overline{\mathbb{L}}(z-2,1,0,0,1)\\ \oplus\\ \overline{\mathbb{L}}(z-1,1,1,1,0)\end{matrix}\xrightarrow{\overline{d}_1}\overline{\mathbb{L}}(z-1,1,0,1,0).$$

If we take bases: $\varphi^{[2]p}\otimes\varphi^{[2]}\otimes 1\otimes 1\otimes\nu^{(t)}$ for $\overline{\mathbb{L}}(p,1,0,s,t)$, and

$$\varphi^{[2]p}\otimes\varphi^{[2]}\otimes f^{[1]}\otimes 1\otimes\nu^{(t)},\quad \varphi^{[2]p}\otimes\varphi^{[2]}\otimes f^{[2]}\otimes 1\otimes\nu^{(t)}$$

for $\overline{\mathbb{L}}(p,1,1,s,t)$, then

$$\overline{d}_3=\begin{bmatrix}0\\-1\\x_2\end{bmatrix},\quad \overline{d}_2=\begin{bmatrix}-u_1x_1&-u_1x_2&-u_1\\-x_2&0&0\\0&-x_2&-1\end{bmatrix},\quad\text{and}\quad \overline{d}_1=\begin{bmatrix}x_2&-u_1x_1&-u_1x_2\end{bmatrix}.$$

It is clear that $\overline{\mathbb{I}}^{(z)}$ is acyclic.

The only modification which is required when $z=2$ is that $\overline{\mathbb{I}}^{(z)}$ now has $\overline{\mathbb{U}}(0,1,1)$ in place of $\overline{\mathbb{L}}(z-3,1,0,0,2)$. We take $1\otimes\omega_{G^*}\otimes\mu^{(1)}$ to be the basis for $\overline{\mathbb{U}}(0,1,1)$. The matrix for $\overline{d}_3$ remains unchanged. $\square$

Lemma 8.3. *Adopt Data 1.2. If $\psi\colon F \to F$ is an isomorphism, then $\mathbb{I}^{(z)}[u, X, v]$ is isomorphic to $\mathbb{I}^{(z)}[u, X \circ \psi, \psi^{-1}(v)]$. If $\psi\colon G \to G$ is an isomorphism, then $\mathbb{I}^{(z)}[u, X, v]$ is isomorphic to $\mathbb{I}^{(z)}[\psi^*(u), \psi^{-1} \circ X, v]$.*

Proof. In the first case, define $\Psi\colon \mathbb{I}^{(z)}[u, X, v] \to \mathbb{I}^{(z)}[u, X \circ \psi, \psi^{-1}(v)]$ by

$$\begin{aligned}
&\Psi(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)}) = (S_p\psi^*)(A_p) \otimes (\textstyle\bigwedge^q \psi^*)(\alpha_q) \otimes (\textstyle\bigwedge^r \psi^{-1})(b_r) \otimes c_s \otimes \nu^{(t)},\\
&\Psi(b_p \otimes \delta_q \otimes \mu^{(r)}) = (\textstyle\bigwedge^p \psi^{-1}) b_p \otimes \delta_q \otimes \mu^{(r)},\\
&\Psi(\alpha_p \otimes c_q \otimes \lambda^{(r)}) = (\textstyle\bigwedge^p \psi^*) \alpha_p \otimes c_q \otimes \lambda^{(r)}, \text{ and}\\
&\Psi(B_p \otimes b_q \otimes \alpha_r \otimes \delta_s \otimes \xi^{(t)} = (D_p\psi^{-1})(B_p) \otimes (\textstyle\bigwedge^q \psi^{-1})(b_q) \otimes (\textstyle\bigwedge^r \psi^*)(\alpha_r) \otimes \delta_s \otimes \xi^{(t)}.
\end{aligned}$$

In the second case, define $\Psi\colon \mathbb{I}^{(z)}[u, X, v] \to \mathbb{I}^{(z)}[\psi^*(u), \psi^{-1} \circ X, v]$ by

$$\begin{aligned}
&\Psi(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)}) = A_p \otimes \alpha_q \otimes b_r \otimes (\textstyle\bigwedge^s \psi^{-1})(c_s) \otimes \nu^{(t)},\\
&\Psi(b_p \otimes \delta_q \otimes \mu^{(r)}) = b_p \otimes (\textstyle\bigwedge^q \psi^*)(\delta_q) \otimes \mu^{(r)},\\
&\Psi(\alpha_p \otimes c_q \otimes \lambda^{(r)}) = \alpha_p \otimes (\textstyle\bigwedge^q \psi^{-1})(c_q) \otimes \lambda^{(r)}, \text{ and}\\
&\Psi(B_p \otimes b_q \otimes \alpha_r \otimes \delta_s \otimes \xi^{(t)}) = B_p \otimes b_q \otimes \alpha_r \otimes (\textstyle\bigwedge^s \psi^*)(\delta_s) \otimes \xi^{(t)}.
\end{aligned}$$

Each map Ψ is an isomorphism of complexes. □

We use the following well-known result many times.

Observation 8.4. *Let I and $J = (r_1, \dots, r_n)$ be ideals in the commutative noetherian ring R, and f be the element $\sum_{j=1}^n r_j x_j$ of the polynomial ring $R[x_1, \dots, x_n]$. If $m \le \operatorname{grade} I$ and $m + 1 \le \operatorname{grade} I + J$, then $m + 1 \le \operatorname{grade} I + (f)$.*

Proof. We may mod out by a regular sequence of length m in I; and therefore, it suffices to treat the case where $m = 0$. Suppose that $1 \le \operatorname{grade} I + J$, but $0 = \operatorname{grade} I + (f)$. Then there is a non-zero polynomial g in $R[x_1, \dots, x_n]$ such that $gI = 0$ and $gf = 0$. Use the argument of [18, (6.13), p. 17] to find a non-zero element of R which annihilates $I + J$, and thereby reach a contradiction. □

We now collect the grade estimates which are used in the proof of Theorem 8.6. Most of these estimates may be found elsewhere in the literature.

Lemma 8.5. *Adopt 1.2 with data which is generic in the sense of Convention 1.5.b. Assume that $0 \le \boldsymbol{f} - 1 \le \boldsymbol{g}$. Let K be the $R-$ideal $I_1(uX) + I_{\boldsymbol{f}}(X) + I_1(Xv)$, and X' be the submatrix of X consisting of columns 2 to $\boldsymbol{f}$. The following statements hold:*

(a) *$\boldsymbol{f} + \boldsymbol{g} - 1 \le \operatorname{grade} K$;*
(b) *$\boldsymbol{f} + \boldsymbol{g} \le \operatorname{grade} K + I_1(v)$;*
(c) *$\boldsymbol{f} + \boldsymbol{g} \le \operatorname{grade} K + (v_1) + I_{\boldsymbol{f}-1}(X')$; and*
(d) *if $1 \le t \le \boldsymbol{f} - 1$, then $2\boldsymbol{g} + 1 - t \le \operatorname{grade} I_1(uX) + I_t(X)$.*

Proof. One can prove (a) by mimicking the proof of [5, Prop. 4.2.a]. (An alternate proof is given in [4, Theorem 1.2], provided $\boldsymbol{f} \le \boldsymbol{g}$.) Assertion (b) is clear because $I_1(uX) + I_{\boldsymbol{f}}(X)$ has grade $\boldsymbol{g}$. We now prove (c). Let v' be v with row 1 deleted. Assertion (a) guarantees that $\boldsymbol{g} + \boldsymbol{f} - 2 \le \operatorname{grade} I_1(uX') + I_{\boldsymbol{f}-1}(X') + I_1(X'v')$. Also, $\boldsymbol{g} + \boldsymbol{f} - 1 \le 2\boldsymbol{g} \le \operatorname{grade} I_1(u) + I_{\boldsymbol{f}-1}(X') + I_1(X'v')$; hence, Observation 8.4 yields that $\boldsymbol{g} + \boldsymbol{f} - 1 \le \operatorname{grade} I_1(uX) + I_{\boldsymbol{f}-1}(X') + I_1(X'v')$. (The entries of the first column of X play the role of the new indetermiates.) The proof of (c) is complete because v_1 is yet another new indeterminate. The argument of [13, Lemma 7.33.b] establishes (d). □

Theorem 8.6. *Adopt Data 1.2 with $\boldsymbol{f}-1\le \boldsymbol{g}$ and $-1\le z$. If the data (u,X,v) is generic (in the sense of Convention 1.5.b), then $\mathbb{I}^{(z)}$ is acyclic and $H_0(\mathbb{I}^{(z)})$ is isomorphic to an ideal of $H_0(\mathbb{I}^{(0)})$.*

Proof. Take $\widetilde{u}$, $\widetilde{X}$, and $\overline{u}$ as in Proposition 3.6. Let $\mathbb{I}^{(z)}$, K, $\mathfrak{b}_3$, $\mathfrak{p}_2$, N, and $\mathfrak{N}$ be the complexes, ideals, and modules which are created in Definitions 2.3 and 3.1 and Observation 3.3 using the data (u,X,v). Use the data $(\widetilde{u},\widetilde{X},v)$ to create $\widetilde{\mathbb{I}}^{(z)}$, $\widetilde{K}$, and $\widetilde{\mathfrak{b}}_3$; and use $(\overline{u},X,v)$ to create $\overline{\mathbb{I}}^{(z)}$ and $\overline{K}$. The proof proceeds by induction on $\boldsymbol{g}$. Theorem 8.1 takes care of the acyclicity of $\mathbb{I}^{(z)}$ when $\boldsymbol{f}-1=\boldsymbol{g}$. We prove that $H_0(\mathbb{I}^{(z)})$ is isomorphic to an ideal of R/K at the end of the proof. In the mean time, we assume that $\boldsymbol{f}\le\boldsymbol{g}$. Let $w=\sum_{i=1}^{\boldsymbol{f}} x_{\boldsymbol{g}\,i}v_i$. Observe that the composition

$$G^*\xrightarrow{X^*}F^*\longrightarrow\mathfrak{b}_3$$

of Observation 3.3 carries $\gamma^{[\boldsymbol{g}]}$ to w.

The induction hypothesis guarantees that $\widetilde{\mathbb{I}}^{(z)}$ is acyclic for all z with $-1\le z$. Therefore, the long exact sequence of Proposition 3.6 yields $H_j(\overline{\mathbb{I}}^{(z)})=0$ for all z and j with $2\le j$ and $0\le z$. The following observations are necessary before we consider $H_1(\overline{\mathbb{I}}^{(z)})$.

(8.7) The $R-$ideal $\widetilde{K}$ is perfect of grade $\boldsymbol{f}+\boldsymbol{g}-2$.
(8.8) The $R/\widetilde{K}-$ideal $\widetilde{\mathfrak{b}}_3$ has positive grade.
(8.9) The element w is regular on $R/\widetilde{K}$.

The induction hypothesis gives that $\widetilde{\mathbb{I}}^{(0)}$ is acyclic; hence $\operatorname{pd}R/\widetilde{K}\le\boldsymbol{f}+\boldsymbol{g}-2$. On the other hand, Lemma 8.5.a gives $\boldsymbol{f}+\boldsymbol{g}-2\le\operatorname{grade}\widetilde{K}$. When we combine these two inequalities we obtain $\operatorname{pd}R/\widetilde{K}\le\boldsymbol{f}+\boldsymbol{g}-2\le\operatorname{grade}\widetilde{K}$; and thereby establish (8.7). Assertion (8.8) follows from (8.7) because of part (b) of Lemma 8.5. Observation 8.4 ensures (8.9).

We saw in Observation 3.3 that there is an $R/\widetilde{K}-$module surjection

$$H_0(\widetilde{\mathbb{I}}^{(z)})=S_z(\widetilde{N})\twoheadrightarrow\widetilde{\mathfrak{b}}_3^z \tag{8.10}$$

for all z with $0\le z$. Since $\widetilde{\mathfrak{b}}_3$ has positive grade, and $H_0(\widetilde{\mathbb{I}}^{(z)})$ is isomorphic to an ideal of $R/\widetilde{K}$ (by induction), we conclude that (8.10) is an isomorphism. When the isomorphism of (8.10) is applied to the exact sequence of Proposition 3.6, we obtain the exact sequence

$$0=H_1(\widetilde{\mathbb{I}}^{(z)})\to H_1(\overline{\mathbb{I}}^{(z)})\to\widetilde{\mathfrak{b}}_3^{z-1}\xrightarrow{w}\widetilde{\mathfrak{b}}_3^{z};$$

thus, $H_1(\overline{\mathbb{I}}^{(z)})=0$ for all z with $1\le z$.

The ideal $\overline{K}$ also contains w; consequently, the same argument as above yields that the surjection $H_0(\widetilde{\mathbb{I}}^{(-1)})\twoheadrightarrow\overline{K}/\widetilde{K}$ of Proposition 3.6 is also an isomorphism. It follows that $H_1(\overline{\mathbb{I}}^{(0)})=0$; and therefore, $\overline{\mathbb{I}}^{(z)}$ is acyclic for all $0\le z$. The complex $\overline{\mathbb{I}}^{(\boldsymbol{g}-\boldsymbol{f}+1)}$ has length $\boldsymbol{g}+\boldsymbol{f}-1$ (see Corollary 4.11), and it resolves a prefect $R-$module of projective dimension $\boldsymbol{g}+\boldsymbol{f}-1$; thus, $\overline{\mathbb{I}}^{(-1)} \cong \left(\overline{\mathbb{I}}^{(\boldsymbol{g}-\boldsymbol{f}+1)}\right)^* [-(\boldsymbol{g}+\boldsymbol{f}-1)]$ is also acyclic.

View R as a graded ring where each element of R_0 has degree zero and every entry of X, u, and v has degree one. The short exact sequence

$$0 \to R \xrightarrow{u_{\boldsymbol{g}}} R \to R/(u_{\boldsymbol{g}}) \to 0$$

induces a short exact sequence of graded complexes

$$0 \to \mathbb{I}^{(z)} \xrightarrow{u_{\boldsymbol{g}}} \mathbb{I}^{(z)} \to \overline{\mathbb{I}}^{(z)} \to 0.$$

The corresponding long exact sequence of homology yields that multiplication by $u_{\boldsymbol{g}}$ is an automorphism of $H_j(\mathbb{I}^{(z)})$ for all i and j with $1 \le j$ and $-1 \le z$. Since the homology of $\mathbb{I}^{(z)}$ is finitely generated and graded, and $u_{\boldsymbol{g}}$ has positive degree, we conclude that $\mathbb{I}^{(z)}$ is acyclic for $-1 \le z$.

It remains to show that $H_0(\mathbb{I}^{(z)})$ is isomorphic to an ideal of R/K whenever $\boldsymbol{f}-1 \le \boldsymbol{g}$. Fix $1 \le z$. It is easy to see that the $R/K-$module $H_0(\mathbb{I}^{(z)})$ has rank one. Indeed, if P is an associated prime of R/K, then (8.8) gives $I_1(v) \nsubseteq P$; hence, Example 8.14 shows that $H_0(\mathbb{I}^{(z)})_P = (R/K)_P$. Let j be an integer with $\boldsymbol{g}+\boldsymbol{f} \le j \le 2\boldsymbol{f}+\boldsymbol{g}-2$, and let F_j be the radical of the $R-$ideal generated by

$$\{x \in R \mid \operatorname{pd}_{R_x} H_0(\mathbb{I}^{(z)})_x < j\}.$$

A quick look at Example 8.15 shows that if Δ is a $t \times t$ minor of X, then

$$\operatorname{pd} H_0(\mathbb{I}^{(z)})_\Delta \le 2\boldsymbol{f}+\boldsymbol{g}-2-t.$$

It follows that $I_1(v)+I_1(uX)+I_{2\boldsymbol{f}+\boldsymbol{g}-1-j}(X) \subseteq F_j$. Apply Lemma 8.5.d to see that

$$j+1 \le j+1+(\boldsymbol{g}+1-\boldsymbol{f}) \le \operatorname{grade} F_j. \tag{8.11}$$

It follows that $H_0(\mathbb{I}^{(z)})$ is a torsion-free $R/K-$module. We conclude that the surjection

$$\mathrm{H}_0(\mathbb{I}^{(z)}) \twoheadrightarrow \mathfrak{b}_3^z \tag{8.12}$$

is an isomorphism. Finally, we consider the case $z=-1$. We have seen that $H_0(\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}+1)})$ is a perfect $R-$module of projective dimension $\boldsymbol{g}+\boldsymbol{f}-1$, and that

$$H_0(\mathbb{I}^{(-1)}) = \operatorname{Ext}_R^{\boldsymbol{g}+\boldsymbol{f}-1}(H_0(\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}+1)}), R). \tag{8.13}$$

It follows that $H_0(\mathbb{I}^{(-1)})$ is a torsion-free $R/K-$module. If $P \in \operatorname{Ass}(R/K)$, then Example 8.14 shows that $H_0(\mathbb{I}^{(\boldsymbol{g}-\boldsymbol{f}+1)})_P$ is obtained from R_P by modding out a regular sequence of length $\boldsymbol{g}+\boldsymbol{f}-1$; thus, (8.13) yields that $H_0(\mathbb{I}^{(-1)})$ has rank one. Recall, from Observation 3.3, that there is an $R/K-$module surjection

$$H_0(\mathbb{I}^{(-1)}) \twoheadrightarrow \mathfrak{p}_2.$$

The $R/K-$ideal on the right side has positive grade. It follows that this surjection is an isomorphism. $\square$

Remarks. (a) If $2 \le \boldsymbol{f} \le \boldsymbol{g}$, then the ideal $\mathfrak{a}_2$ of Theorem 0.3 has positive grade; hence, the argument surrounding (8.12) also yields $H_0(\mathbb{I}^{(z)}) \cong \mathfrak{a}_2^z$.
(b) The inequality of (8.11) is the best possible. Indeed, Example 2.8 shows that if $(\boldsymbol{g},\boldsymbol{f}) = (1,2)$, then $\operatorname{grade} F_3 = 4$.

Example 8.14. Assume $\boldsymbol{f}-1\le\boldsymbol{g}$. If R is local and $I_{\boldsymbol{f}-1}(X)=R$ or $I_1(v)=R$, then $H_0(\mathbb{I}^{(z)})=R/K$ for $0\le z$. Indeed, in the first case, one may choose the bases for F and G so that the matrix of X is

$$\left[\begin{array}{c|c} I_{\boldsymbol{f}-1} & 0\\ \hline & x_{\boldsymbol{ff}}\\ 0 & \vdots\\ & x_{\boldsymbol{gf}}\end{array}\right].$$

It readily follows that $K=(u_1,\dots,u_{\boldsymbol{f}-1},x_{\boldsymbol{ff}},\dots,x_{\boldsymbol{gf}},v_1,\dots,v_{\boldsymbol{f}-1})$ and $N=R/K$. In the second case, one may choose the bases for F and G so that the matrix of v is $\left[\begin{smallmatrix}1&0&\dots&0\end{smallmatrix}\right]^{\mathrm{t}}$. It readily follows that $K=(x_{11},\dots,x_{\boldsymbol{g}1},(uX)_2,\dots(uX)_{\boldsymbol{f}})$, and $N=R/K$.

Example 8.15. Suppose that T_1 and T_2 are indeterminates over R. Let

$$\mathbf{u}=[\,u\quad T_1\,],\quad \mathbf{X}=\left[\begin{smallmatrix}X&0\\0&1\end{smallmatrix}\right],\quad \mathbf{v}=\left[\begin{smallmatrix}v\\T_2\end{smallmatrix}\right],\quad\text{and}\quad \mathbf{N}=N(\mathbf{u},\mathbf{X},\mathbf{v}).$$

It is clear that the $R[T]-$ideal $K(\mathbf{u},\mathbf{X},\mathbf{v})$ is equal to $K+(T_1,T_2)$. Furthermore, if the $R-$module N is viewed as an $R[T_1,T_2]-$module by way of the ring homomorphism

$$R[T_1,T_2]\to R[T_1,T_2]/(T_1,T_2)=R,$$

then $\mathbf{N}$ and N are isomorphic as $R[T_1,T_2]-$modules. Consequently, if the R-module $S_z(N)$ has finite projective dimension, then $\operatorname{pd}_{R[T_1,T_2]}S_z(\mathbf{N})=\operatorname{pd}_R S_z(N)+2$.

Example 8.16. Adopt Data 1.2 with generic data. If $\boldsymbol{g}\le\boldsymbol{f}-2$, then $H_1(\mathbb{I}^{(0)})\neq 0$. Indeed, $z=[(\bigwedge^{\boldsymbol{g}}X^*)(\omega_{G^*})](b_{\boldsymbol{g}+1})\otimes 1\otimes\mu^{(0)}\in\mathbb{U}(1,0,0)$ is a cycle in $\mathbb{I}^{(0)}$. On the other hand, the only summands of $\mathbb{I}^{(0)}$ which might map to this cycle have the form $\mathbb{T}(p,q,r)$, with (p,q,r) in the set $T_{\mathbb{T}}^{(0)}$ from Definition 2.3; in particular, $\boldsymbol{f}-\boldsymbol{g}\le r$. Furthermore, d of $\mathbb{T}(p,q,r)$ is contained in $\mathbb{T}^{(0)}\oplus\bigoplus_t\mathbb{U}(q+r-t,\boldsymbol{g}-\boldsymbol{f}+p-t+r,t)$. Thus, if $\mathbb{T}(p,q,r)$ maps to $\mathbb{U}(1,0,0)$, then $q+r=1$. It follows that z represents a non-zero element of homology whenever $\boldsymbol{g}\le\boldsymbol{f}-2$.

9. The case $\boldsymbol{g}=\boldsymbol{f}-1$.

Theorem 9.1 is the main calculation in the proof of Theorem 8.1.

Theorem 9.1. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be free modules of rank* $\boldsymbol{f}$ *and* $\boldsymbol{g}$*, respectively, over the commutative noetherian ring* R*. Let* $\mathbf{u}\in\mathbf{G}^*$*,* $\mathbf{v}\in\mathbf{F}$*,* $\mathbf{X}\colon\mathbf{F}\to\mathbf{G}$ *be an* $R-$*module homomorphism, and* $(\mathbb{I}^{(z)},\mathbf{d})$ *be the complex which is constructed using the data* $(\mathbf{u},\mathbf{X},\mathbf{v})$*. Suppose that the above data decomposes as* $\mathbf{F}=F\oplus R\mathbf{f}$*,* $\mathbf{F}^*=F^*\oplus R\boldsymbol{\phi}$*,* $\mathbf{G}=G\oplus R\mathbf{g}$*,* $\mathbf{G}^*=G^*\oplus R\boldsymbol{\gamma}$*,*

$$\mathbf{X}=\begin{bmatrix}X&0\\0&1\end{bmatrix},\quad \mathbf{u}=[\,u\quad u_{\boldsymbol{g}}\,],\quad\textit{and}\quad \mathbf{v}=\begin{bmatrix}v\\v_{\boldsymbol{f}}\end{bmatrix},$$

where F *and* G *are free* $R-$*modules of rank* $\boldsymbol{f}-1$ *and* $\boldsymbol{g}-1$*, respectively,* $\boldsymbol{\phi}(F)=0$*,* $\boldsymbol{\phi}(\mathbf{f})=1$*,* $\mathbf{f}(F^*)=0$*,* $\boldsymbol{\gamma}(G)=0$*,* $\boldsymbol{\gamma}(\mathbf{g})=1$*, and* $\mathbf{g}(G^*)=0$*. Let* $(\mathbb{I}^{(z)},d)$ *be the* *complex which is constructed using the data (u,X,v), and let $(\mathbb{Y},y)$, be the total complex which is associated to the double complex*

$$0\to \mathbb{I}^{(z)} \xrightarrow{\begin{bmatrix}-v_{\boldsymbol{f}}\\ u_{\boldsymbol{g}}\end{bmatrix}} \mathbb{I}^{(z)}\oplus\mathbb{I}^{(z)} \xrightarrow{\begin{bmatrix}u_{\boldsymbol{g}} & v_{\boldsymbol{f}}\end{bmatrix}} \mathbb{I}^{(z)}\to 0.$$

If $-1\le z$ and $\boldsymbol{f}-1=\boldsymbol{g}$, then the complexes $(\mathbf{I}^{(z)},\mathbf{d})$ and $(\mathbb{Y},y)$ are homologically equivalent.

Remark. The result remains true if the hypothesis $\boldsymbol{f}-1=\boldsymbol{g}$ is replaced by $\boldsymbol{f}-1\le\boldsymbol{g}$; however the proof becomes more complicated.

Proof. The hypothesis $\boldsymbol{f}-1=\boldsymbol{g}$ affords two immediate simplifications. First of all, every $\mathbb{W}-$type summand of $\mathbb{I}^{(z)}$ and $\mathbf{I}^{(z)}$ is zero. Indeed, if (p,q,r,s,t) is in the set $T_{\mathbb{W}}^{(z)}$ of Definition 2.3, then $2\boldsymbol{f}+z+1\le s+\boldsymbol{f}$; hence, $\boldsymbol{g}<s$. In a similar manner, the set $T_{\mathbb{T}}^{(z)}$ becomes $\{(p,q,r)\mid p+q+r\le \boldsymbol{f}-1 \quad\text{and}\quad 1+z\le r\}$. The complex $\mathbb{I}^{(z)}$ is formed using components $\mathbb{L}(p,q,r,s,t)$, $\mathbb{U}(p,q,r)$, and $\mathbb{T}(p,q,r)$, and constant

$$\sigma_z(p,q,r,t)=(-1)^{r+z+q+q\boldsymbol{f}}\theta_q\theta_p\tbinom{\boldsymbol{f}-2-p-q-r+t}{r-1-z}.$$

The complex $\mathbf{I}^{(z)}$ is formed using components $\mathbf{L}(p,q,r,s,t)$, $\mathbf{U}(p,q,r)$, and $\mathbf{T}(p,q,r)$, and constant

$$\boldsymbol{\sigma}_z(p,q,r,t)=(-1)^{r+z+q\boldsymbol{f}}\theta_q\theta_p\tbinom{\boldsymbol{f}-1-p-q-r+t}{r-1-z}.$$

For integers a, b, c, and d with $0\le a$, and $0\le b,c,d\le 1$, let

$$\begin{aligned}
&\mathbf{L}(p,q,r,s,t;a,b,c,d)=[S_pF^*\cdot\boldsymbol{\phi}^a]\otimes\left[\textstyle\bigwedge^q F^*\wedge\boldsymbol{\phi}^b\right]\otimes\left[\textstyle\bigwedge^r F\wedge\mathbf{f}^c\right]\otimes\left[\textstyle\bigwedge^s G\wedge\mathbf{g}^d\right]\otimes\nu^{(t)},\\
&\mathbf{U}(p,q,r;b,c)=\left[\textstyle\bigwedge^p F\wedge\mathbf{f}^b\right]\otimes\left[\textstyle\bigwedge^q G^*\wedge\boldsymbol{\gamma}^c\right]\otimes\mu^{(r)},\ \text{and}\\
&\mathbf{T}(p,q,r;b,c)=\left[\textstyle\bigwedge^p F^*\wedge\boldsymbol{\phi}^b\right]\otimes\left[\textstyle\bigwedge^q G\wedge\mathbf{g}^c\right]\otimes\lambda^{(r)}.
\end{aligned}$$

Let $\mathbf{U}(b,c)$ be the direct sum of all submodules of $\mathbf{I}^{(z)}$ of the form $\mathbf{U}(p,q,r;b,c)$. The symbols $\mathbf{L}(a,b,c,d)$ and $\mathbf{T}(b,c)$ are given meaning in the analogous manner. Define submodules $\mathbb{A}$, $\mathbb{C}$, $\mathbb{E}$, and $\mathbb{J}$ of $\mathbf{I}^{(z)}$ by

$$\mathbb{A}=\begin{cases}
\bigoplus_{\{(p,q,r,s,t)\mid p+t\le z-1,\quad z\le p+q+t,\quad 2\boldsymbol{f}-3=r+s+t\}}\mathbf{L}(p,q,r,s,t;0,0,0,1)\\
\oplus\bigoplus_{\{(p,q,r,s,t)\mid p+t\le z-1,\quad z-1\le p+q+t,\quad 2\boldsymbol{f}-3=r+s+t\}}\mathbf{L}(p,q,r,s,t;0,1,0,1)\\
\oplus\bigoplus_{\{(p,q,r)\mid q\le z-1,\quad z\le q+r,\quad \boldsymbol{f}-1=p+r\}}\mathbf{U}(p,q,r;0,0)\\
\oplus\bigoplus_{\{(p,q,r)\mid q\le z-1,\quad z-1\le q+r,\quad \boldsymbol{f}-1=p+r\}}\mathbf{U}(p,q,r;0,1).
\end{cases}$$

$$\mathbb{C}=\begin{cases}
\bigoplus_{S_{\mathbb{L}}(\text{not})}\mathbf{L}(p,q,r,s,t;a,b,c,d)\\
\oplus\bigoplus_{\{(p,q,r,s,t)\mid r+s+t\le 2\boldsymbol{f}-4,\quad \boldsymbol{f}-2\le s+t,\quad p+q+t=z-1\}}\mathbf{L}(p,q,r,s,t;0,1,0,1)\\
\oplus\bigoplus_{\{(p,q,r,s,t)\mid r+s+t\le 2\boldsymbol{f}-4,\quad \boldsymbol{f}-2\le s+t,\quad p+q+t=z-1\}}\mathbf{L}(p,q,r,s,t;0,1,1,1)\\
\oplus\bigoplus_{\{(p,q,r)\mid 0\le r,\quad p+r\le\boldsymbol{f}-2,\quad q+r=z-1\}}[\mathbf{U}(p,q,r;0,1)\oplus\mathbf{U}(p,q,r;1,1)],\ \text{where}
\end{cases}$$

$$S_{\mathbb{L}}(\text{not})=\left\{(p,q,r,s,t;a,b,c,d)\;\middle|\;\begin{array}{l}p+t+a\le z-1,\quad \boldsymbol{f}-1\le s+t+d,\\ z\le p+q+t+a+b,\\ r+s+t+c+d\le 2\boldsymbol{f}-2,\quad\text{and}\quad (a,d)\ne(0,1)\end{array}\right\},$$

$$\mathbb{E} = \bigoplus_{\{(p,q)\mid p+q=\boldsymbol{f}+z-1\}} \mathbf{U}(p,q,0;0,0) \oplus \bigoplus_{\{(p,q)\mid p+q=\boldsymbol{f}-2-z\}} \mathbf{T}(p,q,z+1;0,0), \text{ and}$$

$$\mathbb{J} = \begin{cases} \bigoplus\limits_{S_{\mathbb{L}}} \mathbf{L}(p,q,r,s,t;0,0,0,1) \oplus \bigoplus\limits_{S_{\mathbb{L}}} \mathbf{L}(p,q,r,s,t;0,0,1,1) \oplus \bigoplus\limits_{S_{\mathbb{L}}} \mathbf{L}(p,q,r,s,t;0,1,0,1) \\ \oplus \bigoplus\limits_{S_{\mathbb{L}}} \mathbf{L}(p,q,r,s,t;0,1,1,1) \oplus \bigoplus\limits_{S_{\mathbb{U}}(0)} \mathbf{U}(p,q,r;0,0) \oplus \bigoplus\limits_{S_{\mathbb{U}}(1)} \mathbf{U}(p,q,r;0,1) \\ \oplus \bigoplus\limits_{S_{\mathbb{U}}(1)} \mathbf{U}(p,q,r;1,0) \oplus \bigoplus\limits_{S_{\mathbb{U}}(2)} \mathbf{U}(p,q,r;1,1) \oplus \bigoplus\limits_{S_{\mathbb{T}}(0)} \mathbf{T}(p,q,r;0,0) \\ \oplus \bigoplus\limits_{S_{\mathbb{T}}(1)} \mathbf{T}(p,q,r;0,1) \oplus \bigoplus\limits_{S_{\mathbb{T}}(1)} \mathbf{T}(p,q,r;1,0) \oplus \bigoplus\limits_{S_{\mathbb{T}}(2)} \mathbf{T}(p,q,r;1,1), \end{cases}$$

where

$$S_{\mathbb{L}} = \left\{ (p,q,r,s,t) \;\middle|\; \begin{array}{l} p+t \le z-1, \quad \boldsymbol{f}-2 \le s+t, \\ z \le p+q+t, \quad \text{and} \quad r+s+t \le 2\boldsymbol{f}-4 \end{array} \right\},$$

$$S_{\mathbb{U}}(0) = \begin{cases} \{(p,q,0) \mid\; p+q \le \boldsymbol{f}-2+z \quad \text{and} \quad z \le q\} \\ \cup \left\{ (p,q,r) \;\middle|\; \begin{array}{l} 1 \le r, \quad p+q+r \le \boldsymbol{f}-1+z, \\ p+r \le \boldsymbol{f}-2, \quad \text{and} \quad z \le q+r \end{array} \right\} \\ \cup \{(p,q,r) \mid\; 1 \le r, \quad p+r = \boldsymbol{f}-1, \quad \text{and} \quad z = q\}, \end{cases}$$

$$S_{\mathbb{U}}(1) = \left\{ (p,q,r) \;\middle|\; \begin{array}{l} 0 \le r, \quad p+q+r \le \boldsymbol{f}-2+z, \\ p+r \le \boldsymbol{f}-2, \quad \text{and} \quad z \le q+r \end{array} \right\},$$

$$S_{\mathbb{U}}(2) = \left\{ (p,q,r) \;\middle|\; \begin{array}{l} 0 \le r, \quad p+q+r \le \boldsymbol{f}-3+z, \\ p+r \le \boldsymbol{f}-2, \quad \text{and} \quad z \le q+r \end{array} \right\},$$

$$\begin{aligned} S_{\mathbb{T}}(0) &= \{(p,q,z+1) \mid p+q \le \boldsymbol{f}-3-z\} \cup \{(p,q,r) \mid p+q+r \le \boldsymbol{f}-1, \quad z+2 \le r,\}, \\ S_{\mathbb{T}}(1) &= \{(p,q,r) \mid p+q+r \le \boldsymbol{f}-2, \quad z+1 \le r,\}, \text{ and} \\ S_{\mathbb{T}}(2) &= \{(p,q,r) \mid p+q+r \le \boldsymbol{f}-3, \quad z+1 \le r\}. \end{aligned}$$

A straightforward, but long, calculation shows that, as a module, $\mathbb{I}^{(z)}$ is equal to the direct sum $\mathbb{A} \oplus \mathbb{C} \oplus \mathbb{E} \oplus \mathbb{J}$. A short calculation yields that $\mathbb{C}$ is a subcomplex of $\mathbb{I}^{(z)}$. The complex $(\mathbb{C}, \mathbf{d})$ is split exact, by Proposition 9.8; and therefore,

$$(\mathbb{I}^{(z)}, \mathbf{d}) \xrightarrow{\text{proj}^{\mathbb{A}\oplus\mathbb{E}\oplus\mathbb{J}}} (\mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J}, \text{proj}^{\mathbb{A}\oplus\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d})$$

is a quasi-isomorphism of complexes. Define

$$M_1 \colon \bigoplus_{p+r=\boldsymbol{f}-1} \mathbf{U}(p,z,r;0,0) \subseteq \mathbb{E} \oplus \mathbb{J} \to \mathbb{A}, \text{ by}$$

$$M_1(b_p \otimes \delta_z \otimes \mu^{(\boldsymbol{f}-1-p)}) = -\sum_{\substack{\boldsymbol{f}-p\le t \\ |I|=\boldsymbol{f}-1-t}} f_I \otimes \left[(\textstyle\bigwedge^{p-\boldsymbol{f}+1+t} X)(\varphi_I[b_p]) \right] (\delta_z) \otimes \mu^{(t)}.$$

(Define M_1 on all of $\mathbb{I}^{(z)}$ by taking M_1 to be zero on all other summands $\mathbf{L}(*;*)$, $\mathbf{U}(*;*)$, and $\mathbf{T}(*;*)$.) Lemma 9.4 shows that $(\mathbb{A}, \text{proj}^{\mathbb{A}} \circ \mathbf{d})$ is a complex and that

$$(\mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J}, \text{proj}^{\mathbb{A}\oplus\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d}) \xrightarrow{\text{proj}^{\mathbb{A}} + M_1} (\mathbb{A}, \text{proj}^{\mathbb{A}} \circ \mathbf{d})$$

is a map of complexes. The complex $(\mathbb{A}, \operatorname{proj}^{\mathbb{A}} \circ \mathbf{d})$ is split exact, by Proposition 9.8; and therefore, Lemma 9.3 yields that

$$(\mathbb{E}\oplus\mathbb{J}, \operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d}\circ(1-M_1)) \xrightarrow{1-M_1} (\mathbb{A}\oplus\mathbb{E}\oplus\mathbb{J}, \operatorname{proj}^{\mathbb{A}\oplus\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d})$$

is a quasi-isomorphism of complexes. Define

$$M_2\colon \bigoplus_{p+q=\boldsymbol{f}+z-1} \mathbb{U}(p,q,0;0,0) \subseteq \mathbb{E} \to \mathbb{J}$$

by $M_2\left(b_p\otimes\delta_q\otimes\mu^{(0)}\right)$ is equal to

$$\begin{cases} \quad\theta_{\boldsymbol{f}-2-q}\theta_{\boldsymbol{f}-1-p}(-1)^{1+z+q\boldsymbol{f}}(v\wedge b_p)(\omega_{F^*})\otimes\delta_q(\omega_G)\otimes\lambda^{(z+1)} \\ +\chi(1\le p)\theta_{\boldsymbol{f}-2-q}\theta_{\boldsymbol{f}-1-p}(-1)^{z+\boldsymbol{f}+q+q\boldsymbol{f}}b_p(\omega_{F^*})\otimes(u\wedge\delta_q)(\omega_G)\otimes\lambda^{(z+1)} \\ +\chi(p\le\boldsymbol{f}-2)\displaystyle\sum_{\substack{0\le t\\ |I|=p-1-t}}(-1)^{q+t}f_I\wedge\mathbf{f}\otimes\left[(\textstyle\bigwedge^{1+t}X)[\varphi_I(b_p)]\right](\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)} \\ -\chi(p\le\boldsymbol{f}-2)\displaystyle\sum_{\substack{0<t\\ |I|=p-t}}f_I\otimes\left[(\textstyle\bigwedge^{t}X)(\varphi_I(b_p))\right](\delta_q)\otimes\mu^{(t)}. \end{cases}$$

(Extend the domain of M_2 to be all of $\mathbb{I}^{(z)}$.) Let $\mathbf{d}_{\mathbb{E}}\colon\mathbb{E}\to\mathbb{E}$ represent the map $\operatorname{proj}^{\mathbb{E}}\circ\mathbf{d}\circ(1-M_1)\circ(1-M_2)$. Apply Lemma 9.13 to see that $(\mathbb{E},\mathbf{d}_{\mathbb{E}})$ is a complex and that

$$(\mathbb{E},\mathbf{d}_{\mathbb{E}}) \xrightarrow{1-M_2} (\mathbb{E}\oplus\mathbb{J}, \operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1))$$

is a map of complexes. It is easy to see that the complex $(\mathbb{E},\mathbf{d}_{\mathbb{E}})$ is split exact. Indeed, if $x=\alpha_p\otimes c_q\otimes\lambda^{(z+1)}\in\mathbb{T}(p,q,z+1;0,0)\subseteq\mathbb{E}$, then

$$\mathbf{d}_{\mathbb{E}}(x)=(-1)^q\boldsymbol{\sigma}_z(p,q,z+1,0)\alpha_p(\omega_F)\otimes c_q(\omega_{G^*})\otimes\mu^{(0)}\in\mathbb{U}(\boldsymbol{f}-1-p,\boldsymbol{f}-2-q,0;0,0).$$

Thus,

$$\mathbf{d}_{\mathbb{E}}\colon \bigoplus_{p+q=\boldsymbol{f}-2-z}\mathbb{T}(p,q,z+1;0,0)\to\bigoplus_{p+q=\boldsymbol{f}+z-1}\mathbb{U}(p,q,0;0,0)$$

is an isomorphism and $(\mathbb{E},\mathbf{d}_{\mathbb{E}})$ is a split exact complex. Let $\mathbf{d}_{\mathbb{J}}\colon\mathbb{J}\to\mathbb{J}$ be the map $(\operatorname{proj}^{\mathbb{J}}+M_2)\circ\mathbf{d}\circ(1-M_1)$. Lemma 9.3 yields that

$$(\mathbb{E}\oplus\mathbb{J}, \operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1)) \xrightarrow{\operatorname{proj}^{\mathbb{J}}+M_2} (\mathbb{J},\mathbf{d}_{\mathbb{J}}) \tag{9.2}$$

is a quasi-isomorphism of complexes. In Lemma 9.16 we exhibit a map of complexes

$$\widehat{\Psi}\colon(\mathbb{Y},y)\to(\mathbb{E}\oplus\mathbb{J}, \operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1)).$$

Let $\Psi\colon(\mathbb{Y},y)\to(\mathbb{J},\mathbf{d}_{\mathbb{J}})$ be the composition

$$\mathbb{Y}\xrightarrow{\widehat{\Psi}}\mathbb{E}\oplus\mathbb{J}\xrightarrow{\operatorname{proj}^{\mathbb{J}}+M_2}\mathbb{J}.$$

We prove in Lemma 9.17 that Ψ is an isomorphism of complexes. $\square$

Lemma 9.3. *Let* $(\mathbb{F}, f)$ *be a complex such that, as a module,* $\mathbb{F}$ *is equal to the direct sum* $\mathbb{A} \oplus \mathbb{E}$.
(a) *If* $(\mathbb{A}, a)$ *is a complex and* $M \colon \mathbb{E} \to \mathbb{A}$ *is a module homomorphism such that* $\operatorname{proj}^{\mathbb{A}} + M : (\mathbb{F}, f) \longrightarrow (\mathbb{A}, a)$ *is a map of complexes, then*

$$(\mathbb{E}, \operatorname{proj}^{\mathbb{E}} \circ f \circ (1 - M)) \xrightarrow{1-M} (\mathbb{F}, f)$$

is a map of complexes.
(b) *If* $(\mathbb{E}, e)$ *is a complex and* $M \colon \mathbb{E} \to \mathbb{A}$ *is a module homomorphism such that* $1 - M \colon (\mathbb{E}, e) \longrightarrow (\mathbb{F}, f)$ *is a map of complexes, then*

$$(\mathbb{F}, f) \xrightarrow{\operatorname{proj}^{\mathbb{A}} + M} (\mathbb{A}, (\operatorname{proj}^{\mathbb{A}} + M) \circ f)$$

is a map of complexes.

Proof. (a) Observe that $\ker\Big(\operatorname{proj}^{\mathbb{A}} + M \colon \mathbb{F} \to \mathbb{A}\Big) = \operatorname{im}\Big(1 - M \colon \mathbb{E} \to \mathbb{F}\Big)$. If $x \in \mathbb{E}$, then $f \circ (1 - M)(x)$ is in the kernel of $\operatorname{proj}^{\mathbb{A}} + M$; and therefore, $f \circ (1 - M)(x)$ is equal to $(1 - M)(y)$ for some $y \in \mathbb{E}$. It follows that $\operatorname{proj}^{\mathbb{E}} \circ f \circ (1 - M)(x) = y$; thus, $(1 - M) \circ \operatorname{proj}^{\mathbb{E}} \circ f \circ (1 - M)(x) = f \circ (1 - M)(x)$, as desired.
(b) It is clear that $(\operatorname{proj}^{\mathbb{A}} + M) \circ f(x) = (\operatorname{proj}^{\mathbb{A}} + M) \circ f \circ (\operatorname{proj}^{\mathbb{A}} + M)(x)$, if $x \in \mathbb{A}$. We must establish the above equation for $x \in \mathbb{E}$. In other words, we must show that $(\operatorname{proj}^{\mathbb{A}} + M) \circ f \circ (1 - M)$ kills $\mathbb{E}$. However, the hypothesis ensures that $f \circ (1 - M)$ is equal to $(1 - M) \circ e$, and it is clear that $(\operatorname{proj}^{\mathbb{A}} + M) \circ (1 - M)$ kills $\mathbb{E}$. □

Lemma 9.4. *Adopt the notation of Theorem 9.1. Then* $(\mathbb{A}, \operatorname{proj}^{\mathbb{A}} \circ \mathbf{d})$ *is a complex and*

$$(\mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J}, \operatorname{proj}^{\mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J}} \circ \mathbf{d}) \xrightarrow{\operatorname{proj}^{\mathbb{A}} + M_1} (\mathbb{A}, \operatorname{proj}^{\mathbb{A}} \circ \mathbf{d})$$

is a map of complexes.

Proof. It suffices to show that the diagram

$$\begin{array}{ccc} \mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J} & \xrightarrow{\operatorname{proj}^{\mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J}} \circ \mathbf{d}} & \mathbb{A} \oplus \mathbb{E} \oplus \mathbb{J} \\ {\scriptstyle \operatorname{proj}^{\mathbb{A}} + M_1} \downarrow & & \downarrow {\scriptstyle \operatorname{proj}^{\mathbb{A}} + M_1} \\ \mathbb{A} & \xrightarrow{\operatorname{proj}^{\mathbb{A}} \circ \mathbf{d}} & \mathbb{A} \end{array} \tag{9.5}$$

commutes. If $x \in \mathbb{A}$, then both paths around (9.5) send x to $\operatorname{proj}^{\mathbb{A}} \circ \mathbf{d}(x)$. If $x = b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbf{U}(p, q, r; 0, 0) \subseteq \mathbb{E}$, then both paths around (9.5) send x to

$$\chi(q = z) \sum_{\substack{\{(t,s) \mid t \le z-1,\ q \le s+t\} \\ |K| = r+q-s-t \\ |J| = s}} (-1)^{ps+t+p+s} 1 \otimes (\textstyle\bigwedge^s X^*)(\gamma_J) \wedge \varphi_K \otimes f_K \wedge b_p \otimes g_J \wedge \delta_q(\omega_G) \wedge \mathbf{g} \otimes \nu^{(t)}.$$

(The counter-clockwise path involves the argument used to show "$T_3 - T_6 = 0$" in the calculation related to (9.6).) If $x = \alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbf{T}(p, q, r; 0, 0) \subseteq \mathbb{E}$, then the counter-clockwise path around (9.5) sends x to 0 and the clockwise path sends x to

$$\begin{cases} \chi(p=0) \sum\limits_{\substack{1\le t\\ |I|=q-t+r}} (-1)^q \boldsymbol{\sigma}_z(p,q,r,t) f_I \otimes \left[(\bigwedge^{\boldsymbol{f}-p+t-q-r-1} X)\left((\varphi_I \wedge \alpha_p)[\omega_F]\right) \wedge c_q\right](\omega_{G^*}) \otimes \mu^{(t)} \\ \\ + \chi(p=0) M_1 \left((-1)^q \boldsymbol{\sigma}_z(p,q,r,0)\alpha_p(\omega_F) \otimes c_q(\omega_{G^*}) \otimes \mu^{(0)}\right), \end{cases}$$

which is also equal to zero.

The counter-clockwise path around (9.5) kills all of $\mathbb{J}$, except

$$\mathbb{U}(p,q,r;0,0), \text{with } 1 \le r, \quad p+r = \boldsymbol{f}-1, \quad \text{and} \quad z = q. \tag{9.6}$$

The clockwise path kills all of $\mathbb{J}$, except (9.6), and, possibly,

$$\mathbb{T}(p,q,r;0,0), \text{with } p+q+r = \boldsymbol{f}-1 \quad \text{and} \quad z+2 \le r. \tag{9.7}$$

Fix $x = b_p \otimes \delta_q \otimes \mu^{(r)}$ as described in (9.6). The counter-clockwise path around (9.5) sends x to $T_1 + T_2 + T_3$, where

$$\begin{aligned} T_1 &= \sum_{\substack{\boldsymbol{f}-p\le t\\ |I|=\boldsymbol{f}-1-t}} (-1)^{\boldsymbol{f}-t} f_I \otimes [X(v)]\left(\left[(\bigwedge^{p-\boldsymbol{f}+1+t} X)(\varphi_I[b_p])\right](\delta_q)\right) \otimes \mu^{(t)}, \\ T_2 &= -\sum_{\substack{\boldsymbol{f}-p\le t\\ |I|=\boldsymbol{f}-1-t}} v \wedge f_I \otimes \left[(\bigwedge^{p-\boldsymbol{f}+1+t} X)(\varphi_I[b_p])\right](\delta_q) \otimes \mu^{(t-1)}, \text{ and} \\ T_3 &= \sum_{\substack{\boldsymbol{f}-p\le t\\ |I|=\boldsymbol{f}-1-t}} \sum_{\substack{\{(t',s) | t'\le z-1,\ z+\boldsymbol{f}-1-t-p\le s+t'\}\\ |K|=z+\boldsymbol{f}-1-p-s-t'\\ |J|=s}} (-1)^{(\boldsymbol{f}-t)s+t'+\boldsymbol{f}-t} 1 \otimes (\bigwedge^s X^*)(\gamma_J) \wedge \varphi_K \\ &\quad \otimes f_K \wedge f_I \otimes g_J \wedge \left(\left[(\bigwedge^{p-\boldsymbol{f}+1+t} X)(\varphi_I[b_p])\right](\delta_q)\right)(\omega_G) \wedge \mathbf{g} \otimes \nu^{(t')}. \end{aligned}$$

The clockwise path sends x to $T_4 + T_5 + T_6$, where

$$\begin{aligned} T_4 &= (-1)^p b_p \otimes [X(v)](\delta_q) \otimes \mu^{(r)}, \\ T_5 &= \sum_{\substack{\boldsymbol{f}-p-1\le t\\ |I|=\boldsymbol{f}-1-t}} (-1)^1 f_I \otimes \left[(\bigwedge^{p+2-\boldsymbol{f}+t)} X)(\varphi_I[v \wedge b_p])\right](\delta_q) \otimes \mu^{(t)}, \text{ and} \\ T_6 &= \sum_{\substack{\{(t,s) | t\le z-1,\ q\le s+t\}\\ |K|=r+q-s-t\\ |J|=s}} (-1)^{ps+t+p+s} 1 \otimes (\bigwedge^s X^*)(\gamma_J) \wedge \varphi_K \\ &\quad \otimes f_K \wedge b_p \otimes g_J \wedge \delta_q(\omega_G) \wedge \mathbf{g} \otimes \nu^{(t)}. \end{aligned}$$

We see that

$$T_1 - T_4 = \sum_{\substack{\boldsymbol{f}-p-1\le t\\ |I|=\boldsymbol{f}-1-t}} (-1)^{\boldsymbol{f}-t} f_I \otimes \left[(\bigwedge^{p-\boldsymbol{f}+2+t} X)(v \wedge \varphi_I[b_p])\right](\delta_q) \otimes \mu^{(t)}.$$

Apply Lemma 1.9.e to obtain

$$T_2 = \sum_{\substack{\boldsymbol{f}-p-1\le t\\ |I|=\boldsymbol{f}-t-1}} (-1)^{\boldsymbol{f}-t-1} f_I \otimes \left[(\bigwedge^{p-\boldsymbol{f}+2+t} X)([v(\varphi_I)][b_p])\right](\delta_q) \otimes \mu^{(t)}.$$

Proposition 1.1.a now yields that $T_1 - T_4 + T_2 - T_5 = 0$. Apply Lemma 1.9.e and Proposition 1.1 to $T_3 - T_6$, which is equal to

$$\begin{cases} \sum\limits_{\substack{\boldsymbol{f}-p-1\le t\\ |I|=\boldsymbol{f}-1-t}} \sum\limits_{\substack{\{(t',s)|t'\le z-1,\ z+\boldsymbol{f}-1-t-p\le s+t'\}\\ |K|=z+\boldsymbol{f}-1-p-s-t'\\ |J|=s}} (-1)^{(\boldsymbol{f}-t)s+t'+\boldsymbol{f}-t} 1\otimes (\bigwedge^s X^*)(\gamma_J)\wedge\varphi_K \\ \quad \otimes f_K\wedge f_I\otimes g_J\wedge\left(\left[(\bigwedge^{p-\boldsymbol{f}+1+t} X)(\varphi_I[b_p])\right](\delta_q)\right)(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')}, \end{cases}$$

in order to obtain

$$\begin{cases} \sum\limits_{\substack{\boldsymbol{f}-p-1\le t\\ |I|=\boldsymbol{f}-1-t}} \sum\limits_{\substack{\{(t',s)|t'\le z-1,\ z+\boldsymbol{f}-1-t-p\le s+t'\}\\ |K|=z+2\boldsymbol{f}-2-p-s-t'-t\\ |J|=s+p-\boldsymbol{f}+1+t}} (-1)^{(\boldsymbol{f}-t)s+t'+\boldsymbol{f}-t} \\ \quad \otimes(\varphi_I[b_p])\left((\bigwedge^{s+p-\boldsymbol{f}+1+t} X^*)(\gamma_J)\right)\wedge f_I(\varphi_K)\otimes f_K\otimes g_J\wedge\delta_q(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')}. \end{cases}$$

Let $L = s+p-\boldsymbol{f}+1+t$. Lemma 1.9.d now yields that $T_3 - T_6$ is equal to

$$\sum_{\substack{\{(t',L)|t'\le z-1,\ z\le L+t'\}\\ |K|=z+\boldsymbol{f}-1-L-t'\\ |J|=L}} (-1)^{L-p+t'+1} 1\otimes b_p\left((\bigwedge^L X^*)(\gamma_J)\wedge\varphi_K\right)\otimes f_K\otimes g_J\wedge\delta_q(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')},$$

and this sum is zero because $\operatorname{rank} F < \boldsymbol{f} \le L + |K|$.

Fix $x = \alpha_p\otimes c_q\otimes\lambda^{(r)}$, as described in (9.7). The clockwise path around (9.5) sends x to

$$\delta_{p\,0}\begin{cases} \sum\limits_{|I|=q+z+1} (-1)^q\boldsymbol{\sigma}_z(p,q,r,r-z-1) \\ \qquad M_1\left(f_I\otimes\left[(\bigwedge^{r-z-1} X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(r-z-1)}\right) \\ \quad + \sum\limits_{r-z\le t}\sum\limits_{|I|=q-t+r} (-1)^q\boldsymbol{\sigma}_z(p,q,r,t) f_I\otimes\left[(\bigwedge^t X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(t)} \end{cases}$$

$$= \delta_{p\,0}\begin{cases} (-1)^{q+1}\boldsymbol{\sigma}_z(p,q,r,r-z-1) \sum\limits_{|I|=q+z+1} \sum\limits_{\substack{r-z\le t\\ |J|=\boldsymbol{f}-1-t}} f_J \\ \qquad \otimes\left[(\bigwedge^t X)\left(\varphi_J[f_I]\wedge(\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(t)} \\ \quad + \sum\limits_{r-z\le t}\sum\limits_{|I|=q-t+r} (-1)^q\boldsymbol{\sigma}_z(p,q,r,t) f_I\otimes\left[(\bigwedge^t X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\otimes\mu^{(t)}. \end{cases}$$

Apply Lemma 1.9.f in order to conclude that this sum is zero. □

Proposition 9.8. *The complexes* $(\mathbb{A}, \operatorname{proj}^{\mathbb{A}}\circ\mathbf{d})$ *and* $(\mathbb{C}, \mathbf{d})$ *are split exact.*

Proof. Let $\mathbb{D}[[p,q,r;0]]$ be the submodule of $\mathbb{A}$ which is given in Definition 7.11 with

(9.9) $\qquad \operatorname{rank} F$ replaced by $\boldsymbol{f}-1$, $\quad \operatorname{rank} G$ replaced by $\boldsymbol{f}-2$, and

$\mathbb{L}(a,b,c,d,e)$ and $\mathbb{U}(a,b,c)$ replaced by $\mathbf{L}(a,b,c,d,e;0,0,0,1)$ and $\mathbf{U}(a,b,c;0,0)$, respectively. In a similar manner, let $\mathbb{D}[[p,q,r;1]]$ be the submodule of $\mathbb{A}$ which is given in Definition 7.11 when hypothesis (9.9) is in effect, and the modules $\mathbf{L}(a,b,c,d,e;0,1,0,1)$ and $\mathbf{U}(a,b,c;0,1)$ are used. It is not difficult to see that $\mathbb{A}$ decomposes as the direct sum

$$\mathbb{A} = \bigoplus_{\{0\le p,\ 1\le q\}} \mathbb{D}[[p,q,0;0]] \oplus \bigoplus_{\{0\le p,\ 0\le q\}} \mathbb{D}[[p,q,0;1]].$$

Filter $\mathbb{A}$ as follows. Take $\bigoplus \mathbb{D}[[p,q,0;0]] < \bigoplus \mathbb{D}[[p,q,0;1]]$, and

$$\mathbb{D}[[p',q',r';\ell]] < \mathbb{D}[[p,q,r;\ell]] \iff (p',q',r') < (p,q,r) \text{ in the order of Definition 7.11,} \tag{9.10}$$

for $\ell = 0,1$. Let $d^{[0]}$ be the component of $\text{proj}^{\mathbb{A}} \circ \mathbf{d}$ which is homogeneous with respect to the above filtration. It is not difficult to see that if x is a homogeneous element of $\mathbb{A}$, then $\text{proj}^{\mathbb{A}} \circ \mathbf{d}(x) = x' + d^{[0]}(x)$, for some $x' \in \mathbb{A}$ with $x' < x$. Let $(\mathbb{D}[[p,q,r]], d^{[0]})$ be the complex of Proposition 7.14.a with the hypothesis of (9.9) in effect. It is not difficult to establish isomorphisms from $(\mathbb{D}[[p,q,r;0]], d^{[0]})$ and $(\mathbb{D}[[p,q,r;1]], d^{[0]})$ to $(\mathbb{D}[[p,q,r]], d^{[0]})$. Apply Proposition 7.14.b and Theorem 5.7 to see that $(\mathbb{D}[[p,q,r]], d^{[0]})$ is split exact. It follows that the complex $(\mathbb{A}, \text{proj}^{\mathbb{A}} \circ \mathbf{d})$ is split exact.

Let $\mathbb{D}[[p,q,r;1']]$ and $\mathbb{D}[[p,q,r;2]]$ be the submodules of $\mathbb{C}$ which are given in Definition 7.11 when hypothesis (9.9) is in effect. Modules of the form

$\mathbf{L}(a,b,c,d,e;0,1,0,1)$ and $\mathbf{U}(a,b,c;0,1)$ are used in $\mathbb{D}[[p,q,r;1']]$, and
$\mathbf{L}(a,b,c,d,e;0,1,1,1)$ and $\mathbf{U}(a,b,c;1,1)$ are used in $\mathbb{D}[[p,q,r;2]]$.

Let $\mathbb{Q}$ represent $\bigoplus \mathbf{L}(p,q,r,s,t;a,b,c,d)$, where the sum is taken over the set $S_{\mathbb{L}}(\text{not})$. Observe that that $(\mathbb{Q}, \mathbf{d})$ is a subcomplex of $(\mathbb{C}, \mathbf{d})$. It is easy to see that

$$\mathbb{C} = \mathbb{Q} \oplus \bigoplus_{\{0 \le p,\ 1 \le r\}} \mathbb{D}[[p,0,r;1']] \oplus \bigoplus_{\{0 \le p,\ 1 \le r\}} \mathbb{D}[[p,0,r;2]].$$

Filter $\mathbb{C}$ by taking $\mathbb{Q} < \bigoplus \mathbb{D}[[p,0,r;1']] < \bigoplus \mathbb{D}[[p,0,r;2]]$, and (9.10) for $\ell = 1', 2$. Let $d^{[0]}$ be the component of $\mathbf{d}$ which is homogeneous with respect to the above filtration. It is not difficult to see that $\mathbf{d}$ is a non-increasing function on $\mathbb{C}$ and that the complexes $(\mathbb{D}[[p,q,r;1']], d^{[0]})$ and $(\mathbb{D}[[p,q,r;2]], d^{[0]})$ are isomorphic to the split exact complex $(\mathbb{D}[[p,q,r]], d^{[0]})$, which is defined in the first part of the proof. To complete the proof, it suffices to show that $(\mathbb{Q}, \mathbf{d})$ is split exact.

Filter $\mathbb{Q}$ by taking $\mathbf{L}(p',q',r',s',t';a',b',c',d') < \mathbf{L}(p,q,r,s,t;a,b,c,d)$, whenever

$$\begin{cases} q' < q, \text{ or} \\ q' = q \text{ and } r' + s' + 2(t' + c' + d') < r + s + 2(t + c + d), \text{ or} \\ q' = q,\ r' + s' + 2(t' + c' + d') = r + s + 2(t + c + d), \text{ and } a' + b' - c' - d' < a + b - c - d. \end{cases}$$

Let $d^{[0]}$ be the component of $\mathbf{d}$ which preserves this filtration. It is not difficult to see that the restriction of $\mathbf{d}$ to $\mathbb{Q}$ is a non-increasing function and that if

$$x = \boldsymbol{\phi}^a A_p \otimes \alpha_q \wedge \boldsymbol{\phi}^b \otimes b_r \wedge \mathbf{f}^c \otimes c_s \wedge \mathbf{g}^d \otimes \nu^{(t)} \in \mathbf{L}(p,q,r,s,t;a,b,c,d) \subseteq \mathbb{Q},$$

then $d^{[0]}(x)$ is equal to

$$\begin{cases} + (-1)^q \chi(b=1)\chi(p+a+t \le z-2)\boldsymbol{\phi}^{a+1} A_p \otimes \alpha_q \otimes b_r \wedge \mathbf{f}^c \otimes c_s \wedge \mathbf{g}^d \otimes \nu^{(t)} \\ + \chi(c=0)(-1)^{q+r+b}\chi(\boldsymbol{f} \le s+d+t)\boldsymbol{\phi}^{a+1} A_p \otimes \alpha_q \wedge \boldsymbol{\phi}^b \otimes b_r \wedge \mathbf{f} \otimes c_s \wedge \mathbf{g}^d \otimes \nu^{(t-1)} \\ + \chi(d=0)(-1)^{q+b+r+c+1+s}\boldsymbol{\phi}^{a+1} A_p \otimes \alpha_q \wedge \boldsymbol{\phi}^b \otimes b_r \wedge \mathbf{f}^c \otimes c_s \wedge \mathbf{g} \otimes \nu^{(t-1)}. \end{cases}$$

At this point, it suffices to show that the complex $(\mathbb{Q}, d^{[0]})$ is split exact.

Define $\mathbb{F}[p,q,r,s,t,a]$ to be the sum of all $\mathbb{L}(p,q,r,s,t';a',b,c,d) \subseteq \mathbb{Q}$ such that

$$t = t' - 2 + c + d \quad \text{and} \quad a = a' + b + 2 - c - d.$$

Observe that $\mathbb{Q} = \bigoplus \mathbb{F}[p,q,r,s,t,a]$, where the sum is taken over all (p,q,r,s,t,a) with

$$p+t+a \le z, \quad \boldsymbol{f} - 3 \le s+t, \quad z \le p+q+t+a, \quad r+s+t \le 2\boldsymbol{f} - 4, \quad \text{and} \quad 1 \le a. \tag{9.11}$$

It is now obvious that the complex $(\mathbb{Q}, d^{[0]})$ is the direct sum of the subcomplexes $(\mathbb{F}[p,q,r,s,t,a], d^{[0]})$. In the ensuing discussion, fix prameters $p,q,r,s,t,a,b,c,$ and d. Take $<t;a,b,c,d>$ to be the module $\mathbb{L}(p,q,r,s,t;a,b,c,d)$, and $\mathbb{F}$ to be the complex $(\mathbb{F}[p,q,r,s,t,a], d^{[0]})$. If

$$p+t+a \le z-1, \quad \boldsymbol{f} - 2 \le s+t, \tag{9.12}$$

and $3 \le a$, then $\mathbb{F}$ is

$$0 \to <t+2;a-3,1,0,0> \xrightarrow{\delta_3} \begin{matrix} <t+2;a-2,0,0,0> \\ \oplus \\ <t+1;a-2,1,0,1> \\ \oplus \\ <t+1;a-2,1,1,0> \end{matrix} \xrightarrow{\delta_2} \begin{matrix} <t+1;a-1,0,0,1> \\ \oplus \\ <t+1;a-1,0,1,0> \\ \oplus \\ <t;a-1,1,1,1> \end{matrix} \xrightarrow{\delta_1} <t;a,0,1,1> \to 0;$$

where

$$\delta_3 = \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{bmatrix}, \quad \text{and} \quad \delta_1 = \begin{bmatrix} * & * & * \end{bmatrix},$$

and each map which is labeled $*$ is an isomorphism. It is clear that the complex $\mathbb{F}$ is split exact. There are a handful of degenerate versions of this complex, and each of these is also split exact. First, we continue to assume that (9.12) holds. If $2 = a$, then $\mathbb{F}$ is

$$0 \to \begin{matrix} <t+2;a-2,0,0,0> \\ \oplus \\ <t+1;a-2,1,1,0> \end{matrix} \xrightarrow{\delta_2} \begin{matrix} <t+1;a-1,0,0,1> \\ \oplus \\ <t+1;a-1,0,1,0> \\ \oplus \\ <t;a-1,1,1,1> \end{matrix} \xrightarrow{\delta_1} <t;a,0,1,1> \to 0;$$

if $1 = a$, then $\mathbb{F}$ is

$$0 \to <t+1;a-1,0,1,0> \xrightarrow{\delta_1} <t;a,0,1,1> \to 0.$$

Next, we take $p+t+a \le z-1$ and $\boldsymbol{f} - 3 = s+t$. If $3 \le a$, then $\mathbb{F}$ is

$$0 \to <t+2;a-3,1,0,0> \xrightarrow{\delta_3} \begin{matrix} <t+2;a-2,0,0,0> \\ \oplus \\ <t+1;a-2,1,0,1> \end{matrix} \xrightarrow{\delta_2} <t+1;a-1,0,0,1> \to 0;$$

if $2 = a$, then $\mathbb{F}$ is

$$<t+2;a-2,0,0,0> \xrightarrow{\delta_2} <t+1;a-1,0,0,1>;$$

and if $1 = a$, then $\mathbb{F}$ is the zero complex. Next, we take $p+t+a = z$ and $\boldsymbol{f}-2 \le s+t$. If $3 \le a$, then $\mathbb{F}$ is

$$0 \to \text{<}t+2; a-3,1,0,0\text{>} \xrightarrow{\delta_3} \begin{matrix} \text{<}t+1; a-2,1,0,1\text{>} \\ \oplus \\ \text{<}t+1; a-2,1,1,0\text{>} \end{matrix} \xrightarrow{\delta_2} \text{<}t; a-1,1,1,1\text{>} \to 0;$$

if $2 = a$, then $\mathbb{F}$ is

$$0 \to \text{<}t+1; a-2,1,1,0\text{>} \xrightarrow{\delta_2} \text{<}t; a-1,1,1,1\text{>} \to 0;$$

if $1 = a$, then $\mathbb{F}$ is the zero complex. Finally, we take $p+t+a = z$ and $\boldsymbol{f}-3 = s+t$. If $3 \le a$, then $\mathbb{F}$ is

$$0 \to \text{<}t+2; a-3,1,0,0\text{>} \xrightarrow{\delta_3} \text{<}t+1; a-2,1,0,1\text{>} \to 0;$$

and if $a \le 2$, then $\mathbb{F}$ is the zero complex. We conclude that $(\mathbb{F}[p,q,r,s,t,a], d^{[0]})$ is a split exact complex, whenever (9.11) is satisfied. $\square$

Lemma 9.13. *In the notation of Theorem 9.1,* $(\mathbb{E}, \mathbf{d}_{\mathbb{E}})$ *is a complex and*

$$(\mathbb{E}, \mathbf{d}_{\mathbb{E}}) \xrightarrow{1-M_2} (\mathbb{E} \oplus \mathbb{J}, \operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d} \circ (1-M_1))$$

is a map of complexes.

Proof. It suffices to show that the diagram

$$\begin{array}{ccc} \mathbb{E} & \xrightarrow{\mathbf{d}_{\mathbb{E}}} & \mathbb{E} \\ {\scriptstyle 1-M_2}\downarrow & & {\scriptstyle 1-M_2}\downarrow \\ \mathbb{E}\oplus\mathbb{J} & \xrightarrow{\operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d} \circ (1-M_1)} & \mathbb{E}\oplus\mathbb{J} \end{array}$$

commutes. In other words, we must show that $\tau = (\operatorname{proj}^{\mathbb{J}} + M_2) \circ \mathbf{d} \circ (1 - M_1 - M_2)$ kills $\mathbb{E}$. If $x = \alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbf{T}(p,q,r;0,0) \subseteq \mathbb{E}$, then $\tau(x)$ is equal to

$$\begin{cases} (-1)^{p-1} v(\alpha_p) \otimes c_q \otimes \lambda^{(r)} \\ + (-1)^{p+q} \chi(1 \le q+r) \alpha_p \otimes u(c_q) \otimes \lambda^{(r)} \\ + \chi(1 \le p) \sum\limits_{\substack{0 \le t \\ |I| = q-t+r-1}} (-1)^{p-q-r+t} \boldsymbol{\sigma}_z(p,q,r,t) f_I \wedge \mathbf{f} \\ \qquad \otimes \left[(\bigwedge^{t+1} X)\left((\varphi_I \wedge \alpha_p)[\omega_F] \right) \wedge c_q \right] (\omega_{G^*}) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)} \\ + \chi(1 \le p) \sum\limits_{\substack{1 \le t \\ |I| = q-t+r}} (-1)^q \boldsymbol{\sigma}_z(p,q,r,t) f_I \otimes \left[(\bigwedge^t X)\left((\varphi_I \wedge \alpha_p)[\omega_F] \right) \wedge c_q \right] (\omega_{G^*}) \otimes \mu^{(t)} \\ + (-1)^q \boldsymbol{\sigma}_z(p,q,r,0) M_2 \left(\alpha_p[\omega_F] \otimes c_q(\omega_{G^*}) \otimes \mu^{(0)} \right), \end{cases}$$

and this is easily seen to be zero. If $x = b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbf{U}(p,q,r;0,0) \subseteq \mathbb{E}$, then $\tau(x)$ is equal to

$$(\operatorname{proj}^{\mathbb{J}} + M_2) \circ \mathbf{d} \begin{cases} \theta_{\boldsymbol{f}-2-q} \theta_{\boldsymbol{f}-1-p} (-1)^{z+q\boldsymbol{f}} (v \wedge b_p)(\omega_{F^*}) \otimes \delta_q(\omega_G) \otimes \lambda^{(z+1)} \\ - \chi(1 \le p) \theta_{\boldsymbol{f}-2-q} \theta_{\boldsymbol{f}-1-p} (-1)^{z+\boldsymbol{f}+q+q\boldsymbol{f}} b_p(\omega_{F^*}) \otimes (u \wedge \delta_q)(\omega_G) \otimes \lambda^{(z+1)} \\ + \chi(p \le \boldsymbol{f}-2) \sum\limits_{\substack{0 \le t \\ |I| = p-1-t}} (-1)^{q+t+1} f_I \wedge \mathbf{f} \otimes \left[(\bigwedge^{1+t} X)[\varphi_I(b_p)] \right] (\delta_q) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)} \\ + \sum\limits_{\substack{0 \le t \\ |I| = p-t}} f_I \otimes \left[(\bigwedge^t X)(\varphi_I(b_p)) \right] (\delta_q) \otimes \mu^{(t)}. \end{cases}$$

After $\tau(x)$ is expanded, and the easy cancellations are made, one is left with a sum of 13 terms. The technique which was used on $T_3 - T_6$ in the proof of Lemma 9.4 shows that the term in $\mathbf{L}(0,0,0,1)$ is zero. Six terms are in $\mathbf{U}(0,0)$: three of these involve u, the other three involve v. Six terms also are in $\mathbf{U}(1,1)$: once again, three involve u, and the other three involve v. Each of these four triples adds to zero by way of Proposition 1.1.a. $\square$

Remark 9.14. The complex $\mathbb{Y}$ is defined in the statement of Theorem 9.1. The summand of $\mathbb{Y}$ in position i is equal to $\mathbb{Y}_i = \mathbb{I}_{i-2}^{(z)} \oplus \mathbb{I}_{i-1}^{(z)} \oplus \mathbb{I}_{i-1}^{(z)} \oplus \mathbb{I}_{i}^{(z)}$. We think of the elements of $\mathbb{Y}$ as column vectors. In particular, the element $b_p \otimes \delta_q \otimes \mu^{(r)}$ of $\mathbb{U}[3] \subseteq \mathbb{Y}$ is

$$\begin{bmatrix} 0 \\ 0 \\ b_p \otimes \delta_q \otimes \mu^{(r)} \\ 0 \end{bmatrix}.$$

The differential $y_i \colon \mathbb{Y}_i \to \mathbb{Y}_{i-1}$ is given by

$$y_i = \begin{bmatrix} d & 0 & 0 & 0 \\ (-1)^{i+1} v_{\boldsymbol{f}} & d & 0 & 0 \\ (-1)^i u_{\boldsymbol{f}-1} & 0 & d & 0 \\ 0 & (-1)^{i+1} u_{\boldsymbol{f}-1} & (-1)^{i+1} v_{\boldsymbol{f}} & d \end{bmatrix}.$$

Definition 9.15. Define $\widehat{\Psi} \colon \mathbb{Y} \to \mathbb{E} \oplus \mathbb{J}$ by

$$\begin{aligned}
&\widehat{\Psi}\left(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)} \in \mathbb{L}[4]\right) = A_p \otimes \alpha_q \otimes b_r \otimes c_s \wedge \mathbf{g} \otimes \nu^{(t)}, \\
&\widehat{\Psi}\left(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)} \in \mathbb{L}[3]\right) = (-1)^{r+s+\boldsymbol{f}-1} A_p \otimes \alpha_q \wedge \boldsymbol{\phi} \otimes b_r \otimes c_s \wedge \mathbf{g} \otimes \nu^{(t)}, \\
&\widehat{\Psi}\left(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)} \in \mathbb{L}[2]\right) = (-1)^{s+\boldsymbol{f}-1} A_p \otimes \alpha_q \otimes b_r \wedge \mathbf{f} \otimes c_s \wedge \mathbf{g} \otimes \nu^{(t)}, \\
&\widehat{\Psi}\left(A_p \otimes \alpha_q \otimes b_r \otimes c_s \otimes \nu^{(t)} \in \mathbb{L}[1]\right) = (-1)^{r+1} A_p \otimes \alpha_q \wedge \boldsymbol{\phi} \otimes b_r \wedge \mathbf{f} \otimes c_s \wedge \mathbf{g} \otimes \nu^{(t)}, \\
&\widehat{\Psi}\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[4]\right) = \begin{cases} \chi(1 \le r)\chi(z+1 \le q+r)(-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r-1)} \\ \quad + b_p \otimes \delta_q \otimes \mu^{(r)}, \end{cases} \\
&\widehat{\Psi}\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[3]\right) = b_p \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)}, \\
&\widehat{\Psi}\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[2]\right) = (-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \otimes \mu^{(r)},
\end{aligned}$$

$\widehat{\Psi}\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[1]\right)$ is equal to

$$\begin{cases}
\chi(p+q+r \le \boldsymbol{f} - 3 + z)(-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)} \\
- \chi(p+q+r = \boldsymbol{f} - 2 + z)\chi(p+r \le \boldsymbol{f} - 3) \sum\limits_{\substack{r+1 \le t \\ |I| = p+1+r-t}} f_I \otimes \left[\left(\bigwedge^{t-1-r} X\right)(\varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t)} \\
- \chi(p+q+r = \boldsymbol{f} - 2 + z)\chi(p+r = \boldsymbol{f} - 2) b_p \otimes \delta_q \otimes \mu^{(r+1)} \\
+ \chi(p+q+r = \boldsymbol{f} - 2 + z) \sum\limits_{\substack{1+r \le t \\ |I| = p+r-t}} (-1)^{r+t+1+q} f_I \wedge \mathbf{f} \otimes \left[\left(\bigwedge^{t-r} X\right)(\varphi_I(b_p))\right](\delta_q) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)} \\
+ \chi(r=0)\chi(p+q = \boldsymbol{f} + z - 2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+p}(v \wedge b_p)(\omega_{F^*}) \otimes \delta_q(\omega_G) \otimes \lambda^{(z+1)} \\
+ \chi(r=0)\chi(p+q = \boldsymbol{f} + z - 2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+z+1} b_p(\omega_{F^*}) \otimes (u \wedge \delta_q)(\omega_G) \otimes \lambda^{(z+1)},
\end{cases}$$

$$\widehat{\Psi}\left(\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}[4]\right)=\begin{cases}\alpha_p\otimes c_q\otimes\lambda^{(r)}\\ +(-1)^{p+1}\chi(z+2\le r)\alpha_p\wedge\boldsymbol{\phi}\otimes c_q\wedge\mathbf{g}\otimes\lambda^{(r-1)},\end{cases}$$
$$\widehat{\Psi}\left(\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}[3]\right)=(-1)^{p+q}\alpha_p\wedge\boldsymbol{\phi}\otimes c_q\otimes\lambda^{(r)},$$
$$\widehat{\Psi}\left(\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}[2]\right)=(-1)^{1+q}\alpha_p\otimes c_q\wedge\mathbf{g}\otimes\lambda^{(r)},\ \text{and}$$
$$\widehat{\Psi}\left(\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}[1]\right)=\begin{cases}\alpha_p\otimes c_q\otimes\lambda^{(r+1)}\\ -\chi(p+q+2r\le\boldsymbol{f}+z-2)\sum\limits_{|J|=r-z}\varphi_J\wedge\alpha_p\\ \qquad\otimes c_q\wedge(\bigwedge^{r-z}X)(f_J)\otimes\lambda^{(z+1)}.\end{cases}$$

Lemma 9.16. *If the notation of Theorem 9.1 and Definition 9.15 are adopted, then*
$$\widehat{\Psi}\colon(\mathbb{Y},y)\to(\mathbb{E}\oplus\mathbb{J},\operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1))$$
is a map of complexes.

Proof. Most of the calculation that $\widehat{\Psi}\circ y(x)=\operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1)\circ\widehat{\Psi}(x)$, for $x\in\mathbb{Y}$ proceeds without much difficulty. The two interesting cases are $x\in\mathbb{U}[1]$ and $x\in\mathbb{T}[1]$. We first treat $x=b_p\otimes\delta_q\otimes\mu^{(r)}\in\mathbb{U}(p,q,r)\subseteq\mathbb{U}[1]$, with

$$0\le r,\quad p+q+r\le\boldsymbol{f}-2+z,\quad p+r\le\boldsymbol{f}-2,\quad\text{and}\quad z\le q+r.$$

One can calculate that $\widehat{\Psi}\circ y(x)=\sum_{i=1}^{12}S_i$, where

$$S_1=(-1)^q[X^*(u)](b_p)\wedge\mathbf{f}\otimes\delta_q\wedge\boldsymbol{\gamma}\otimes\mu^{(r)},$$

$$S_2=\chi(1\le r)\chi(p+q+r\le\boldsymbol{f}-3+z)(-1)^{p+q}b_p\wedge\mathbf{f}\otimes u\wedge\delta_q\wedge\boldsymbol{\gamma}\otimes\mu^{(r-1)},$$

$$S_3=\chi(1\le r)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r\le t\\|I|=p+r-1-t}}(-1)^{r+t+q+p}f_I\wedge\mathbf{f}$$
$$\otimes\left[(\textstyle\bigwedge^{t-r+1}X)\left(\varphi_I(b_p)\right)\right](u\wedge\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$S_4=\chi(z+1\le q+r)(-1)^{p+q+1}b_p\wedge\mathbf{f}\otimes[X(v)](\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(r)},$$

$$S_5=\chi(1\le r)\chi(z+1\le q+r)\chi(p+q+r\le\boldsymbol{f}-3+z)(-1)^q v\wedge b_p\wedge\mathbf{f}\otimes\delta_q\wedge\boldsymbol{\gamma}\otimes\mu^{(r-1)},$$

$$S_6=\chi(1\le r)\chi(z+1\le q+r)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r\le t\\|I|=p+r-t}}(-1)^{r+t+q}f_I\wedge\mathbf{f}$$
$$\otimes\left[(\textstyle\bigwedge^{t-r+1}X)\left(\varphi_I(v\wedge b_p)\right)\right](\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$S_7=-\chi(1\le r)\chi(z+1\le q)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r\le t\\|I|=p+1+r-t}}f_I$$
$$\otimes\left[(\textstyle\bigwedge^{t-r}X)\left(\varphi_I(v\wedge b_p)\right)\right](\delta_q)\otimes\mu^{(t)},$$

$$S_8=-\chi(1\le r)\chi(q=z)\chi(p+r=\boldsymbol{f}-2)v\wedge b_p\otimes\delta_q\otimes\mu^{(r)},$$

$$S_9=\chi(1\le r)\chi(p+q+r=\boldsymbol{f}-2+z)(-1)^p\sum_{\substack{r\le t\\|I|=p+r-t}}f_I\otimes\left[(\textstyle\bigwedge^{t-r}X)\left(\varphi_I(b_p)\right)\right](u\wedge\delta_q)\otimes\mu^{(t)},$$

$$S_{10} = \sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |J|=s\\ |I|=r+q-s-t}} (-1)^{1+r+q+ps} \otimes (\textstyle\bigwedge^s X^*)(\gamma_J) \wedge \varphi_I \wedge \boldsymbol{\phi} \otimes f_I \wedge b_p \wedge \mathbf{f} \otimes g_J \wedge \delta_q(\omega_G) \wedge \mathbf{g} \otimes \nu^{(t)},$$

$$S_{11} = (-1)^{p+1} v_{\boldsymbol{f}} b_p \wedge \mathbf{f} \otimes \delta_q \otimes \mu^{(r)}, \text{ and}$$

$$S_{12} = (-1)^{p+q} u_{\boldsymbol{f}-1} b_p \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)}.$$

On the other hand, $(1-M_1)\circ\widehat{\Psi}(x)$ is equal to

$$\begin{cases} \chi(p+q+r\le \boldsymbol{f}-3+z)(-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)} \\ -\chi(p+q+r=\boldsymbol{f}-2+z)\chi(p+r\le \boldsymbol{f}-2) \displaystyle\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}} f_I \otimes \left[\left(\textstyle\bigwedge^{t-1-r} X\right)(\varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t)} \\ +\chi(p+q+r=\boldsymbol{f}-2+z) \displaystyle\sum_{\substack{1+r\le t\\ |I|=p+r-t}} (-1)^{r+t+1+q} f_I \wedge \mathbf{f} \otimes \left[\left(\textstyle\bigwedge^{t-r} X\right)(\varphi_I(b_p))\right](\delta_q) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)} \\ +\chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+p}(v\wedge b_p)(\omega_{F^*}) \otimes \delta_q(\omega_G) \otimes \lambda^{(z+1)} \\ +\chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{z+q\boldsymbol{f}+1} b_p(\omega_{F^*}) \otimes (u\wedge \delta_q)(\omega_G) \otimes \lambda^{(z+1)}. \end{cases}$$

So, $\operatorname{proj}^{\mathbb{E}\oplus\mathbb{J}} \circ \mathbf{d} \circ (1-M_1)\circ\widehat{\Psi}(x) = \sum_{i=1}^{28} T_i$, with

$$T_1 = \chi(p+q+r\le \boldsymbol{f}-3+z)(-1)^q [X^*(u)](b_p) \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)},$$

$$T_2 = \chi(z+1\le q+r)\chi(p+q+r\le \boldsymbol{f}-3+z)(-1)^{q+p+1} b_p \wedge \mathbf{f} \otimes [X(v)](\delta_q) \wedge \boldsymbol{\gamma} \otimes \mu^{(r)},$$

$$T_3 = \chi(z+1\le q+r)\chi(p+q+r\le \boldsymbol{f}-3+z)\chi(1\le r)(-1)^q v \wedge b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r-1)},$$

$$T_4 = \chi(p+q+r\le \boldsymbol{f}-3+z)\chi(1\le r)(-1)^{p+q} b_p \wedge \mathbf{f} \otimes u \wedge \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r-1)},$$

$$T_5 = \chi(p+q+r\le \boldsymbol{f}-3+z)(-1)^{q+p} u_{\boldsymbol{f}-1} b_p \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)},$$

$$T_6 = \chi(p+q+r\le \boldsymbol{f}-3+z)(-1)^{p+1} v_{\boldsymbol{f}} b_p \wedge \mathbf{f} \otimes \delta_q \otimes \mu^{(r)},$$

$$\begin{aligned} T_7 = \chi(p+q+r\le \boldsymbol{f}-3+z) &\sum_{\substack{\{(t,s)\mid t\le z-1,\ q\le s+t\}\\ |K|=r+q-s-t\\ |J|=s}} (-1)^{ps+r+1+q} \\ &\otimes (\textstyle\bigwedge^s X^*)(\gamma_J) \wedge \varphi_K \wedge \boldsymbol{\phi} \otimes f_K \wedge b_p \wedge \mathbf{f} \otimes g_J \wedge \delta_q(\omega_G) \wedge \mathbf{g} \otimes \nu^{(t)}, \end{aligned}$$

$$T_8 = -\chi(p+q+r=\boldsymbol{f}-2+z) \sum_{\substack{r+1\le t\\ |I|=p+1+r-t}} [X^*(u)](f_I) \otimes \left[\left(\textstyle\bigwedge^{t-1-r} X\right)(\varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t)},$$

$$\begin{aligned} T_9 = \chi(p+q+r=\boldsymbol{f}-2+z)\chi(p+r\le \boldsymbol{f}-3) &\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}} (-1)^{p+r-t} f_I \\ &\otimes \left[\left(\textstyle\bigwedge^{t-r} X\right)(v\wedge \varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t)}, \end{aligned}$$

$$\begin{aligned} T_{10} = -\chi(p+q+r=\boldsymbol{f}-2+z)\chi(p+r\le \boldsymbol{f}-3) &\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}} v\wedge f_I \\ &\otimes \left[\left(\textstyle\bigwedge^{t-1-r} X\right)(\varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t-1)}, \end{aligned}$$

$$T_{11} = -\chi(q=z)\chi(p+r=\boldsymbol{f}-2) v \wedge b_p \otimes \delta_q \otimes \mu^{(r)},$$

$$T_{12}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}}(-1)^{p+r-t+1}f_I$$
$$\otimes u\wedge\Big[(\textstyle\bigwedge^{t-1-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\otimes\mu^{(t-1)},$$

$$T_{13}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}}\sum_{\substack{\{(t',s)|t'\le z-1,\ q+1+r-t\le s+t'\}\\ |K|=q+r-s-t'\\ |J|=s}}(-1)^{ps+s+rs+ts+q+r+1}$$
$$\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_K\wedge\boldsymbol{\phi}\otimes f_K\wedge f_I\wedge\mathbf{f}\otimes g_J\wedge\Big(\Big[(\textstyle\bigwedge^{t-1-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\Big)\,(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')},$$

$$T_{14}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}}(-1)^{p+r+t}$$
$$v_{\boldsymbol{f}}f_I\wedge\mathbf{f}\otimes\Big[(\textstyle\bigwedge^{t-1-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\otimes\mu^{(t-1)},$$

$$T_{15}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}}(-1)^{p+q}u_{\boldsymbol{f}-1}f_I$$
$$\otimes\Big[(\textstyle\bigwedge^{t-1-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t-1)},$$

$$T_{16}=\chi(z+1\le q)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{r+1\le t\\ |I|=p+1+r-t}}\sum_{\substack{\{(t',s)|t'\le z-1,\ q+1+r-t\le s+t'\}\\ |K|=q+1+r-s-t'\\ |J|=s}}(-1)^{sp+rs+ts+t'+p+r+t}$$
$$\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_K\otimes f_K\wedge f_I\otimes g_J\wedge\Big(\Big[(\textstyle\bigwedge^{t-1-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\Big)\,(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')},$$

$$T_{17}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{r+t+1+q}[X^*(u)](f_I)\wedge\mathbf{f}$$
$$\otimes\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$T_{18}=\chi(z+1\le q+r)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{q+p}f_I\wedge\mathbf{f}$$
$$\otimes[X(v)](\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q))\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$T_{19}=\chi(z+1\le q+r)\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{r+t+1+q}v\wedge f_I\wedge\mathbf{f}$$
$$\otimes\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t-1)},$$

$$T_{20}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{1+q+p}f_I\wedge\mathbf{f}$$
$$\otimes u\wedge\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t-1)},$$

$$T_{21}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{1+q+p}u_{\boldsymbol{f}-1}f_I\otimes\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$T_{22}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}(-1)^{t+p+r}v_{\boldsymbol{f}}f_I\wedge\mathbf{f}\otimes\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\otimes\mu^{(t)},$$

$$T_{23}=\chi(p+q+r=\boldsymbol{f}-2+z)\sum_{\substack{1+r\le t\\ |I|=p+r-t}}\sum_{\substack{\{(t',s)|t'\le z-1,\ q+r-t\le s+t'\}\\ |K|=q+r-s-t'\\ |J|=s}}(-1)^{sp+rs+ts+r+q}$$
$$\otimes(\textstyle\bigwedge^s X^*)(\gamma_J)\wedge\varphi_K\wedge\boldsymbol{\phi}\otimes f_K\wedge f_I\wedge\mathbf{f}\otimes g_J\wedge\Big(\Big[(\textstyle\bigwedge^{t-r}X)\,(\varphi_I(b_p))\Big]\,(\delta_q)\Big)\,(\omega_G)\wedge\mathbf{g}\otimes\nu^{(t')},$$

$$T_{24} = \chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\chi(z+1\le q)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}\sum_{0\le t}\sum_{|I|=p-t}(-1)^{q+z+1+t+q\boldsymbol{f}}$$
$$\boldsymbol{\sigma}_z(\boldsymbol{f}-2-p,\boldsymbol{f}-2-q,z+1,t)f_I\wedge\mathbf{f}\otimes\left[(\textstyle\bigwedge^{t+1}X)\left(\varphi_I(v\wedge b_p)\right)\right](\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$T_{25} = \chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\chi(p\le\boldsymbol{f}-3)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+p+\boldsymbol{f}+q}\sum_{0\le t}\sum_{|I|=p-t+1}$$
$$\boldsymbol{\sigma}_z(\boldsymbol{f}-2-p,\boldsymbol{f}-2-q,z+1,t)f_I\otimes\left[(\textstyle\bigwedge^{t}X)\left(\varphi_I(v\wedge b_p)\right)\right](\delta_q)\otimes\mu^{(t)},$$

$$T_{26} = \chi(r=0)\chi(q=z)\chi(p=\boldsymbol{f}-2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+p+\boldsymbol{f}+q}$$
$$\boldsymbol{\sigma}_z(\boldsymbol{f}-2-p,\boldsymbol{f}-2-q,z+1,0)v\wedge b_p\otimes\delta_q\otimes\mu^{(0)},$$

$$T_{27} = \chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}\sum_{0\le t}\sum_{|I|=p-t-1}(-1)^{z+t+\boldsymbol{f}+q\boldsymbol{f}}$$
$$\boldsymbol{\sigma}_z(\boldsymbol{f}-1-p,\boldsymbol{f}-3-q,z+1,t)f_I\wedge\mathbf{f}\otimes\left[(\textstyle\bigwedge^{1+t}X)[\varphi_I(b_p)]\right](u\wedge\delta_q)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},\ \text{and}$$

$$T_{28} = \chi(r=0)\chi(p+q=\boldsymbol{f}+z-2)\theta_{\boldsymbol{f}-1-p}\theta_{\boldsymbol{f}-3-q}(-1)^{q\boldsymbol{f}+p}\sum_{0\le t}\sum_{|I|=p-t}$$
$$\boldsymbol{\sigma}_z(\boldsymbol{f}-1-p,\boldsymbol{f}-3-q,z+1,t)f_I\otimes\left[(\textstyle\bigwedge^{t}X)[\varphi_I(b_p)]\right](u\wedge\delta_q)\otimes\mu^{(t)}.$$

Observe that $S_2=T_4$, $S_5=T_3$, $S_8=T_{11}+T_{26}$, $S_{10}=T_7+T_{13}+T_{23}$, S_{11} is equal to $T_6+T_{14}+T_{22}$, and $S_{12}=T_5+T_{15}+T_{21}$. Use Proposition 1.1.a to see that $S_7=T_9+T_{10}+T_{25}$, $S_9=T_8+T_{12}+T_{28}$, $S_1+S_3=T_1+T_{17}+T_{20}+T_{27}$, and $S_4+S_6=T_2+T_{18}+T_{19}+T_{24}$. The argument which shows that $T_3-T_6=0$ in the proof of Lemma 9.4 yields that $T_{16}=0$.

Finally, we take $x=\alpha_p\otimes c_q\otimes\lambda^{(r)}\in\mathbb{T}(p,q,r)\subseteq\mathbb{Y}[1]$, with
$$p+q+r\le\boldsymbol{f}-2\quad\text{and}\quad z+1\le r.$$
We have $\widehat{\Psi}\circ y(x)=\sum_{i=1}^{16}S_i$, with

$$S_1=(-1)^{p+1}v(\alpha_p)\otimes c_q\otimes\lambda^{(r+1)},$$

$$S_2=\chi(1\le q+r)(-1)^{p+q}\alpha_p\otimes u(c_q)\otimes\lambda^{(r+1)},$$

$$S_3=\chi(z+2\le r)\chi(1\le q+r)\alpha_p\wedge[X^*(u)]\otimes c_q\otimes\lambda^{(r)},$$

$$S_4=\chi(z+2\le r)(-1)^p\alpha_p\otimes c_q\wedge[X(v)]\otimes\lambda^{(r)},$$

$$S_5=(-1)^p\chi(p+q+2r\le\boldsymbol{f}+z-1)\sum_{|J|=r-z}\varphi_J\wedge v(\alpha_p)\otimes c_q\wedge(\textstyle\bigwedge^{r-z}X)(f_J)\otimes\lambda^{(z+1)},$$

$$S_6=\chi(1\le q+r)\chi(p+q+2r\le\boldsymbol{f}+z-1)(-1)^{p+q+1}\sum_{|J|=r-z}\varphi_J\wedge\alpha_p$$
$$\otimes u(c_q)\wedge(\textstyle\bigwedge^{r-z}X)(f_J)\otimes\lambda^{(z+1)},$$

$$S_7=-\chi(z+2\le r)\chi(p+q+2r\le\boldsymbol{f}+z-1)\sum_{|J|=r-z-1}\varphi_J\wedge\alpha_p\wedge[X^*(u)]$$
$$\otimes c_q\wedge(\textstyle\bigwedge^{r-1-z}X)(f_J)\otimes\lambda^{(z+1)},$$

$$S_8=\chi(z+2\le r)\chi(p+q+2r\le\boldsymbol{f}+z-1)(-1)^{p+1}\sum_{|J|=r-1-z}\varphi_J\wedge\alpha_p$$
$$\otimes c_q\wedge[X(v)]\wedge(\textstyle\bigwedge^{r-1-z}X)(f_J)\otimes\lambda^{(z+1)},$$

$$S_9=\sum_{\substack{0\le t\\|I|=q+r-t}}\sigma_z(p,q,r,t)\chi(p+q+2r-t\le\boldsymbol{f}-2+z)(-1)^{p+r-t-1}f_I\wedge\mathbf{f}$$
$$\otimes\left[(\textstyle\bigwedge^{\boldsymbol{f}-1-p+t-q-r}X)\left((\varphi_I\wedge\alpha_p)[\omega_F]\right)\wedge c_q\right](\omega_{G^*})\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$S_{10} = -\sigma_z(p,q,r,q+2r-\boldsymbol{f}-z+1+p)\chi(\boldsymbol{f}+z-1\le p+q+2r)\chi(q+r\le \boldsymbol{f}-3)$$
$$\sum_{\substack{p+q+2r-\boldsymbol{f}-z+2\le t\\ |J|=q+r+1-t\\ |I|=\boldsymbol{f}+z-1-p-r}} f_J\otimes\Big[(\textstyle\bigwedge^{t+\boldsymbol{f}-2-p-q-r}X)\Big(\varphi_J(f_I)\wedge(\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\Big](\omega_{G^*})\otimes\mu^{(t)},$$

$$S_{11} = -\sum_{|I|=q-p+1+z}\sigma_z(p,q,r,r+p-1-z)\chi(q+r=\boldsymbol{f}-2)f_I$$
$$\otimes\Big[(\textstyle\bigwedge^{\boldsymbol{f}-2-z-q}X)\Big((\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\Big](\omega_{G^*})\otimes\mu^{(r+p-z)},$$

$$S_{12} = \sigma_z(p,q,r,p+q+2r-\boldsymbol{f}-z+1)\chi(\boldsymbol{f}+z-1\le p+q+2r)\sum_{\substack{p+q+2r-\boldsymbol{f}-z+2\le t\\ |J|=q+r-t\\ |I|=\boldsymbol{f}+z-1-p-r}}(-1)^{t+p+r}$$
$$f_J\wedge\mathbf{f}\otimes\Big[(\textstyle\bigwedge^{t-p-q-r+\boldsymbol{f}-1}X)\Big(\varphi_J(f_I)\wedge(\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\Big](\omega_{G^*})\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$S_{13} = \sum_{|I|=q+r}\sigma_z(p,q,r,0)\chi(p+q+2r=\boldsymbol{f}+z-1)\theta_{\boldsymbol{f}-1-q-r}\theta_{\boldsymbol{f}-2-p-r}(-1)^{p\boldsymbol{f}+r\boldsymbol{f}+\boldsymbol{f}}$$
$$(-1)^{q+r}(v\wedge f_I)(\omega_{F^*})\otimes(\textstyle\bigwedge^{\boldsymbol{f}-1-p-q-r}X)\Big((\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\otimes\lambda^{(z+1)},$$

$$S_{14} = \sum_{|I|=q+r}\sigma_z(p,q,r,0)\chi(p+q+2r=\boldsymbol{f}+z-1)\theta_{\boldsymbol{f}-1-q-r}\theta_{\boldsymbol{f}-2-p-r}(-1)^{p\boldsymbol{f}+r\boldsymbol{f}+p+q}$$
$$f_I(\omega_{F^*})\otimes u\Big[(\textstyle\bigwedge^{\boldsymbol{f}-1-p-q-r}X)\Big((\varphi_I\wedge\alpha_p)[\omega_F]\Big)\wedge c_q\Big]\otimes\lambda^{(z+1)},$$

$$S_{15} = (-1)^p v_{\boldsymbol{f}}\alpha_p\otimes c_q\wedge\mathbf{g}\otimes\lambda^{(r)},\ \text{and}$$

$$S_{16} = u_{\boldsymbol{f}-1}\alpha_p\wedge\boldsymbol{\phi}\otimes c_q\otimes\lambda^{(r)}.$$

On the other hand, $\mathrm{proj}^{\mathbb{E}\oplus\mathbb{J}}\circ\mathbf{d}\circ(1-M_1)\circ\widehat{\Psi}(x)$ is equal to $\sum_{i=1}^{13}T_i$, with

$$T_1 = (-1)^{p-1}v(\alpha_p)\otimes c_q\otimes\lambda^{(r+1)},$$

$$T_2 = (-1)^{p+q}\alpha_p\otimes u(c_q)\otimes\lambda^{(r+1)},$$

$$T_3 = \alpha_p\wedge[X^*(u)]\otimes c_q\otimes\lambda^{(r)},$$

$$T_4 = (-1)^p\alpha_p\otimes c_q\wedge[X(v)]\otimes\lambda^{(r)},$$

$$T_5 = \sum_{0\le t}\sum_{|I|=q-t+r}(-1)^{p+q+r+1+t}\boldsymbol{\sigma}_z(p,q,r+1,t)f_I\wedge\mathbf{f}$$
$$\otimes\Big(\Big[(\textstyle\bigwedge^{\boldsymbol{f}-p+t-q-r-1}X)((\varphi_I\wedge\alpha_p)[\omega_F])\wedge c_q\Big](\omega_{G^*})\Big)\wedge\boldsymbol{\gamma}\otimes\mu^{(t)},$$

$$T_6 = u_{\boldsymbol{f}-1}\alpha_p\wedge\boldsymbol{\phi}\otimes c_q\otimes\lambda^{(r)},$$

$$T_7 = (-1)^p v_{\boldsymbol{f}}\alpha_p\otimes c_q\wedge\mathbf{g}\otimes\lambda^{(r)},$$

$$T_8 = \chi(q+r\le\boldsymbol{f}-3)\sum_{0\le t}\sum_{|I|=q-t+r+1}(-1)^q\boldsymbol{\sigma}_z(p,q,r+1,t)f_I$$
$$\otimes\Big[(\textstyle\bigwedge^{\boldsymbol{f}-p+t-q-r-2}X)((\varphi_I\wedge\alpha_p)[\omega_F])\wedge c_q\Big](\omega_{G^*})\otimes\mu^{(t)},$$

$$T_9 = \chi(q+r=\boldsymbol{f}-2)\sum_{|I|=q+z+1}(-1)^q\boldsymbol{\sigma}_z(p,q,r+1,r-z)f_I$$
$$\otimes\Big[(\textstyle\bigwedge^{\boldsymbol{f}-p-z-q-2}X)((\varphi_I\wedge\alpha_p)[\omega_F])\wedge c_q\Big](\omega_{G^*})\otimes\mu^{(r-z)},$$

$$T_{10} = \chi(p+q+2r\le\boldsymbol{f}+z-2)\sum_{|J|=r-z}(-1)^{p+r+z}v(\varphi_J\wedge\alpha_p)\otimes c_q\wedge(\textstyle\bigwedge^{r-z}X)(f_J)\otimes\lambda^{(z+1)},$$

$$T_{11} = \chi(p+q+2r \le \boldsymbol{f}+z-2) \sum_{|J|=r-z} (-1)^{p+q+1}\varphi_J \wedge \alpha_p \otimes u(c_q \wedge (\textstyle\bigwedge^{r-z} X)(f_J)) \otimes \lambda^{(z+1)},$$

$$T_{12} = \chi(p+q+2r \le \boldsymbol{f}+z-2) \sum_{\substack{|J|=r-z\\ 0\le t\\ |I|=q+r-t}} (-1)^{p+z+q+t}\boldsymbol{\sigma}_z(p+r-z, q+r-z, z+1, t) f_I \wedge \mathbf{f} \otimes$$
$$\Big[(\textstyle\bigwedge^{\boldsymbol{f}+z-1-p-q-2r+t} X)\left((\varphi_I \wedge \varphi_J \wedge \alpha_p)[\omega_F]\right) \wedge c_q \wedge (\textstyle\bigwedge^{r-z} X)(f_J)\Big](\omega_{G^*}) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)},$$

and

$$T_{13} = \chi(p+q+2r \le \boldsymbol{f}+z-2) \sum_{|J|=r-z} \sum_{\substack{0\le t\\ |I|=q+r-t+1}} (-1)^{q+r+z+1}\boldsymbol{\sigma}_z(p+r-z, q+r-z, z+1, t)$$
$$f_I \otimes \Big[(\textstyle\bigwedge^{\boldsymbol{f}+z-2-p-q-2r+t} X)\left((\varphi_I \wedge \varphi_J \wedge \alpha_p)[\omega_F]\right) \wedge c_q \wedge (\textstyle\bigwedge^{r-z} X)(f_J)\Big](\omega_{G^*}) \otimes \mu^{(t)}.$$

Observe that $S_1 = T_1$, $S_2 = T_2$, $S_{11} = T_9$ $S_{15} = T_7$, and $S_{16} = T_6$. Use parts (e) and (g) of Lemma 1.9 to see that $S_4 + S_5 + S_8 - T_4 - T_{10}$ and $-S_{13}$ are both equal to

$$(-1)^{p+r+z}\chi(p+q+2r = \boldsymbol{f}+z-1) \sum_{|J|=r-z} v(\varphi_J \wedge \alpha_p) \otimes c_q \wedge (\textstyle\bigwedge^{r-z} X)(f_J) \otimes \lambda^{(z+1)};$$

and that $S_3 + S_6 + S_7 - T_3 - T_{11}$ and $-S_{14}$ are both equal to

$$(-1)^{p+q+1}\chi(p+q+2r = \boldsymbol{f}+z-1) \sum_{|J|=r-z} \varphi_J \wedge \alpha_p \otimes u\left(c_q \wedge (\textstyle\bigwedge^{r-z} X)(f_J)\right) \otimes \lambda^{(z+1)}.$$

Apply part (c) of Lemma 1.9 to T_{12} and T_{13}, and part (f) to S_{12} and S_{10}, in order to see that $T_{12} - S_9 - S_{12}$ is equal to $-T_5$, and that $S_{10} - T_8$ and T_{13} are both equal to

$$\begin{cases} \chi(q+r \le \boldsymbol{f}-3)\chi(p+q+2r \le \boldsymbol{f}+z-2) \displaystyle\sum_{\substack{0\le t\\ |I|=q-t+r+1}} (-1)^{r+q+z+q\boldsymbol{f}}\theta_q\theta_p\tbinom{\boldsymbol{f}-2-p-q-r+t}{r-z} f_I \otimes \\ \qquad \Big[(\textstyle\bigwedge^{\boldsymbol{f}-p+t-q-r-2} X)\left((\varphi_I \wedge \alpha_p)[\omega_F]\right) \wedge c_q\Big](\omega_{G^*}) \otimes \mu^{(t)}. \quad \square \end{cases}$$

Lemma 9.17. *The map Ψ from Theorem 9.1 is an isomorphism of complexes.*

Proof. In light of Lemma 9.16 and (9.2) it suffices to show that $\Psi\colon \mathbb{Y} \to \mathbb{J}$ is a module isomorphism. The assertion is clear on the $\mathbb{L}$ level. We calculate that

$$\Psi\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[4]\right) = \begin{cases} \chi(1 \le r)\chi(z+1 \le q+r)(-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r-1)} \\ \qquad \in \mathbf{U}(p,q,r-1;1,1) \\ + b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbf{U}(p,q,r;0,0), \end{cases}$$
$$\Psi\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[3]\right) = b_p \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)} \in \mathbf{U}(p,q,r;0,1),$$
$$\Psi\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[2]\right) = (-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \otimes \mu^{(r)} \in \mathbf{U}(p,q,r;1,0),$$

and $\Psi\left(b_p \otimes \delta_q \otimes \mu^{(r)} \in \mathbb{U}[1]\right)$ is equal to

$$\begin{cases} \chi(p+q+r \le \boldsymbol{f}-3+z)(-1)^q b_p \wedge \mathbf{f} \otimes \delta_q \wedge \boldsymbol{\gamma} \otimes \mu^{(r)} \in \mathbf{U}(p,q,r;1,1) \\ -\chi(p+q+r=\boldsymbol{f}-2+z)\chi(p+r \le \boldsymbol{f}-3) \sum\limits_{\substack{r+1\le t \\ |I|=p+1+r-t}} f_I \otimes \left[\left(\bigwedge^{t-1-r} X\right)(\varphi_I(b_p))\right](\delta_q) \otimes \mu^{(t)} \\ \qquad \in \mathbf{U}(p+1+r-t, q+1+r-t, t; 0,0) \\ -\chi(p+q+r=\boldsymbol{f}-2+z)\chi(p+r=\boldsymbol{f}-2) b_p \otimes \delta_q \otimes \mu^{(r+1)} \in \mathbf{U}(p,q,r+1;0,0) \\ +\chi(p+q+r=\boldsymbol{f}-2+z) \sum\limits_{\substack{1+r\le t \\ |I|=p+r-t}} (-1)^{r+t+1+q} f_I \wedge \mathbf{f} \otimes \left[\left(\bigwedge^{t-r} X\right)(\varphi_I(b_p))\right](\delta_q) \wedge \boldsymbol{\gamma} \otimes \mu^{(t)} \\ \qquad \in \mathbf{U}(p+r-t, q+r-t, t; 1,1). \end{cases}$$

Observe that the set $S_{\mathbb{U}}(0)$ is equal to the disjoint union $A \cup B \cup C$ for

$$A = \{(p,q,r) \mid 0 \le r, \quad p+q+r \le f-2+z, \quad p+r \le f-2, \quad \text{and} \quad z \le q+r\},$$

$$B = \{(p,q,r) \mid 1 \le r, \quad p+q+r = f-1+z, \quad \text{and} \quad p+r \le f-2\}, \text{ and}$$

$$C = \{(p,q,r) \mid 1 \le r, \quad p+r = f-1, \quad \text{and} \quad z = q\}.$$

It follows that the map Ψ carries $\bigoplus \mathbb{U}[k] \to \mathbf{U}$ as follows:

$$\begin{array}{lcccccc} & \mathbb{U}[3] & \mathbb{U}[2] & \mathbb{U}[1] & \mathbb{U}[1] & \mathbb{U}[4] & \mathbb{U}[1] \\ & & & {\scriptstyle p+q+r\le \boldsymbol{f}-3+z} & {\scriptstyle p+q+r=\boldsymbol{f}-2+z} & & {\scriptstyle p+q+r=\boldsymbol{f}-2+z} \\ & & & & {\scriptstyle p+r=\boldsymbol{f}-2} & & {\scriptstyle p+r\le \boldsymbol{f}-3} \\ \mathbf{U}(0,1) & \cong & & & & & \\ \mathbf{U}(1,0) & & \cong & & & & \\ \mathbf{U}(1,1) & & & \cong & * & * & * \\ \mathbf{U}(0,0)\cap C & & & & \cong & & \\ \mathbf{U}(0,0)\cap A & & & & & \cong & * \\ \mathbf{U}(0,0)\cap B & & & & & & \cong, \end{array}$$

and this is an isomorphism. We calculate that

$$\Psi\left(\alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbb{T}[4]\right) = \begin{cases} \alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbf{T}(p,q,r;0,0) \\ +(-1)^{p+1}\chi(z+2 \le r)\alpha_p \wedge \boldsymbol{\phi} \otimes c_q \wedge \mathbf{g} \otimes \lambda^{(r-1)} \\ \qquad \in \mathbf{T}(p,q,r-1;1,1), \end{cases}$$

$$\Psi\left(\alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbb{T}[3]\right) = (-1)^{p+q}\alpha_p \wedge \boldsymbol{\phi} \otimes c_q \otimes \lambda^{(r)} \in \mathbf{T}(p,q,r;1,0),$$

$$\Psi\left(\alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbb{T}[2]\right) = (-1)^{1+q}\alpha_p \otimes c_q \wedge \mathbf{g} \otimes \lambda^{(r)} \in \mathbf{T}(p,q,r;0,1),$$

$$\Psi\left(\alpha_p \otimes c_q \otimes \lambda^{(r)} \in \mathbb{T}[1]\right) = \begin{cases} \alpha_p \otimes c_q \otimes \lambda^{(r+1)} \in \mathbf{T}(p,q,r+1;0,0) \\ -\chi(p+q+2r \le \boldsymbol{f}+z-3) \sum\limits_{|J|=r-z} \varphi_J \wedge \alpha_p \otimes c_q \wedge \left(\bigwedge^{r-z} X\right)(f_J) \\ \qquad \otimes \lambda^{(z+1)} \in \mathbf{T}(p+r-z, q+r-z, z+1; 0,0). \end{cases}$$

The map Ψ carries $\bigoplus \mathbb{T}[k] \to \mathbf{T}$ as follows:

$$\begin{array}{lccccc} & \mathbb{T}[3] & \mathbb{T}[2] & \mathbb{T}[4] & \mathbb{T}[1] & \mathbb{T}[4] \\ & & & z+1=r & & z+2\le r \\ \mathbf{T}(1,0) & \cong & & & & \\ \mathbf{T}(0,1) & & \cong & & & \\ \mathbf{T}(0,0),\ z+1=r & & & \cong & * & \\ \mathbf{T}(0,0),\ z=2\le r & & & & \cong & * \\ \mathbf{T}(1,1) & & & & & \cong, \end{array}$$

and this also is an isomorphism. $\square$

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